FRACTIONAL SKEW MONOID RINGS

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Dedicated to the memory of Dmitry Tyukavkin

Abstract. Given an action α of a monoid T on a ring A by ring endomorphisms, and an Ore subset S of T, a general construction of a fractional skew monoid ring $S^{\mathrm{op}} *_\alpha A *_\alpha T$ is given, extending the usual constructions of skew group rings and of skew semigroup rings. In case S is a subsemigroup of a group G such that $G = S^{-1}S$, we obtain a G-graded ring $S^{\mathrm{op}} *_\alpha A *_\alpha S$ with the property that, for each $s \in S$, the s-component contains a left invertible element and the s^{-1}-component contains a right invertible element. In the most basic case, where $G = \mathbb{Z}$ and $S = T = \mathbb{Z}^+$, the construction is fully determined by a single ring endomorphism α of A. If α is an isomorphism onto a proper corner pAp, we obtain an analogue of the usual skew Laurent polynomial ring, denoted by $A[t_+, t_-; \alpha]$. Examples of this construction are given, and it is proven that several classes of known algebras, including the Leavitt algebras of type $(1, n)$, can be presented in the form $A[t_+, t_-; \alpha]$. Finally, mild and reasonably natural conditions are obtained under which $S^{\mathrm{op}} *_\alpha A *_\alpha S$ is a purely infinite simple ring.

Introduction

Let $\alpha\colon G \to \mathrm{Aut}(A)$, $g \mapsto \alpha_g$, be an action of a group G on a unital ring A. A useful construction attached to these data is the *skew group ring* $A *_\alpha G$, see [18] and [20]. This is the ring of formal expressions $\sum_{g\in G} a_g g$, where $a_g \in A$ and almost all the coefficients a_g are 0. Addition is defined componentwise and multiplication is defined according to the rule $(ag)(bh) = (a\alpha_g(b))(gh)$. The skew group ring $A *_\alpha G$ can also be defined as the unital ring R such that there are a unital ring homomorphism $\phi\colon A \to R$ and a unital monoid homomorphism $i\colon G \to R$ from G to the multiplicative structure of R, universal with respect to the property that $i(g)\phi(a) = \phi(\alpha_g(a))i(g)$ for all $a \in A$ and all $g \in G$. In his pioneering paper [19], Paschke gave a construction of a C^*-algebraic crossed product $A \rtimes_\alpha \mathbb{N}$ associated to a not necessarily unital C^*-algebra endomorphism α on a C^*-algebra A. Paschke's C^*-algebraic construction has been generalized to other semigroups, see [13], [14], [15] and [16]. Moreover, Rørdam [22] used Paschke's construction together with the Pimsner-Voiculescu exact sequence associated to an automorphism [6, Theorem 10.2.1] to realize any pair of countable abelian groups (G_0, G_1) as $(K_0(B), K_1(B))$ for a certain purely infinite, simple, nuclear separable C^*-algebra B.

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In this paper, we develop a systematic purely algebraic theory of fractional skew monoid rings with respect to monoid actions on rings by not necessarily unital ring endomorphisms, in which an Ore submonoid is inverted. (Recall that a monoid is a semigroup with a neutral element.) More precisely, we assume the following data are given (see (1.1) for the detailed definitions of the properties):

(1) A monoid T acting on a unital ring A by endomorphisms;
(2) A submonoid S of T satisfying the left denominator conditions, and such that S is left saturated in T.

Then a fractional skew monoid ring $S^{\text{op}} *_\alpha A *_\alpha T$ is constructed, with suitable maps from A, S^{op} and T to $S^{\text{op}} *_\alpha A *_\alpha T$, which satisfy a universal property analogous to the one for the skew group ring described above, see Definition 1.2. It is not difficult to show that such a ring exists by using a construction with generators and relations, but it is rather non-obvious to determine the algebraic structure of $S^{\text{op}} *_\alpha A *_\alpha T$. The ring $S^{\text{op}} *_\alpha A *_\alpha T$ is best understood by means of its $S^{-1}T$-graded structure, obtained in Proposition 1.6. The structure is completely pinned down in (1.12) in the case where T acts by injective endomorphisms.

The general construction of $S^{\text{op}} *_\alpha A *_\alpha T$ is given in Section 1. In the other sections, we specialize the construction to the case of a submonoid S of a group G such that $G = S^{-1}S$ (taking $T = S$), and to an action α of S on A by corner isomorphisms, meaning that α_s is an isomorphism from A onto the corner ring $\alpha_s(1)A\alpha_s(1)$ for all $s \in S$. Several examples of interest are considered in Section 2 in the case where $S = T = \mathbb{Z}^+$. In particular, the Leavitt algebras $V_{1,n}(k)$ and $U_{1,n}(k)$, already considered by Leavitt, Skornyakov, Cohn, Bergman and others, are seen here to be particular cases of our construction.

For $S = T = \mathbb{Z}^+$, the construction is determined by a single corner isomorphism α, and the elements of the fractional skew monoid ring $R = \mathbb{Z}^+ *_\alpha A *_\alpha \mathbb{Z}^+$ can all be written as 'polynomials' of the form

$$r = a_n t_+^n + \ldots + a_1 t_+ + a_0 + t_- a_{-1} + \ldots t_-^m a_{-m},$$

with coefficients $a_i \in A$. Because of this similarity of R with a skew-Laurent polynomial ring, we shall use the notation $R = A[t_+, t_-; \alpha]$. Using this construction and the Bass-Heller-Swan-Farrell-Hsiang-Siebenmann Theorem, the K_1 group of these algebras is computed in [2].

A general source of interesting examples is provided in Section 3. Namely, assume that G is a group acting on a ring A by automorphisms, and that there are a submonoid S of G such that $G = S^{-1}S$ and a non-trivial idempotent e in A such that $\alpha_s(e) \in eAe$ for all $s \in S$. Then the corner ring $e(A *_\alpha G)e$ of the skew group ring $A *_\alpha G$ is isomorphic as a G-graded ring to a fractional skew monoid ring $S^{\text{op}} *_{\alpha'} (eAe) *_{\alpha'} S$ (Proposition 3.3). Under the standing assumption that S acts by corner isomorphisms, we prove that all $S^{\text{op}} *_\alpha A *_\alpha S$ can be exhibited in the form $e(A *_\alpha G)e$ (Proposition 3.8).

Sections 4 and 5 deal with actions on simple rings. Using a suitable definition of outer action of a monoid S on a ring A, we prove in Theorem 4.1 that $S^{\text{op}} *_\alpha A *_\alpha S$ is a simple ring for any outer action α of S on a simple ring A. This is a generalization of a well-known sufficient condition for simplicity of skew group rings, see [18, Theorem 2.3]. Section 5 shows that, under mild conditions on A and on the outer action α of S on A, the fractional skew monoid ring $S^{\text{op}} *_\alpha A *_\alpha S$ is a purely infinite simple ring (Theorem 5.3). In particular, this holds whenever A is either a simple ultramatricial algebra over some field or a purely infinite simple ring. The class of purely infinite simple rings has been recently studied by the first, third and fourth authors in [3], and constitute an important and large class of relatively well-behaved simple rings. They can be thought of as the nice rings in the wild universe of the directly infinite simple rings; see specially [3, Corollary 2.2 and Theorem 2.3] for the good behaviour of K-theory of purely infinite simple rings. A further nice property of them has been recently established by the first author in [1]: Every purely infinite simple ring satisfies the exchange property.

All rings and modules in this paper will be assumed to be unital unless explicitly noted. (The main exception is the ring $S^{-1}A$ constructed in Section 3.) However, many of the subrings we deal with will have units different from the unit of the larger ring; specifically, we will deal with many *corners* pAp in a ring A, where p is an idempotent. Note that any ring endomorphism ε of A, even if not unital when considered as a map $A \to A$, is unital when viewed as a ring homomorphism $A \to \varepsilon(1)A\varepsilon(1)$.

We will use the standard order structure on the set of idempotents in a ring A; that is, for idempotents e and f in A, we have $e \le f$ if and only if $e = ef = fe$. Two idempotents e and f are said to be *equivalent*, written $e \sim f$, if there are elements $x, y \in A$ such that $e = xy$ and $f = yx$. This is equivalent to saying that the right A-modules eA and fA are isomorphic. We write $e \lesssim f$ in case $e \sim f'$ for some idempotent $f' \le f$. We say that an idempotent e is *infinite* if there are nonzero orthogonal idempotents e' and g such that $e = e' + g$ and $e \sim e'$. If no such decomposition exists, e is called a *finite* idempotent.

1. The general construction

We present the construction of a fractional skew monoid ring in full generality in this section, and establish the precise graded structure of this ring. The basic data consist of a ring A, a monoid T acting on A by ring endomorphisms, and a left denominator set $S \subseteq T$; the fractional skew monoid ring we construct is graded by $S^{-1}T$, and its identity component is the quotient of A modulo the union of the kernels of the endomorphisms by which S acts.

1.1. We begin by fixing the basic data needed for our construction; these data and conventions will remain in force throughout the paper. Let A be a (unital) ring, and $\mathrm{Endr}(A)$ the monoid of non-unital (i.e., not necessarily unital) ring endomorphisms of A.

Let T be a monoid and $\alpha : T \to \mathrm{Endr}(A)$ a monoid homomorphism, written $t \mapsto \alpha_t$. In general, we will write T multiplicatively, with its identity element denoted 1, but in some applications it will be convenient to switch to additive notation for T. For $t \in T$, set $p_t = \alpha_t(1)$, an idempotent in A. Then α_t can be viewed as a unital ring homomorphism from A to the corner p_tAp_t. For $s, t \in T$, we have $p_{st} = \alpha_{st}(1) = \alpha_s\alpha_t(1) = \alpha_s(p_t)$.

Let $S \subseteq T$ be a submonoid satisfying the left denominator conditions, i.e., the left Ore condition and the monoid version of left reversibility: whenever $t, u \in T$ with $ts = us$ for some $s \in S$, there exists $s' \in S$ such that $s't = s'u$. Then there exists a monoid of fractions, $S^{-1}T$, with the usual properties (e.g., see [7, §1.10] or [8, §0.8]).

We shall also assume that S is *left saturated* in T: whenever $s \in S$ and $t \in T$ such that $ts \in S$, we must have $t \in S$. This assumption means that equality in $S^{-1}T$ can be described as follows: if $s_1^{-1}t_1 = s_2^{-1}t_2$ for some $s_i \in S$ and $t_i \in T$, there exist $u_1, u_2 \in S$ such that $u_1s_1 = u_2s_2$ and $u_1t_1 = u_2t_2$. (The usual denominator conditions only yield the latter equations for, say, some $u_1 \in S$ and $u_2 \in T$. But then $u_2s_2 = u_1s_1 \in S$, and left saturation implies $u_2 \in S$.)

Definition 1.2. The label $S^{\mathrm{op}} *_\alpha A *_\alpha T$ stands for a (unital) ring R equipped with a (unital) ring homomorphism $\phi : A \to R$ and monoid homomorphisms $s \mapsto s_-$ from $S^{\mathrm{op}} \to R$ and $t \mapsto t_+$ from $T \to R$, universal with respect to the following relations:

(1) $t_+\phi(a) = \phi\alpha_t(a)t_+$ for all $a \in A$ and $t \in T$;
(2) $\phi(a)s_- = s_-\phi\alpha_s(a)$ for all $a \in A$ and $s \in S$;
(3) $s_-s_+ = 1$ for all $s \in S$;
(4) $s_+s_- = \phi(p_s)$ for all $s \in S$.

Note that condition (2) follows from the others. Given $a \in A$ and $s \in S$, we have $s_+\phi(a) = \phi\alpha_s(a)s_+$ by (1), and on multiplying each term of this equation on the left and on the right by s_-, we obtain $\phi(a)s_- = s_-\phi\alpha_s(a)\phi(p_s) = s_-\phi(\alpha_s(a)p_s)$ from (3) and (4), whence (2) follows because $\alpha_s(a)p_s = \alpha_s(a)$.

1.3. At this point, we sketch the *existence* of the ring $R = S^{\mathrm{op}}*_\alpha A*_\alpha T$. The existence of a ring satisfying the universal property of Definition 1.2 follows from a construction with generators and relations, which does not use at all any property of S; in fact, S can be an arbitrary subset of T. Take $B = A * \mathbb{Z}\langle t_+, s_- \mid t \in T,\ s \in S\rangle$ to be the free product of A and the free ring on the disjoint union $T \sqcup S$, and let $i_1 \colon A \to B$ and $i_2 \colon \mathbb{Z}\langle t_+, s_- \mid t \in T,\ s \in S\rangle \to B$ be the canonical maps. Let J be the two-sided ideal of B generated by

(a) $i_2(t_+)i_1(a) - i_1(\alpha_t(a))i_2(t_+)$ for all $a \in A$ and $t \in T$;
(b) $i_2((tt')_+) - i_2(t_+)i_2(t'_+)$ for all $t, t' \in T$;
(c) $i_2(s_-)i_2(s_+) - i_1(1)$ for all $s \in S$;
(d) $i_2(s_+)i_2(s_-) - i_1(p_s)$ for all $s \in S$.

Then $R = B/J$ is the ring we are looking for, and ϕ is the composite map $\pi \circ i_1$, where $\pi \colon B \to B/J$ is the canonical projection. (Here we identify the elements s_-, for $s \in S$, with their images $\pi i_2(s_-)$, and similarly for the elements t_+.) Note that the relations $(ss')_- = s'_-s_-$, for all $s, s' \in S$ such that $ss' \in S$, hold automatically from (a)–(d) above. Also, we have already observed that condition (2) in 1.2 follows from conditions (1),(3) and (4), and so it follows from (a)–(d) too.

Rather than introduce a notation for the product in S^{op}, we view the map $(-)_-$ as a monoid anti-homomorphism $S \to R$, so that $(su)_- = u_-s_-$ for $s, u \in S$.

The construction above will also be applied when A is an algebra over a field k and the ring endomorphisms α_t for $t \in T$ are k-linear. In this case, it is easily checked that ϕ maps $k = k\cdot 1$ into the center of B (use relations (1),(2) above and part (c) of the following lemma to see that $\phi(k)$ commutes with each s_- and t_+), so that B becomes a k-algebra and ϕ becomes a k-algebra homomorphism. The universal property of B then holds also in the category of k-algebras.

The following lemma and subsequent results pin down the structure of $R = S^{\mathrm{op}} *_\alpha A *_\alpha T$. This structure simplifies considerably when the maps α_s are injective – see (1.12).

Lemma 1.4. *Let $a, b \in A$, $s, u \in S$, and $t, v \in T$.*

(a) $s_+\phi(a)s_- = \phi\alpha_s(a)$.
(b) $s_-\phi\alpha_s(a)s_+ = \phi(a)$.
(c) $s_- = s_-\phi(p_s)$ *and* $t_+ = \phi(p_t)t_+$.

(d) $s_-\phi(a)t_+ = s_-\phi(p_s a p_t)t_+$.
(e) $s_-\phi(a)t_+ = (us)_-\phi\alpha_u(a)(ut)_+$.
(f) *There exist* $x \in S$ *and* $y \in T$ *such that* $xt = yu$. *For any such* x, y,

$$\bigl[s_-\phi(a)t_+\bigr]\bigl[u_-\phi(b)v_+\bigr] = (xs)_-\phi\bigl(\alpha_x(ap_t)\alpha_y(b)\bigr)(yv)_+.$$

In particular, $t_+u_- = x_-p_{xt}y_+$.

Proof. (a) $s_+\phi(a)s_- = \phi\alpha_s(a)s_+s_- = \phi\alpha_s(a)\phi(p_s) = \phi(\alpha_s(a)p_s) = \phi\alpha_s(a)$.

(b) This follows from (a) because $s_-s_+ = 1$.

(c) $s_- = \phi(1)s_- = s_-\phi\alpha_s(1) = s_-\phi(p_s)$. Similarly, $t_+ = t_+\phi(1) = \phi\alpha_t(1)t_+ = \phi(p_t)t_+$.

(d) This is clear from (c).

(e) From (b), we have $\phi(a) = u_-\phi\alpha_u(a)u_+$, and the desired equation follows because $s_-u_- = (us)_-$.

(f) Note that $(xt)_+(yu)_- = (xt)_+(xt)_- = \phi(p_{xt}) = \phi\alpha_x(p_t)$. Using (e), we get

$$\begin{aligned}\bigl[s_-\phi(a)t_+\bigr]\bigl[u_-\phi(b)v_+\bigr] &= \bigl[(xs)_-\phi\alpha_x(a)(xt)_+\bigr]\bigl[(yu)_-\phi\alpha_y(b)(yv)_+\bigr]\\ &= (xs)_-\phi\alpha_x(a)\phi\alpha_x(p_t)\phi\alpha_y(b)(yv)_+\\ &= (xs)_-\phi\bigl(\alpha_x(ap_t)\alpha_y(b)\bigr)(yv)_+. \qquad \square\end{aligned}$$

Corollary 1.5. $R = \sum_{s\in S,\, t\in T} s_-\phi(A)t_+ = \sum_{s\in S,\, t\in T} s_-\phi(p_sAp_t)t_+$.

Proof. The second equality is clear from Lemma 1.4(d). Let R' denote the sum in question. Clearly R' is closed under addition, and it is closed under multiplication by Lemma 1.4(f). Also, $1_R = 1_-\phi(1_A)1_+ \in R'$. Thus, R' is a unital subring of R.

Since the images of ϕ, $s \mapsto s_-$, and $t \mapsto t_+$ are contained in R', we can view these as maps into R'. The universal property for R then implies that there is a unique unital ring homomorphism $\psi : R \to R'$ such that $\psi\phi = \phi$ while $\psi(s_-) = s_-$ for $s \in S$ and $\psi(t_+) = t_+$ for $t \in T$. Consequently, ψ acts as the identity on R', whence $\psi(R) = R'$. Moreover, if we view ψ as a ring homomorphism $R \to R$, we have $\psi\phi = \mathrm{id}_R\phi$ while $\psi(s_-) = \mathrm{id}_R(s_-)$ for $s \in S$ and $\psi(t_+) = \mathrm{id}_R(t_+)$ for $t \in T$. Now the universal property for R implies that $\psi = \mathrm{id}_R$, and therefore $R = \psi(R) = R'$. $\square$

We next exhibit the graded ring structure of R. As the reader will note, this result can also be obtained from the proof of Proposition 1.10 below, and so Propositions 1.6 and 1.10 could have been combined. However, we think that separating the two results is helpful in orienting the reader.

Proposition 1.6. *The ring* R *has an* $S^{-1}T$*-grading* $R = \bigoplus_{x\in S^{-1}T} R_x$ *where each* $R_x = \bigcup_{s^{-1}t=x} s_-\phi(A)t_+$.

Proof. We can view R as a left A-module via ϕ, and the relations in R imply that each $s_-\phi(A)t_+$ is a left A-submodule. If $s_1, s_2 \in S$ and $t_1, t_2 \in T$ such that $s_1^{-1}t_1 = s_2^{-1}t_2$, there exist $u_1, u_2 \in S$ such that $u_1s_1 = u_2s_2$ and $u_1t_1 = u_2t_2$, whence Lemma 1.4(e) implies that $(s_i)_-\phi(A)(t_i)_+ \subseteq (u_1s_1)_-\phi(A)(u_1t_1)_+$ for $i = 1, 2$. Thus, each R_x is a directed union of left A-submodules of R, and so is a left A-submodule itself.

It is clear from Corollary 1.5 that $R = \sum_{x\in S^{-1}T} R_x$, and from Lemma 1.4(f) that $R_xR_y \subseteq R_{xy}$ for all $x, y \in S^{-1}T$. Hence, it only remains to show that the sum of the R_x is a direct sum.

Let R' denote the external direct sum of the R_x, and set $E' = \mathrm{End}_{\mathbb{Z}}(R')$. There is a unital ring homomorphism $\lambda : A \to E'$ such that each $\lambda(a)$ is the left A-module multiplication by $a \in A$.

Given $s \in S$, observe that $s_- R_x \subseteq R_{s^{-1}x}$ for all $x \in S^{-1}T$. Hence, there exists $\mu_s \in E'$ such that $\mu_s(b)_y = s_- b_{sy}$ for all $b \in R'$ and $y \in S^{-1}T$. Since $\phi(a)s_- = s_-\phi\alpha_s(a)$ for $a \in A$, we see that $\lambda(a)\mu_s = \mu_s\lambda\alpha_s(a)$ for $a \in A$. Observe also that $s \mapsto \mu_s$ is a monoid homomorphism $S^{\mathrm{op}} \to E'$.

Given $t \in T$, it follows from Lemma 1.4(f) that $t_+ R_x \subseteq R_{tx}$ for all $x \in S^{-1}T$. Hence, there exists $\nu_t \in E'$ such that $\nu_t(b)_y = \sum_{tx=y} t_+ b_x$ for $b \in R'$ and $y \in S^{-1}T$. Since $t_+\phi(a) = \phi\alpha_t(a)t_+$ for $a \in A$, we see that $\nu_t\lambda(a) = \lambda\alpha_t(a)\nu_t$ for $a \in A$. Observe also that $t \mapsto \nu_t$ is a monoid homomorphism $T \to E'$.

Since $s_- s_+ = 1$ and $s_+ s_- = \phi(p_s)$ for $s \in S$, we see that $\mu_s\nu_s = \mathrm{id}_{R'} = 1_{E'}$ and $\nu_s\mu_s = \lambda(p_s)$ for $s \in S$. Now by the universal property of R, there exists a unital ring homomorphism $\psi : R \to E'$ such that $\psi\phi = \lambda$ while $\psi(s_-) = \mu_s$ for $s \in S$ and $\psi(t_+) = \nu_t$ for $t \in T$.

Note that $1_R = 1_-\phi(1)1_+ \in R_1$, so there exists $e \in R'$ such that $e_1 = 1$ while $e_z = 0$ for all $z \neq 1$. Given $s \in S$, $a \in A$, and $t \in T$, we observe that

$$\big[\psi(s_-\phi(a)t_+)(e)\big]_{s^{-1}t} = \big[\mu_s\lambda(a)\nu_t(e)\big]_{s^{-1}t} = s_-\phi(a)t_+$$

and all other components of $\psi(s_-\phi(a)t_+)(e)$ are zero. Hence, for $x \in S^{-1}T$ and $b \in R_x$, we have $[\psi(b)(e)]_x = b$ while $[\psi(b)(e)]_y = 0$ for all $y \neq x$. Consequently, if $b_1 + \cdots + b_n = 0$ for some $b_i \in R_{x_i}$ where the x_i are distinct elements of $S^{-1}T$, then $b_i = [\psi(b_1 + \cdots + b_n)(e)]_{x_i} = 0$ for all i. Therefore $\sum_{x\in S^{-1}T} R_x = \bigoplus_{x\in S^{-1}T} R_x$, as desired. □

To completely pin down the elements of R, we need to know the relations holding in each homogeneous component R_x. In particular, if $p_s a p_t \in \ker(\phi)$, then $s_-\phi(a)t_+ = 0$ by Lemma 1.4(d), and we would like to show that $s_-\phi(a)t_+ = 0$ only when $p_s a p_t \in \ker(\phi)$. For this purpose, we set up another representation of R on a left A-module.

Lemma 1.7. *Let $u, s \in S$ and $t \in T$.*

(a) *The map $* : A \times p_sAp_t \to p_sAp_t$ given by the rule $a * b := \alpha_s(a)b$ turns the abelian group p_sAp_t into a left A-module.*

(b) *The restriction of α_u to p_sAp_t is a left A-module homomorphism $p_sAp_t \to p_{us}Ap_{ut}$.*

Proof. Part (a) is clear because α_s is a unital ring homomorphism from A to p_sAp_s, while part (b) follows because $\alpha_{us} = \alpha_u\alpha_s$. □

Each homogeneous component R_x of R turns out to be a direct limit of the rectangular corners p_sAp_t over pairs (s,t) such that $s^{-1}t = x$. However, there is no natural partial order on the set of these pairs – the limit has to be taken over a small category.

Definition 1.8. For $x \in S^{-1}T$, let $\mathcal{D}_x$ be the small category in which the objects are all pairs $(s,t) \in S \times T$ such that $s^{-1}t = x$, the morphisms from an object (s,t) to an object (s',t') are those elements $u \in S$ such that $us = s'$ and $ut = t'$, and composition of morphisms is given by the multiplication in S. The Ore and saturation conditions on S imply that $\mathcal{D}_x$ is directed: given any objects (s_1,t_1) and (s_2,t_2) in $\mathcal{D}_x$, there exist an object (s,t) and morphisms $u_i : (s_i,t_i) \to (s,t)$ in $\mathcal{D}_x$ for $i = 1,2$. Consequently, colimits based on $\mathcal{D}_x$ are directed colimits.

Taking account of Lemma 1.7, there is a functor $F_x : \mathcal{D}_x \to A$-Mod such that $F_x(s,t) = p_s A p_t$ for all objects (s,t) in $\mathcal{D}_x$ and $F_x(u) = \alpha_u|_{p_s A p_t}$ for all morphisms $u : (s,t) \to (us, ut)$ in $\mathcal{D}_x$. Let M_x denote the colimit of F_x, with natural maps $\eta_{s,t} : p_s A p_t \to M_x$ for objects (s,t) in $\mathcal{D}_x$. Since M_x is a directed colimit, it is the union of its submodules $\eta_{s,t}(p_s A p_t)$ for $(s,t) \in \mathcal{D}_x$. Note that if $b_i \in p_{s_i} A p_{t_i}$ for $i = 1, 2$, where $(s_i, t_i) \in D_x$, then $\eta_{s_1,t_1}(b_1) = \eta_{s_2,t_2}(b_2)$ if and only if there exist $u_1, u_2 \in S$ such that $u_1 s_1 = u_2 s_2$ and $u_1 t_1 = u_2 t_2$ while also $\alpha_{u_1}(b_1) = \alpha_{u_2}(b_2)$.

Lemma 1.9. *Let $s \in S$, $t \in T$, and $x \in S^{-1}T$.*

(a) *There exists an additive map $\sigma_s : M_x \to M_{s^{-1}x}$ such that $\sigma_s \eta_{u,v}(b) = \eta_{us,v}(p_{us} b)$ for $u^{-1}v = x$ and $b \in p_u A p_v$.*

(b) *$a\sigma_s(m) = \sigma_s(\alpha_s(a)m)$ for $a \in A$ and $m \in M_x$.*

(c) *There exists an additive map $\tau_t : M_x \to M_{tx}$ such that $\tau_t \eta_{u,v}(b) = \eta_{w,zv}\alpha_z(b)$ for $u^{-1}v = x$, $b \in p_u A p_v$, and $w \in S$, $z \in T$ such that $wt = zu$.*

(d) *$\tau_t(am) = \alpha_t(a)\tau_t(m)$ for $a \in A$ and $m \in M_x$.*

Proof. (a) For each $(u,v) \in \mathcal{D}_x$, we have $(us, v) \in \mathcal{D}_{s^{-1}x}$, and there is an additive map $p_u A p_v \to M_{s^{-1}x}$ given by $b \mapsto \eta_{us,v}(p_{us} b)$. Moreover, if $w \in S$ then $\eta_{wus,wv}(p_{wus}\alpha_w(b)) = \eta_{wus,wv}\alpha_w(p_{us}b) = \eta_{us,v}(p_{us}b)$. Thus, our maps to $M_{s^{-1}x}$ are compatible with the functor F_x, and so there exists a unique additive map σ_s as described.

(b) If $m = \eta_{u,v}(b)$ for u, v, b as in (a), then

$$\begin{aligned} a\sigma_s(m) &= a\eta_{us,v}(p_{us}b) = \eta_{us,v}(a * (p_{us}b)) = \eta_{us,v}(\alpha_{us}(a)p_{us}b) = \eta_{us,v}(p_{us}\alpha_{us}(a)b) \\ &= \eta_{us,v}(p_{us}(\alpha_s(a) * b)) = \sigma_s\eta_{u,v}(\alpha_s(a) * b) = \sigma_s(\alpha_s(a)m). \end{aligned}$$

(c) Fix $(u,v) \in \mathcal{D}_x$, choose $w \in S$, $z \in T$ such that $wt = zu$, and note that $tx = w^{-1}zv$. Since $\alpha_z(p_u A p_v) \subseteq p_{zu} A p_{zv} \subseteq p_w A p_{zv}$, the composition of $\eta_{w,zv}$ with the restriction of α_z to $p_u A p_v$ gives an additive map $p_u A p_v \to M_{tx}$. Suppose also $w_1 \in S$ and $z_1 \in T$ such that $w_1 t = z_1 u$. Then $w_1^{-1} z_1 = tu^{-1} = w^{-1}z$, so there exist $r_1, r \in S$ such that $r_1 w_1 = rw$ and $r_1 z_1 = rz$. Since also $r_1 z_1 v = rzv$ and $\alpha_{r_1}\alpha_{z_1} = \alpha_r \alpha_z$, it follows that $\eta_{w_1, z_1 v}\alpha_{z_1} = \eta_{w,zv}\alpha_z$ on $p_u A p_v$. Thus, we obtain a well-defined additive map $f_{u,v} : p_u A p_v \to M_{tx}$ which agrees with $\eta_{w,zv}\alpha_z$ for any $w \in S$ and $z \in T$ with $wt = zu$.

Now consider a morphism $r : (u,v) \to (ru, rv)$ in $\mathcal{D}_x$. There exist $w \in S$ and $z \in T$ such that $wt = z(ru)$, so that $f_{ru,rv}$ is given by $\eta_{w,zrv}\alpha_z$. Since $wt = (zr)u$, we also have that $f_{u,v}$ is given by $\eta_{w,zrv}\alpha_{zr}$, and so $f_{u,v}$ equals the composition of $f_{ru,rv}$ with the restriction of α_r to $p_u A p_v$. Thus, the maps $f_{\cdot,\cdot}$ are compatible with F_x, and so there exists a unique additive map τ_t as described.

(d) If $m = \eta_{u,v}(b)$ with u, v, b, w, z as in (c), then

$$\begin{aligned} \tau_t(am) &= \tau_t\eta_{u,v}(a * b) = \tau_t\eta_{u,v}(\alpha_u(a)b) = \eta_{w,zv}\alpha_z(\alpha_u(a)b) = \eta_{w,zv}(\alpha_w\alpha_t(a)\alpha_z(b)) \\ &= \eta_{w,zv}(\alpha_t(a) * \alpha_z(b)) = \alpha_t(a)\eta_{w,zv}\alpha_z(b) = \alpha_t(a)\tau_t(m). \qquad \square \end{aligned}$$

Proposition 1.10. *For each $x \in S^{-1}T$, there is a left A-module isomorphism $\theta_x : M_x \to R_x$ such that $\theta_x\eta_{u,v}(b) = u_-\phi(b)v_+$ for $u^{-1}v = x$ and $b \in p_u A p_v$.*

Proof. In view of Lemma 1.4(e), for each $x \in S^{-1}T$ there is a unique additive map $\theta_x : M_x \to R_x$ as described. If $m = \eta_{u,v}(b)$ with u, v, b as above, then for $a \in A$ we have

$$\theta_x(am) = \theta_x\eta_{u,v}(a * b) = \theta_x\eta_{u,v}(\alpha_u(a)b) = u_-\phi\alpha_u(a)\phi(b)v_+ = \phi(a)u_-\phi(b)v_+ = a\theta_x(m).$$

Thus, θ_x is a left A-module homomorphism. It is surjective by definition of R_x, and so it only remains to show that $\ker(\theta_x) = 0$.

Form the left A-module $M := \bigoplus_{x \in S^{-1}T} M_x$, set $E = \mathrm{End}_{\mathbb{Z}}(M)$, and for each $a \in A$ let $\lambda(a) \in E$ be the map given by left multiplication by a. Then we have a unital ring homomorphism $\lambda : A \to E$.

For all $x \in S^{-1}T$, use the same notations σ_s and τ_t for the additive maps $M_x \longrightarrow M_{s^{-1}x}$ and $M_x \longrightarrow M_{tx}$ described in Lemma 1.9, and also for the corresponding homogeneous maps on M. Thus, for $s \in S$ and $t \in T$ we have additive maps $\sigma_s, \tau_t \in E$ such that $\sigma_s(m)_y = \sigma_s(m_{sy})$ and $\tau_t(m)_y = \sum_{tx=y} \tau_t(m_x)$ for $m \in M$ and $y \in S^{-1}T$. Lemma 1.9 also shows that $\lambda(a)\sigma_s = \sigma_s \lambda \alpha_s(a)$ and $\tau_t \lambda(a) = \lambda \alpha_t(a) \tau_t$ for $a \in A$.

It is easily checked that $s \mapsto \sigma_s$ and $t \mapsto \tau_t$ are monoid homomorphisms $S^{\mathrm{op}} \to E$ and $T \to E$. Now consider $m = \eta_{u,v}(b) \in M_x$ for x, u, v, b as in Lemma 1.9. There exist $w \in S$ and $z \in T$ such that $ws = zu$, and

$$\begin{aligned} \sigma_s \tau_s(m) &= \sigma_s \eta_{w,zv} \alpha_z(b) = \eta_{ws,zv}(p_{ws} \alpha_z(b)) = \eta_{zu,zv}(p_{zu} \alpha_z(b)) \\ &= \eta_{zu,zv} \alpha_z(p_u b) = \eta_{u,v}(b) = m. \end{aligned}$$

It follows that $\sigma_s \tau_s = 1_E$ in E. Next, note that $u \in S$ and $1 \in T$ with $u \cdot s = 1 \cdot us$. Hence,

$$\tau_s \sigma_s(m) = \tau_s \eta_{us,v}(p_{us} b) = \eta_{u,v} \alpha_1(p_{us} b) = \eta_{u,v}(p_s * b) = p_s m.$$

It follows that $\tau_s \sigma_s = \lambda(p_s)$ in E.

By the universal property of R, there is a unital ring homomorphism $\psi : R \to E$ such that $\psi\phi = \lambda$ while $\psi(s_-) = \sigma_s$ for $s \in S$ and $\psi(t_+) = \tau_t$ for $t \in T$.

Define $e \in M$ so that $e_1 = \eta_{1,1}(1)$ while $e_z = 0$ for all $z \neq 1$. We claim that $[(\psi\theta_x(m))(e)]_x = m$ for $x \in S^{-1}T$ and $m \in M_x$. Write $m = \eta_{u,v}(b)$ where $u^{-1}v = x$ and $b \in p_u A p_v$. Then $\psi\theta_x(m) = \psi(u_- \phi(b) v_+) = \sigma_u \lambda(b) \tau_v$ and so

$$\begin{aligned} [(\psi\theta_x(m))(e)]_x &= \sigma_u \lambda(b) \tau_v \eta_{1,1}(1) = \sigma_u \lambda(b) \eta_{1,v} \alpha_v(1) = \sigma_u \eta_{1,v}(b * p_v) \\ &= \sigma_u \eta_{1,v}(b) = \eta_{u,v}(p_u b) = \eta_{u,v}(b) = m, \end{aligned}$$

as claimed.

The claim immediately implies that $\ker(\theta_x) = 0$ for all $x \in S^{-1}T$, as desired. □

Corollary 1.11. (a) *Let $s \in S$, $t \in T$, and $a \in A$. Then $s_- \phi(a) t_+ = 0$ if and only if $p_s a p_t \in \ker(\alpha_{s'})$ for some $s' \in S$. In particular, $\ker(\phi) = \bigcup_{s' \in S} \ker(\alpha_{s'})$.*

(b) *The ideal $I = \ker(\phi)$ satisfies $\alpha_s^{-1}(I) = I$ for all $s \in S$ and $\alpha_t(I) \subseteq I$ for all $t \in T$.*

(c) *α induces a monoid homomorphism $\alpha' : T \to \mathrm{End}_{\mathbb{Z}}(A/I)$, and α'_s is injective for all $s \in S$.*

(d) *$S^{\mathrm{op}} *_\alpha A *_\alpha T = S^{\mathrm{op}} *_{\alpha'} (A/I) *_{\alpha'} T$.*

Proof. (a) By Lemma 1.4(d), $s_- \phi(a) t_+ = s_- \phi(b) t_+$ where $b = p_s a p_t$. Then Proposition 1.10 yields $\theta_x \eta_{s,t}(b) = s_- \phi(a) t_+$ where $x = s^{-1}t$. Since θ_x is an isomorphism, $s_- \phi(a) t_+ = 0$ if and only if $\eta_{s,t}(b) = 0$, which happens if and only if $\alpha_{s'}(b) = 0$ for some $s' \in S$. This verifies the first statement in (a). The second follows on taking $s = t = 1$.

(b) If $t \in T$ and $s \in S$, there exist $s' \in S$ and $t' \in T$ such that $s't = t's$. Then $\alpha_{s'} \alpha_t(\ker(\alpha_s)) = 0$, and so $\alpha_t(\ker(\alpha_s)) \subseteq \ker(\alpha_{s'}) \subseteq I$. This shows that $\alpha_t(I) \subseteq I$ for all $t \in T$.

Now if $s \in S$, the previous paragraph implies that $I \subseteq \alpha_s^{-1}(I)$. If $a \in \alpha_s^{-1}(I)$, then $\alpha_s(a) \in \ker(\alpha_{s'})$ for some $s' \in S$, whence $a \in \ker(\alpha_{s's}) \subseteq I$. Therefore $\alpha_s^{-1}(I) = I$.

(c)(d) These are clear from (a) and (b). $\square$

1.12. As Corollary 1.11 shows, we can always reduce to the case where α_s is injective for all $s \in S$. In that case, ϕ is injective by Corollary 1.11(a), and so we can identify A with the unital subring $\phi(A) \subseteq R$. All of the relations in R simplify in this case:

(1) $t_+a = \alpha_t(a)t_+$ for all $a \in A$ and $t \in T$;
(2) $as_- = s_-\alpha_s(a)$ for all $a \in A$ and $s \in S$;
(3) $s_-s_+ = 1$ for all $s \in S$;
(4) $s_+s_- = p_s$ for all $s \in S$;
(5) R has an $S^{-1}T$-grading $R = \bigoplus_{x \in S^{-1}T} R_x$ where each $R_x = \bigcup_{s^{-1}t=x} s_-At_+$;
(6) $s_-at_+ = s_-p_sap_tt_+$ for $s \in S$, $t \in T$, and $a \in A$, and $s_-at_+ = 0$ if and only if $p_sap_t = 0$;
(7) Let $x = s_1^{-1}t_1 = s_2^{-1}t_2 \in S^{-1}T$ for some $s_1, s_2 \in S$, $t_1, t_2 \in T$, and let $a_1, a_2 \in A$. Then $(s_1)_-a_1(t_1)_+ = (s_2)_-a_2(t_2)_+$ if and only if there exist $u_1, u_2 \in S$ such that $u_1s_1 = u_2s_2$ and $u_1t_1 = u_2t_2$ while also $\alpha_{u_1}(p_{s_1}a_1p_{t_1}) = \alpha_{u_2}(p_{s_2}a_2p_{t_2})$.

2. The case $S = T = \mathbb{Z}^+$; Examples

2.1. For the remainder of the paper, we take advantage of Corollary 1.11 and assume that α_s is injective for all $s \in S$. Thus, the relations in $R = S^{\mathrm{op}} *_\alpha A *_\alpha T$ take the simplified form given in (1.12). Moreover, we assume that the maps α_s are *corner isomorphisms*, that is, each α_s is an isomorphism of A onto p_sAp_s. Finally, we assume that $S = T$ is a submonoid of a group G which is its group of left fractions, that is, $G = S^{-1}S$. These conventions are to remain in effect for the rest of the paper.

2.2. A particularly nice setting is the case when G is a left totally ordered group with positive cone $G^+ = S$ (thus $G = S^{-1} \cup S$ and $S^{-1} \cap S = \{1\}$). In this case, the elements of R can be expressed in a simpler way, namely in the form $\sum_{s \in S} s_-a_s + \sum_{t \in S} a_tt_+$. To achieve this, we need to be able to simplify individual terms s_-at_+, for $s, t \in S$ and $a \in A$. If $s \le t$, then $s^{-1}t \ge 1$, whence $u := s^{-1}t \in S$. Then $s_-at_+ = s_-a(su)_+ = s_-p_sap_ss_+u_+$. Because of our current convention that $\alpha_s : A \to p_sAp_s$ is an isomorphism, $p_sap_s = \alpha_s(b)$ for some $b \in A$, and therefore $s_-at_+ = s_-\alpha_s(b)s_+u_+ = bs_-s_+u_+ = bu_+$. On the other hand, if $s \ge t$, then $v := t^{-1}s \in S$ and $s_-at_+ = v_-c$ where $c = \alpha_t^{-1}(p_tap_t)$.

2.3. We now specialize to the case where S is the additive monoid $\mathbb{Z}^+$, so that $G = \mathbb{Z}$. Here the monoid homomorphism $\alpha : S \to \mathrm{Endr}(A)$ is determined by α_1, and so we change notation, writing α and p for α_1 and p_1. Thus, α is now an isomorphism $A \to pAp$, and the monoid homomorphism $S \to \mathrm{Endr}(A)$ is given by the rule $n \mapsto \alpha^n$. Let t denote the generator $1 \in \mathbb{Z}^+ = S$. Since the maps $s \mapsto s_\pm$ are monoid homomorphisms into the multiplicative structure of R, we have $n_\pm = (t_\pm)^n =: t_\pm^n$ for $n \in \mathbb{Z}^+$, and

$$at_-^n = t_-^n\alpha^n(a) \qquad \text{and} \qquad t_+^na = \alpha^n(a)t_+^n$$

for all $a \in A$ and $n \in \mathbb{Z}^+$.

In view of (2.2), the elements $r \in R = \mathbb{Z}^+ *_\alpha A *_\alpha \mathbb{Z}^+$ can all be written as 'polynomials' of the form

$$r = a_nt_+^n + \ldots + a_1t_+ + a_0 + t_-a_{-1} + \ldots t_-^ma_{-m},$$

with coefficients $a_i \in A$. Because of this similarity of R with a skew-Laurent polynomial ring, we shall use the notation $R = A[t_+, t_-; \alpha]$. Proposition 1.6 shows that R is a $\mathbb{Z}$-graded ring $R = \bigoplus_{i\in\mathbb{Z}} R_i$, and from the discussion above we see that $R_i = At_+^i$ for $i > 0$ and $R_i = t_-^{-i}A$ for $i < 0$, while $A_0 = A$.

Our construction of $\mathbb{Z}^+ *_\alpha A *_\alpha \mathbb{Z}^+$ is an exact algebraic analog of the construction of the crossed product of a C*-algebra by an endomorphism introduced by Paschke [19]. In fact, if A is a C*-algebra and the corner isomorphism α is a *-homomorphism, then Paschke's C*-crossed product, which he denotes $A \rtimes_\alpha \mathbb{N}$, is just the completion of $\mathbb{Z}^+ *_\alpha A *_\alpha \mathbb{Z}^+$ in a suitable norm.

Note again that any ring $R = A[t_+, t_-; \alpha]$ is $\mathbb{Z}$-graded, with $A = R_0$. Moreover, t_+ is a left invertible element of R_1 with a particular left inverse $t_- \in R_{-1}$, and α can be recovered from the rule $\alpha(a) = t_+at_-$. These observations allow us to recognize rings of the form $A[t_+, t_-; \alpha]$ among $\mathbb{Z}$-graded rings, as follows.

Lemma 2.4. *Let $D = \bigoplus_{i\in\mathbb{Z}} D_i$ be a $\mathbb{Z}$-graded ring containing elements $t_+ \in D_1$ and $t_- \in D_{-1}$ such that $t_-t_+ = 1$. Then there is a corner isomorphism $\alpha : D_0 \to t_+t_-D_0t_+t_-$ given by the rule $\alpha(d) = t_+dt_-$, and $D = D_0[t_+, t_-; \alpha]$.*

Proof. It is clear that t_+t_- is an idempotent in D_0, and that the given rule defines an isomorphism $\alpha : D_0 \to t_+t_-D_0t_+t_-$. Hence, there exists a fractional skew monoid ring $\widetilde{D} = D_0[\tilde{t}_+, \tilde{t}_-; \alpha]$. Since $t_+d = \alpha(d)t_+$ and $dt_- = t_-\alpha(d)$ for all $d \in D$, the identity map on D_0 extends uniquely to a ring homomorphism $\phi : \widetilde{D} \to D$ such that $\phi(\tilde{t}_\pm) = t_\pm$. It remains to show that ϕ is an isomorphism. Note that since $t_+^i \in D_i$ and $t_-^i \in D_{-i}$ for all $i \in \mathbb{N}$, the map ϕ is a homomorphism of graded rings. Thus, we need only show that ϕ maps each homogeneous component $\widetilde{D}_i$ isomorphically onto D_i. This is already given when $i = 0$.

Now let $i > 0$. If $x \in \widetilde{D}_i$, then $x = d\tilde{t}_+^i$ for some $d \in D_0$, and $\phi(x) = dt_+^i$. If $\phi(x) = 0$, then $d\alpha^i(1) = dt_+^it_-^i = 0$ in D_0, whence $x = d\alpha^i(1)\tilde{t}_+^i = 0$ in $\widetilde{D}$. Thus, the restriction of ϕ to $\widetilde{D}_i$ is injective. Further, if $y \in D_i$, then $yt_-^i \in D_0$ and $\phi\big((yt_-^i)\tilde{t}_+^i\big) = yt_-^it_+^i = y$. Therefore ϕ maps $\widetilde{D}_i$ isomorphically onto D_i. A symmetric argument shows that this also holds for $i < 0$, completing the proof. □

Example 2.5. *An algebraic version of the Cuntz-Krieger algebras.* We give an algebraic version of the C*-algebras $\mathcal{O}_A$ introduced in [10] (now called "Cuntz-Krieger algebras" in the literature), and show that they may be expressed in the form $B[t_+, t_-; \alpha]$ for ultramatricial algebras B and proper corner isomorphisms α. The latter statement is parallel to the corresponding C*-algebra result: $\mathcal{O}_A = B \rtimes_\alpha \mathbb{N}$ for a suitable approximately finite dimensional C*-algebra B (essentially in [10]; discussed explicitly in [22, Example 2.5]).

Let k be an arbitrary field and $A = (a_{ij})$ an $n\times n$ matrix over k, with $a_{ij} \in \{0, 1\}$ for all i, j. To avoid degenerate and trivial cases, we assume that no row or column of A is identically zero, and that A is not a permutation matrix. We define the *algebraic Cuntz-Krieger algebra associated to* A to be the k-algebra $C = \mathcal{CK}_A(k)$ with generators $x_1, y_1, \dots, x_n, y_n$ and relations

(1) $x_iy_ix_i = x_i$ and $y_ix_iy_i = y_i$ for all i;
(2) $x_iy_j = 0$ for all $i \neq j$;
(3) $x_iy_i = \sum_{j=1}^n a_{ij}y_jx_j$ for all i;
(4) $\sum_{j=1}^n y_jx_j = 1$.

Note that all the x_iy_i and y_jx_j are idempotents, and that the y_jx_j are pairwise orthogonal. The free algebra $k\langle X_1, Y_1, \dots, X_n, Y_n\rangle$ can be given a $\mathbb{Z}$-grading in which the X_i have degree -1 while the Y_i have degree 1, and the relators $X_iY_iX_i - X_i$ etc. corresponding to (1)–(4) are all homogeneous. Hence, C inherits a $\mathbb{Z}$-grading such that each $x_i \in C_{-1}$ and each $y_i \in C_1$.

Now set $N = \{1, \dots, n\}$. Given $\mu = (\mu_1, \dots, \mu_\ell) \in N^\ell$ for some ℓ, we set $x_\mu = x_{\mu_1}x_{\mu_2}\cdots x_{\mu_\ell}$ and $y_\mu = y_{\mu_1}y_{\mu_2}\cdots y_{\mu_\ell}$. The case $\ell = 0$ is allowed, with the conventions that $N^0 = \{\varnothing\}$ and $x_\varnothing = y_\varnothing = 1$. The subalgebra $B = C_0$ of C is the k-linear span of the set

$$\{y_\mu x_\nu \mid \mu, \nu \in N^\ell,\ \ \ell \in \mathbb{Z}^+\}.$$

As in [10, Proposition 2.3 and following discussion], B is an ultramatricial k-algebra, and $K_0(B)$ is isomorphic (as an ordered group) to the direct limit of the sequence

$$\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \cdots,$$

with the class $[B] \in K_0(B)$ corresponding to the image of the order-unit $(1, 1, \dots, 1)^{\mathrm{tr}}$ in the first $\mathbb{Z}^n$. (See [11, Chapter 15] for a development of ultramatricial algebras and their classification via K_0.)

For $i = 1, \dots, n$, let e_i denote the sum of those y_jx_j for which $y_jx_j \le x_iy_i$ but $y_jx_j \not\le x_my_m$ for any $m < i$. These e_i are pairwise orthogonal idempotents in B, with each $e_i \le x_iy_i$. Since the matrix A has no identically zero columns, each y_jx_j lies below some x_iy_i, and so each y_jx_j lies below some e_i. In fact, $y_jx_j \le e_i$ where i is the least index such that $a_{ij} = 1$. From relation (4), it follows that $\sum_{i=1}^n e_i = 1$. Next, note that the elements $y_ie_ix_i$ are pairwise orthogonal idempotents in B (because $e_ix_iy_i = e_i$ for all i), whence the sum $p := y_1e_1x_1 + \cdots + y_ne_nx_n$ is an idempotent in B. Moreover, $x_ip = e_ix_i$ and $py_i = y_ie_i$ for all i. We claim that $p \ne 1$.

If $p = 1$, then each $x_i = e_ix_i$, whence each $x_iy_i = e_i$. Then the x_iy_i are pairwise orthogonal. In view of the relations (3), it follows that each column of A has only one nonzero entry. Since A has no identically zero rows, it must be a permutation matrix, contradicting our assumptions. Therefore $p \ne 1$, as claimed.

Now set $t_- = e_1x_1 + \cdots + e_nx_n \in C_{-1}$ and $t_+ = y_1e_1 + \cdots + y_ne_n \in C_1$. Then $t_+t_- = p$, and

$$t_-t_+ = \sum_{i=1}^n e_ix_iy_ie_i = \sum_{i,j=1}^n a_{ij}e_iy_jx_je_i = \sum_{j=1}^n y_jx_j = 1,$$

because each $y_jx_j \le e_i$ for precisely one i, and $a_{ij} = 1$ for that i. Hence, there is a proper corner isomorphism $\alpha : B \to pBp$ given by the rule $\alpha(b) = t_+bt_-$, and we conclude from Lemma 2.4 that

$$C = \mathcal{CK}_A(k) = B[t_+, t_-; \alpha]. \qquad \square$$

In case the matrix A in Example 2.5 has all of its entries equal to 1, the relations for the algebra $\mathcal{CK}_A(k)$ reduce to

(1) $x_iy_j = \delta_{i,j}$ for all i, j;
(2) $\sum_{j=1}^n y_jx_j = 1$.

Thus in this case, $\mathcal{CK}_A(k)$ is the *Leavitt algebra* $V_{1,n}(k)$ first studied in [17]. (The notation $V_{1,n}$ was introduced in [5].) There is a related Leavitt algebra $U_{1,n}(k)$ which, as we now show, can also be presented as a fractional skew monoid ring.

Example 2.6. Let k be a field and $n \in \mathbb{N}$. The algebra $U = U_{1,n}(k)$ is the k-algebra with generators $x_1, y_1, \dots, x_n, y_n$ and relations $x_iy_j = \delta_{i,j}$ for all i, j. (Thus, $V_{1,n}(k)$ is the factor algebra of $U_{1,n}(k)$ modulo the ideal generated by $1-\sum_{j=1}^{n} y_jx_j$.) The elements $y_1x_1,\dots,y_nx_n$ are pairwise orthogonal idempotents in U. As in Example 2.5, there is a $\mathbb{Z}$-grading on U such that each $x_i \in U_{-1}$ and each $y_i \in U_1$.

Set $N=\{1,\dots,n\}$ and define $x_\mu, y_\mu \in U$ for $\mu \in N^\ell$ as in Example 2.5. In U, the set

$$\{y_\mu x_\nu \mid \mu \in N^\ell,\ \nu \in N^m,\ \ell, m \in \mathbb{Z}^+\}$$

forms a k-basis. We again set $B=U_0$, which is the k-linear span of the set

$$\{y_\mu x_\nu \mid \mu,\nu \in N^\ell,\ \ell \in \mathbb{Z}^+\},$$

and as before, B is ultramatricial. It is isomorphic to a direct limit of the algebras

$$M_{n^i}(k) \times M_{n^{i-1}}(k) \times \cdots \times M_n(k) \times k,$$

the ordered group $K_0(B)$ is isomorphic to the direct limit of a sequence $\mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \cdots$ where each transition map $\mathbb{Z}^i \to \mathbb{Z}^{i+1}$ is given by an $(i+1)\times i$ matrix of the form

$$\begin{pmatrix} n & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

and the class $[B] \in K_0(B)$ corresponds to the image of $1 \in \mathbb{Z}$.

Set $p=y_1x_1 \in B$, a proper idempotent. Then set $t_-=x_1 \in U_{-1}$ and $t_+=y_1 \in U_1$, so that $t_+t_-=p$ and $t_-t_+=1$. Hence, the rule $b \mapsto t_+bt_-$ gives a proper corner isomorphism $\alpha : B \to pBp$, and Lemma 2.4 shows that

$$U=U_{1,n}(k)=B[t_+,t_-;\alpha]. \qquad \square$$

Example 2.7. Let k be a field, and note that there are natural inclusions

$$U_{1,1}(k) \subset U_{1,2}(k) \subset U_{1,3}(k) \subset \cdots$$

among the algebras $U_{1,n}(k)$. Set $U_\infty(k)=\bigcup_{n=1}^{\infty} U_{1,n}(k)$, which is a simple algebra (e.g., [3, Theorem 4.3]). We may also view $U_\infty(k)$ as the k-algebra with an infinite sequence of generators $x_1,y_1,x_2,y_2,\dots$ and relations $x_iy_j=\delta_{i,j}$ for all i,j. This algebra is $\mathbb{Z}$-graded as before, with the x_i having degree -1 and the y_i degree 1. Set $B=U_\infty(k)_0$, which is the k-linear span of the set

$$\{y_\mu x_\nu \mid \mu,\nu \in \{1,\dots,n\}^\ell,\ n \in \mathbb{N},\ \ell \in \mathbb{Z}^+\}.$$

In the present case, B is an ultramatricial k-algebra isomorphic to a direct limit of the algebras

$$M_{n^n}(k) \times M_{n^{n-1}}(k) \times \cdots \times M_n(k) \times k.$$

Here $K_0(B)$ is isomorphic to the direct limit of a sequence $\mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z}^4 \to \cdots$ with transition maps

$$\begin{pmatrix} n & n & n^2 & n^3 & \cdots & n^{n-2} & n^{n-1} \\ 1 & 1 & n & n^2 & \cdots & n^{n-3} & n^{n-2} \\ 0 & 1 & 1 & n & \cdots & n^{n-4} & n^{n-3} \\ & & & & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

and $[B]$ corresponds to $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \in \mathbb{Z}^2$. If we define p, $t_\pm$, α exactly as in Example 2.6, we conclude from Lemma 2.4 that

$$U_\infty(k) = B[t_+, t_-; \alpha]. \qquad \square$$

3. Fractional skew monoid rings versus corners of skew group rings

Paschke [19] and Rørdam [22, Section 2] have shown that a C*-algebra crossed product by an endomorphism corresponds naturally to a corner in a crossed product by an automorphism. In other words, the C*-algebra versions of fractional skew monoid rings $\mathbb{Z}^+ *_\alpha A *_\alpha \mathbb{Z}^+$ are isomorphic to corners $e(B *_{\alpha'} \mathbb{Z})e$ in certain skew group rings. This leads us to ask whether, in general, our rings $S^{\mathrm{op}} *_\alpha A *_\alpha S$ should appear as corner rings $e(B * G)e$, where $B * G$ is some skew group ring over the group $G = S^{-1}S$. This is indeed the case, as we prove in Proposition 3.8. We prepare the way by studying corner rings of the form $e(A * G)e$ (for $G = S^{-1}S$ as above), and showing that they fall into the class of fractional skew monoid rings under appropriate conditions on the action.

3.1. Let A be a unital ring, G a group, and $\alpha : G \to \mathrm{Aut}(A)$ an action. Assume that S is a submonoid of G with $G = S^{-1}S$, and let $R = A *_\alpha G$. Suppose that there exists a nontrivial idempotent $e \in A$ such that $\alpha_s(e) \leq e$ for all $s \in S$.

Lemma 3.2. *Under the above assumptions, the following hold:*

(a) *The action α restricts to an action $\alpha' : S \to \mathrm{Endr}(eAe)$ by corner isomorphisms.*

(b) *There are natural monoid morphisms $S^{\mathrm{op}} \to eRe$, given by $s \mapsto es^{-1}$, and $S \to eRe$, given by $t \mapsto te$, satisfying the conditions* (1)–(4) *in Definition* 1.2 *with respect to α' and the inclusion map $\phi : eAe \to eRe$.*

Proof. (a) This is clear from the hypothesis on e.

(b) Notice that, since $e \leq \alpha_s^{-1}(e)$ for all $s \in S$, we have $es^{-1} = es^{-1}\alpha_s(e) \in eRe$ and $(es^{-1})(et^{-1}) = e(ts)^{-1}$ for $s, t \in S$. Similarly, $se \in eRe$ and $(se)(te) = (st)e$. So, the defined maps are monoid morphisms. It is straightforward to check conditions (1)–(4) in Definition 1.2. $\square$

Because of Lemma 3.2, we have the data to construct a fractional skew monoid ring of the form $S^{\mathrm{op}} *_{\alpha'} (eAe) *_{\alpha'} S$. Since the maps $\alpha'_s = \alpha_s|_{eAe}$ are injective for all $s \in S$, the ring homomorphism $eAe \to S^{\mathrm{op}} *_{\alpha'} (eAe) *_{\alpha'} S$ going with the construction of $S^{\mathrm{op}} *_{\alpha'} (eAe) *_{\alpha'} S$ is injective by Corollary 1.11. Hence, we identify eAe with its image in $S^{\mathrm{op}} *_{\alpha'} (eAe) *_{\alpha'} S$, as in (1.12).

Proposition 3.3. *Under the assumptions of* (3.1), *the rings $S^{\mathrm{op}} *_{\alpha'} (eAe) *_{\alpha'} S$ and $e(A *_\alpha G)e$ are isomorphic as G-graded rings.*

Proof. By the universal property of $S^{\mathrm{op}} *_{\alpha'} (eAe) *_{\alpha'} S$, there exists a unique ring homomorphism $\psi : S^{\mathrm{op}} *_{\alpha'} (eAe) *_{\alpha'} S \to e(A *_\alpha G)e$ such that $\psi(s_- a t_+) = (es^{-1})a(te)$ for all $s, t \in S$ and $a \in eAe$. Clearly, ψ is G-graded. To see that ψ is onto, consider $e(ag)e \in e(A * G)e$ where $a \in A$ and $g \in G$, and write $g = s^{-1}t$ for some $s, t \in S$. Then we have

$$e(ag)e = eas^{-1}te = (es^{-1})(\alpha_s(ea)\alpha_t(e))(te) \in \psi(S^{\mathrm{op}} *_{\alpha'} (eAe) *_{\alpha'} S),$$

which proves that ψ is onto. It only remains to check that ψ is one-to one.

Since ψ is G-graded, we only have to check that $\psi(s_- a t_+) = 0$ implies $a = 0$, when $s, t \in S$ and $a \in p_s(eAe)p_t$. Note that $p_s = \alpha'_s(1_{eAe}) = \alpha_s(e)$, and likewise $p_t = \alpha_t(e)$, so that $a = \alpha_s(e)a\alpha_t(e)$. Now

$$0 = (es^{-1})a(te) = e\alpha_s^{-1}(a\alpha_t(e))(s^{-1}t) = \alpha_s^{-1}(\alpha_s(e)a\alpha_t(e))(s^{-1}t) = \alpha_s^{-1}(a)(s^{-1}t),$$

whence $\alpha_s^{-1}(a) = 0$ and $a = 0$, as desired. □

The following procedure gives a generic way to obtain a situation as in (3.1).

Example 3.4. *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of an abelian group G on a unital ring A, and let e be an idempotent in A. Set $S := \{s \in G \mid \alpha_s(e) \le e\}$. Then S is a submonoid of G and $G' := S^{-1}S$ is a subgroup of G acting on A via α. Moreover, $e(A *_\alpha G')e \cong S^{\mathrm{op}} *_{\alpha'} (eAe) *_{\alpha'} S$, where $\alpha'\colon S \to \mathrm{Endr}(eAe)$ is the induced action of S on eAe by corner isomorphisms.*

Proof. It is clear that S is a submonoid of G, and we can apply Proposition 3.3 to get the result. □

Now we go in the reverse direction, looking for a representation of a fractional skew monoid ring $S^{\mathrm{op}} *_\alpha A *_\alpha S$ as a corner ring of a skew group ring. Our original approach utilized a direct limit construction based on ideas of Rørdam [22]; that approach required S to be abelian. In the meantime, we learned of the work of Picavet [21], whose construction we can make use of without needing S to be abelian.

3.5. Let A be a unital ring, G a group and S a submonoid of G such that $G = S^{-1}S$. Thus, S satisfies the left Ore condition, and left reversibility holds trivially because S has cancellation. Let $\alpha : S \to \mathrm{Endr}(A)$ be an action of S on A by corner isomorphisms, and for $s \in S$ let p_s denote the idempotent $\alpha_s(1)$. We construct a ring $S^{-1}A$ as in [21], but with some changes of notation to fit our situation. As written, the development in [21] would require S to act on A by unital ring endomorphisms. However, almost all the results we shall quote do not make use of this assumption, the exception being the question of an identity – in our situation, $S^{-1}A$ can be a non-unital ring.

First, define a relation $\sim$ on $S \times A$ as follows:

> $(s_1, a_1) \sim (s_2, a_2)$ if and only if there exist $t_1, t_2 \in S$ such that $t_1 s_1 = t_2 s_2$ and $\alpha_{t_1}(a_1) = \alpha_{t_2}(a_2)$.

This is an equivalence relation [21, Lemma 2.1], and we write $[s, a]$ for the equivalence class of a pair (s, a). Let $S^{-1}A = (S \times A)/\!\sim$ be the set of these equivalence classes. The left Ore condition guarantees "common denominators" in $S^{-1}A$: Given any $x_1, x_2 \in S^{-1}A$, there exist $s \in S$ and $a_1, a_2 \in A$ such that each $x_i = [s, a_i]$. By [21, Lemma 2.2 ff.], there is a well-defined associative multiplication on $S^{-1}A$ as follows:

> Given any $[s_1, a_1], [s_2, a_2] \in S^{-1}A$, choose $t_1, t_2 \in S$ such that $t_1 s_1 = t_2 s_2$, and set $[s_1, a_1] \cdot [s_2, a_2] = [t_1 s_1, \alpha_{t_1}(a_1)\alpha_{t_2}(a_2)]$.

(This multiplication rule is simpler than the Ore-Asano rule for multiplication of noncommutative fractions, because the classes $[s,a]$ model elements that would have the form $\alpha_s^{-1}(a)$ if α_s extended to an automorphism of an overring of A.) It is routine to build a well-defined, commutative, associative addition on $S^{-1}A$ by the corresponding rule:

Given any $[s_1,a_1],[s_2,a_2]\in S^{-1}A$, choose $t_1,t_2\in S$ such that $t_1s_1=t_2s_2$, and set $[s_1,a_1]+[s_2,a_2]=[t_1s_1,\alpha_{t_1}(a_1)+\alpha_{t_2}(a_2)]$.

The distributive law is also routine, and so $S^{-1}A$ becomes a (possibly non-unital) ring. In fact, for $[s,a]\in S^{-1}A$ we have $[1,1]\cdot[s,a]=[s,p_sa]$ and $[s,a]\cdot[1,1]=[a,ap_s]$.

Next, we extend α to an action of S on $S^{-1}A$. Since this is done without proof in [21, Theorem 2.4 ff.], we sketch the details.

Lemma 3.6. *The action of α on A extends to an action $\alpha: S\to \mathrm{Aut}(S^{-1}A)$ as follows:*

Given any $s\in S$ and $[t,a]\in S^{-1}A$, choose $s',t'\in S$ such that $s's=t't$, and set $\alpha_s([t,a])=[s',\alpha_{t'}(a)]$.

Proof. First, let $s\in S$ and $[t_1,a_1]=[t_2,a_2]$ in $S^{-1}A$. Let $s_1,u_1,s_2,u_2\in S$ such that $s_1s=u_1t_1$ and $s_2s=u_2t_2$; we must show that $[s_1,\alpha_{u_1}(a_1)]=[s_2,\alpha_{u_2}(a_2)]$. There exist $r_1,r_2\in S$ such that $r_1s_1=r_2s_2$, and each $[s_i,\alpha_{u_i}(a_i)]=[r_is_i,\alpha_{r_iu_i}(a_i)]$. Hence, we may assume that $s_1=s_2$. Note that now $u_1t_1=u_2t_2$.

Since $[t_1,a_1]=[t_2,a_2]$, there exist $v_1,v_2\in S$ such that $v_1t_1=v_2t_2$ and $\alpha_{v_1}(a_1)=\alpha_{v_2}(a_2)$. Further, there are $p,q\in S$ with $pv_1=qu_1$. Then $pv_2t_2=pv_1t_1=qu_1t_1=qu_2t_2$, and so $pv_2=qu_2$. After replacing s_1,u_1,s_2,u_2 by qs_1,qu_1,qs_2,qu_2, we may assume that $pv_i=u_i$ for $i=1,2$. Consequently, $\alpha_{u_1}(a_1)=\alpha_{u_2}(a_2)$, whence $[s_1,\alpha_{u_1}(a_1)]=[s_2,\alpha_{u_2}(a_2)]$. Therefore $\alpha_s([t,a])$ is well-defined.

Consider $s\in S$ and $[t,a_1],[t,a_2]\in S^{-1}A$. Choose $s',t'\in S$ such that $s's=t't$; then

$$\begin{aligned}\alpha_s([t,a_1]\cdot[t,a_2])&=\alpha_s([t,a_1a_2])=[s',\alpha_{t'}(a_1a_2)]\\&=[s',\alpha_{t'}(a_1)]\cdot[s',\alpha_{t'}(a_2)]=\alpha_s([t,a_1])\cdot\alpha_s([t,a_2]),\end{aligned}$$

and similarly for addition. This shows that α_s is a ring endomorphism of $S^{-1}A$. If $\alpha_s([t,a_1])=\alpha_s([t,a_2])$, there exist $u_1,u_2\in S$ such that $u_1s'=u_2s'$ and $\alpha_{u_1t'}(a_1)=\alpha_{u_2t'}(a_2)$. Since then $u_1=u_2$, it follows that $[t,a_1]=[t,a_2]$. Thus, α_s is injective. Moreover, for any $[t,a]\in S^{-1}A$ we see that $\alpha_s([ts,a])=[t,a]$. Therefore $\alpha_s\in\mathrm{Aut}(S^{-1}A)$.

It is clear that α_1 is the identity map. Finally, consider $s_1,s_2\in S$ and $[t,a]\in S^{-1}A$. There exist $s_2',t_2\in S$ such that $s_2's_2=t_2t$, so that $\alpha_{s_2}([t,a])=[s_2',\alpha_{t_2}(a)]$. There exist $s_1',t_1\in S$ such that $s_1's_1=t_1s_2'$, so that $\alpha_{s_1}([s_2',\alpha_{t_2}(a)])=[s_1',\alpha_{t_1t_2}(a)]$. But $s_1's_1s_2=t_1t_2t$, and so $\alpha_{s_1s_2}([t,a])=[s_1',\alpha_{t_1t_2}(a)]=\alpha_{s_1}(\alpha_{s_2}([t,a]))$. Therefore the map $\alpha: S\to\mathrm{Aut}(S^{-1}A)$ is a monoid homomorphism. □

There is a shortcut that can be taken for part of the above work. The given action of S induces on A the structure of a left module over the monoid ring $\mathbb{Z}S$. Moreover, S is a left denominator set in $\mathbb{Z}S$, and the Ore localization $S^{-1}(\mathbb{Z}S)$ is just the group ring $\mathbb{Z}G$. By standard localization theory, there exists a module of fractions $S^{-1}A$, which is a left $\mathbb{Z}G$-module. Thus, one obtains the construction of $S^{-1}A$ as an additive group and the action of S on $S^{-1}A$ by $\mathbb{Z}$-module automorphisms.

Lemma 3.7. *The rule $a\mapsto[1,a]$ defines an S-equivariant ring embedding $\phi: A\to S^{-1}A$ with image $[1,1]\cdot S^{-1}A\cdot[1,1]$.*

Proof. It is clear that ϕ is a ring homomorphism and that it is S-equivariant, i.e., $\phi(\alpha_s(a)) = \alpha_s(\phi(a))$ for $s \in S$ and $a \in A$. If $a \in \ker(\phi)$, then $[1,a] = [1,0]$, and so $\alpha_s(a) = 0$ for some $s \in S$. Since α_s is injective, $a = 0$. Thus, ϕ is an embedding.

Set $e = [1,1] = \phi(1)$, and note that $\phi(a) = e\phi(a)e$ for $a \in A$. Recall that $e[s,a]e = [s, p_s a p_s]$ for any $s \in S$ and $a \in A$. Since $\alpha_s(A) = p_s A p_s$, there exists $b \in A$ with $\alpha_s(b) = p_s a p_s$, whence $e[s,a]e = [s, \alpha_s(b)] = [1,b]$. Therefore the image of ϕ equals $e(S^{-1}A)e$. □

Proposition 3.8. *Let G be a group and S a submonoid of G such that $G = S^{-1}S$. Let $\alpha : S \to \mathrm{Endr}(A)$ be an action of S on A by corner isomorphisms. Then there exist a unital ring B, an action $\widehat{\alpha} : G \to \mathrm{Aut}(B)$, and an idempotent e in B such that $\widehat{\alpha}_s(e) \le e$ for all $s \in S$ and $S^{\mathrm{op}} *_\alpha A *_\alpha S \cong e(B *_{\hat{\alpha}} G)e$ (as G-graded rings).*

Proof. Construct $S^{-1}A$ as above, set $e = [1,1]$, and identify A with the corner $e(S^{-1}A)e$ via Lemma 3.7. Let B be the unitization of $S^{-1}A$; then also $A = eBe$. In view of Lemma 3.6, α extends to an action $G \to \mathrm{Aut}(S^{-1}A)$, and thus to an action $\widehat{\alpha} : G \to \mathrm{Aut}(B)$. It is clear that $\widehat{\alpha}_s(e) \le e$ for $s \in S$, and we conclude from Proposition 3.3 that $e(B *_{\hat{\alpha}} G)e \cong S^{\mathrm{op}} *_{\hat{\alpha}} (eBe) *_{\hat{\alpha}} S = S^{\mathrm{op}} *_\alpha A *_\alpha S$ as G-graded rings. □

4. Simplicity

We continue the general assumptions of (1.1) and (2.1), and seek conditions on A, S, and α under which $R = S^{\mathrm{op}} *_\alpha A *_\alpha S$ is a simple ring. In the case of a group action (i.e., $S = G$ and $\alpha : G \to \mathrm{Aut}(A)$), sufficient conditions for simplicity are well known [18, Theorem 2.3]: If A is simple and the action α is outer, then the skew group ring $A *_\alpha G$ is simple. It turns out that a suitable modification of the notion of an outer action also leads to simplicity in our more general situation.

We shall say that a pair (α_s, α_t), where $s, t \in S$, is *inner* provided there exist elements $u \in p_s A p_t$ and $v \in p_t A p_s$ such that $uv = p_s$, $vu = p_t$ and $\alpha_s(x) = u\alpha_t(x)v$ for all $x \in A$. Note that then $\alpha_s\alpha_t^{-1}(x) = uxv$ for every $x \in p_t A p_t$, and $\alpha_t\alpha_s^{-1}(x) = vxu$ for all $x \in p_s A p_s$. Let us say that α is *outer* in case (α_s, α_t) is not inner for any distinct $s, t \in S$.

We will use the following standard terminology. The *support* of an element $r = \sum_x r_x$ in $R = \bigoplus_{x \in G} R_x$ is the set $\mathrm{Supp}(r) = \{x \in G \mid r_x \neq 0\}$. The *length* of r is the number of elements in the support of r, and is denoted $\mathrm{len}(r)$.

Theorem 4.1. *If A is simple and α is outer, then $R = S^{\mathrm{op}} *_\alpha A *_\alpha S$ is simple.*

Proof. Suppose that R is not simple. Let I be a proper nonzero ideal of R, and let $\rho \in I$ be a nonzero element with minimal length, say length n. Write $\rho = \sum_{i=1}^n (s_i)_- a_i (t_i)_+$ where the $s_i^{-1}t_i$ are distinct elements of $S^{-1}S$ and each a_i is a nonzero element of $p_{s_i} A p_{t_i}$. Observe that $(s_1)_+ \rho (t_1)_- = a_1 + \sum_{i=2}^n \rho_i$ where each ρ_i lies in the $s_1 s_i^{-1} t_i t_1^{-1}$-component of R. Hence, $(s_1)_+ \rho (t_1)_- = a_1 + \sum_{i=2}^n (u_i)_- b_i (v_i)_+$ where the $u_i^{-1}v_i$ are distinct elements of $S^{-1}S$, different from 1, and each $b_i \in p_{u_i} A p_{v_i}$. Moreover, $a_1 \neq 0$ implies $(s_1)_+ \rho (t_1)_- \neq 0$, and so $(s_1)_+ \rho (t_1)_-$ has length n by minimality. Thus, after replacing ρ by $(s_1)_+ \rho (t_1)_-$, we may assume that $s_1 = t_1 = 1$.

Since A is simple, $\sum_{j=1}^m c_j a_1 d_j = 1$ for some $c_j, d_j \in A$. Then we can replace ρ by

$$\sum_{j=1}^m c_j \rho d_j = 1 + \sum_{i=2}^n (s_i)_- \big(\sum_{j=1}^m \alpha_{s_i}(c_j) a_i \alpha_{t_i}(d_j) \big) (t_i)_+,$$

and so we may now assume that $a_1 = 1$. Of course $\rho \neq 1$ because $I \neq R$, whence $n \geq 2$. Set $s = s_2$, $t = t_2$, and $a = a_2 \in p_s A p_t$, so that

$$\rho = 1 + s_- a t_+ + \sum_{i=3}^{n} (s_i)_- a_i (t_i)_+.$$

For any $x \in A$, we have $x\rho - \rho x \in I$ and

$$x\rho - \rho x = s_-\big(\alpha_s(x)a - a\alpha_t(x)\big)t_+ + \sum_{i=3}^{n} (s_i)_- \Diamond_i (t_i)_+$$

for some elements $\Diamond_i \in p_{s_i} A p_{t_i}$ that we need not specify. Thus $x\rho - \rho x$ has length less than n, and so $x\rho - \rho x = 0$ by the minimality of n. Therefore

$$\alpha_s(x)a = a\alpha_t(x)$$

for all $x \in A$. In particular, $p_s A a = p_s A p_s a = \alpha_s(A)a = a\alpha_t(A) = aAp_t$.

Since A is simple, $Ap_tA = AaA = A$, and so

$$aAp_s = aAp_tAp_s = p_sAaAp_s = p_sAp_s,$$

whence there is some $b \in p_tAp_s$ such that $ab = p_s$. Similarly, there is some $c \in p_tAp_s$ such that $ca = p_t$. But $c = cp_s = cab = p_tb = b$, so that $ba = p_t$. Now

$$a\alpha_t(x)b = \alpha_s(x)ab = \alpha_s(x)p_s = \alpha_s(x)$$

for all $x \in A$, and so we conclude that the pair (α_s, α_t) is inner. Since α is assumed to be outer, we must have $s = t$. But then $s_2^{-1}t_2 = s^{-1}t = 1 = s_1^{-1}t_1$, contradicting the distinctness of the $s_i^{-1}t_i$. Therefore R is simple. □

Corollary 4.2. *If A is simple and $p_s \not\sim p_t$ for all distinct $s, t \in S$, then R is simple.* □

Corollary 4.3. *If A is a directly finite simple ring, $p \in A$ is a proper idempotent (i.e., $p \neq 1$), and $\alpha : A \to pAp$ is a corner isomorphism, then $\mathbb{Z}^+ *_\alpha A *_\alpha \mathbb{Z}^+$ is simple.*

Proof. The idempotents corresponding to the monoid homomorphism $\mathbb{Z}^+ \to \mathrm{Endr}(A)$ in this case are the $\alpha^i(1)$ for $i \in \mathbb{Z}^+$. Since $\alpha(1) = p \neq 1$, we have $1 > \alpha(1) > \alpha^2(1) > \cdots$, and it follows from the direct finiteness of A that $\alpha^i(1) \not\sim \alpha^j(1)$ for all distinct $i, j \in \mathbb{Z}^+$. □

5. Purely infinite simplicity

We recall from [3] that a simple ring T is said to be *purely infinite* if every nonzero right ideal of T contains an infinite idempotent. This concept is left-right symmetric, as the following characterization shows: T is purely infinite if and only if (1) T is not a division ring; (2) for every nonzero element $a \in T$, there exist elements $x, y \in T$ such that $xay = 1$ [3, Theorem 1.6]. For instance, the Leavitt algebras $V_{1,n}(k)$ and $U_\infty(k)$ are purely infinite simple rings [3, Theorems 4.2, 4.3]. As we have seen above (Examples 2.5 and 2.7), the $V_{1,n}(k)$ and $U_\infty(k)$ can be presented in the form $\mathbb{Z}^+ *_\alpha B *_\alpha \mathbb{Z}^+$. This suggests that fractional skew monoid rings might be purely infinite simple in some generality. Our goal in this section is to establish sufficient conditions for a fractional skew monoid ring $R = S^{\mathrm{op}} *_\alpha A *_\alpha S$ to be a purely infinite simple ring, under the general assumptions of (1.1) and (2.1).

The following concept will be needed. A ring T is said to be *strictly unperforated* provided the finitely generated projective right (or left) T-modules enjoy the following property: If $mA \prec mB$ for some $m \in \mathbb{N}$, then $A \prec B$. (Here mA denotes the direct sum of m copies of A, and the notation $X \prec Y$ means that X is isomorphic to a proper direct summand of Y. Similarly, $e \prec f$, for idempotents $e, f \in T$, means that $e \sim e' < f$ for some idempotent e' in T.) Stated in terms of idempotents in matrix rings over T, strict unperforation is the condition ($m{\cdot}p \prec m{\cdot}q \implies p \prec q$), where $m{\cdot}p$ denotes the orthogonal sum of m copies of an idempotent p. For instance, ultramatricial algebras are strictly unperforated [11, Theorem 15.24(a)]. Also, any purely infinite simple ring T is strictly unperforated, because $A \prec B$ for all nonzero finitely generated projective T-modules A and B [3, Proposition 1.5].

Lemma 5.1. *, Assume that A is simple and strictly unperforated, and that there exists $u \in S$ such that $p_u \neq 1$. For any nonzero idempotent $e \in A$, there exists $v = u^j \in S$ for some $j \in \mathbb{N}$ such that $p_v \lesssim e$.*

Proof. Set $p_i = p_{u^i} = \alpha_u^i(1)$ for $i \geq 0$. Since A is simple, there exists $m \in \mathbb{N}$ such that $1 \prec m{\cdot}e$ and $1 \lesssim m{\cdot}(1 - p_1)$. Note that

$$(m+1){\cdot}p_1 \lesssim m{\cdot}p_1 \oplus 1 \lesssim m{\cdot}p_1 \oplus m{\cdot}(1-p_1) \sim m{\cdot}1.$$

Applying the isomorphisms $\alpha_u^i : A \to p_iAp_i$, we obtain that $(m+1){\cdot}p_{i+1} \lesssim m{\cdot}p_i$ for all i. It follows by induction that $(m+1)^i{\cdot}p_i \lesssim m^i{\cdot}1$ for all i.

Now choose $j \in \mathbb{N}$ such that $m^{j+1} < (m+1)^j$, and observe that

$$m^{j+1}{\cdot}p_j \prec (m+1)^j{\cdot}p_j \lesssim m^j{\cdot}1 \prec m^{j+1}{\cdot}e,$$

whence $m^{j+1}{\cdot}p_j \prec m^{j+1}{\cdot}e$. Therefore $p_j \prec e$, because A is strictly unperforated. □

The following lemma is a variation on results such as [11, Proposition 3.3].

Lemma 5.2. *If T is a simple ring containing an idempotent $p \neq 0, 1$, then T is generated (as a ring) by its idempotents.*

Proof. Let T' be the subring of T generated by the idempotents. Since $p + pt(1-p)$ is idempotent for any $t \in T$, we see that $pT(1-p) \subseteq T'$, and likewise $(1-p)Tp \subseteq T'$. The simplicity of T implies that $T(1-p)T = T$, whence $pTp = [pT(1-p)][(1-p)Tp] \subseteq T'$, and similarly $(1-p)T(1-p) \subseteq T'$. Therefore $T' = T$. □

Theorem 5.3. *Assume that A is a simple, strictly unperforated ring, in which every nonzero right (left) ideal contains a nonzero idempotent. Assume also that α is outer, and that there exists $u \in S$ with $p_u \neq 1$. Then $R = S^{\mathrm{op}} *_\alpha A *_\alpha S$ is a purely infinite simple ring.*

Proof. The hypothesis that $p_u \neq 1$ will allow us later to apply Lemma 5.1. Moreover, it implies that R is not a division ring.

Let ρ be an arbitrary nonzero element of R. Choose $\rho', \rho'' \in R$ such that $\rho'\rho\rho''$ is nonzero and has minimal length for such nonzero products, say length n. Since it suffices to find $x, y \in R$ such that $x\rho'\rho\rho''y = 1$, we may replace ρ by $\rho'\rho\rho''$. Thus, without loss of generality, all nonzero products $\sigma\rho\sigma'$ in R have length at least n. Now write $\rho = \sum_{i=1}^n (s_i)_- a_i (t_i)_+$ where the $s_i^{-1}t_i$ are distinct elements of $S^{-1}S$ and each a_i is a nonzero element of $p_{s_i}Ap_{t_i}$. As in the proof of Theorem 4.1, after replacing ρ by $(s_1)_+\rho(t_1)_-$ we may assume that $s_1 = t_1 = 1$, so that $\rho = a_1 + \sum_{i=2}^n (s_i)_- a_i (t_i)_+$.

By our hypothesis on idempotents, there exists $a_1' \in A$ such that a_1a_1' is a nonzero idempotent. By Lemma 5.1, there exist $x, y \in A$ such that $xa_1a_1'y = p_v$ for some $v \in S$. Note that $v_-xa_1a_1'yv_+ = 1$. Hence, after replacing ρ by $v_-x\rho a_1'yv_+$, we may assume that $a_1 = 1$. We are thus done in case $n = 1$.

Suppose that $n \geq 2$, and set $s = s_2$, $t = t_2$, and $a = a_2 \in p_sAp_t$. Thus,

$$\rho = 1 + s_-at_+ + \sum_{i=3}^{n}(s_i)_-a_i(t_i)_+$$

at this point. For any idempotent $e \in A$, we have

$$e\rho(1-e) = s_-\alpha_s(e)a\big(p_t - \alpha_t(e)\big)t_+ + \sum_{i=3}^{n}(s_i)_-\Diamond_i(t_i)_+.$$

Since $e\rho(1-e)$ has length less than n, it must be zero, whence $\alpha_s(e)a(p_t - \alpha_t(e)) = 0$. Thus, $\alpha_s(e)a = \alpha_s(e)a\alpha_t(e)$. A symmetric argument involving $(1-e)\rho e$ shows that $a\alpha_t(e) = \alpha_s(e)a\alpha_t(e)$, and so $\alpha_s(e)a = a\alpha_t(e)$.

By Lemma 5.2, A is generated by its idempotents. Hence, it follows from the equations $\alpha_s(e)a = a\alpha_t(e)$ that $\alpha_s(x)a = a\alpha_t(x)$ for all $x \in A$. As in the proof of Theorem 4.1, this implies that the pair (α_s, α_t) is inner, yielding $s = t$ and $s_2^{-1}t_2 = s_1^{-1}t_1$, which contradicts our assumptions. Therefore $n = 1$, and the proof is complete. □

It is perhaps not so surprising that the purely infinite simple property carries over from A to R under suitable conditions. More interesting is that R can be purely infinite simple even when A is directly finite. We single out an important case of this phenomenon in the following corollary.

Corollary 5.4. *Suppose that A is either a purely infinite simple ring or a simple ultramatricial algebra over some field. Assume also that α is outer, and that there exists $u \in S$ with $p_u \neq 1$. Then $R = S^{\mathrm{op}} *_\alpha A *_\alpha S$ is a purely infinite simple ring.* □

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