THE GRADED VERSION OF GOLDIE'S THEOREM

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Abstract. The analogue of Goldie's Theorem for prime rings is proved for rings graded by abelian groups, eliminating unnecessary additional hypotheses used in earlier versions.

Introduction

In recent years, rings with a group-graded structure have become increasingly important and, consequently, the graded analogues of Goldie's Theorems have been widely utilized. Unfortunately, the graded result requires an awkward extra condition: given a semiprime Goldie, $\mathbb{Z}$-graded ring R, one cannot assert that R has a graded-semisimple ring of quotients unless one makes some extra assumption, typically about the existence of homogeneous regular elements (see [**5**, Theorem C.I.1.6], for example), or about the nondegeneracy of products of homogeneous elements (see [**4**, Proposition 1.4], for instance). The standard counter-example [**5**, Example C.I.1.1] is the ring $R = k[x] \oplus k[y]$, graded by giving x degree 1 and y degree -1. Note that this ring has no homogeneous regular elements other than units, yet it is neither graded-semisimple nor graded-artinian.

There seems to be a misconception that Goldie's Theorem for prime rings also requires such an extra condition (see, for example, [**5**, §C.I.1] or [**3**, p. 42]) and this is awkward in applications, as is illustrated by [**2**, §6.1] and [**1**, §5.4]. The purpose of this note is to correct this misconception by showing that, at least for prime rings graded by abelian groups, no such extra hypotheses are required.

A Graded Goldie Theorem

Fix an abelian group G. Throughout, "graded" will mean "G-graded" and, following standard practice, the graded analogue of a standard definition will be denoted by the prefix "gr-". Thus, for example, a gr-uniform module means a graded module that does not contain the direct sum of two nonzero, graded submodules. The formal definitions can be found in [**5**].

The main aim of this note is to prove:

Theorem 1. *Let G be an abelian group and R a G-graded, gr-prime, right gr-Goldie ring. Then, R has a gr-simple, gr-artinian right ring of fractions.*

In the conclusion of Theorem 1, it is tacitly assumed that the right ring of fractions is taken with denominators which are homogeneous regular elements of R, so that the G-grading on R extends (uniquely) to a G-grading on the ring of fractions.

The only place where our proof differs significantly from its ungraded predecessors is in the last paragraph of the proof of Theorem 4, where we need an extra trick to ensure that our chosen regular element is also homogeneous. In the next three results, we only require that G be an abelian semigroup.

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Lemma 2. *Keep the hypotheses of Theorem 1. Then:*

(i) *Any nonzero, graded right ideal I of R contains a non-nilpotent homogeneous element.*

(ii) *The right gr-singular ideal of R is nilpotent, and hence zero.*

Proof. Use the proof of [**5**, Lemma C.I.1.4], respectively [**5**, Lemma C.I.1.2]. (Those results are stated for $\mathbb{Z}$-graded rings, but the structure of the group does not enter into the proof.) □

Lemma 3. *Keep the hypotheses of Theorem 1. Suppose that $a \in R$ is a homogeneous element such that aR is gr-uniform. Then, its right annihilator,* r-ann(a)*, is maximal among right annihilators of nonzero homogeneous elements of R.*

Proof. Suppose that $\text{r-ann}(a) \subsetneq J = \text{r-ann}(b)$, for some homogeneous element $b \in R$. Then $aJ \neq 0$, and so the gr-uniformity of aR implies that aJ is gr-essential in aR. Therefore aR/aJ is a gr-singular right R-module. But $aR/aJ \cong R/J \cong bR$ because $J \supseteq \text{r-ann}(a)$, and so bR is gr-singular. Thus, by Lemma 1(ii), $b = 0$. □

Theorem 4. *Keep the hypotheses of Theorem 1. Then, any essential, graded right ideal I of R contains a homogeneous regular element.*

Proof. Define a homogeneous element $a \in R$ to be *gr-uniform* if the right ideal aR is gr-uniform. By Lemma 2(i) and the Goldie hypothesis, there exists a non-nilpotent, gr-uniform element $a_1 \in I$. By induction, suppose that we have found non-nilpotent, gr-uniform elements $a_1, \dots, a_m \in I$ such that $a_i \in \bigcap_{1\le j\le i-1} \text{r-ann}(a_j)$ for $1 < i \le m$. If $X = \bigcap_{1\le j\le m} \text{r-ann}(a_j) \neq 0$, then $I \cap X \neq 0$ and so Lemma 2(i) again produces a non-nilpotent, gr-uniform element $a_{m+1} \in I \cap X$. Since $a_i \in \text{r-ann}(a_j) = \text{r-ann}(a_j^2)$ for $i > j$, it is easy to see that the sum $\sum_{i\ge 1} a_iR$ is an internal direct sum. Thus, as R has finite gr-uniform dimension, the inductive procedure is finite. In other words, there exists some index n with $\bigcap_{i\le n} \text{r-ann}(a_i) = 0$.

Since R is gr-prime and the a_i are not nilpotent, $a_1^2Ra_2^2R\cdots a_n^2R \neq 0$. Thus, we may find homogeneous elements $s_1, \dots, s_{n-1} \in R$ such that $a_1^2s_1a_2^2s_2\cdots s_{n-1}a_n^2 \neq 0$. Then, by Lemma 2(i), there exists a homogeneous element s_n such that

$$c = a_1^2s_1a_2^2s_2\cdots s_{n-1}a_n^2s_n$$

is not nilpotent. Set

$$d_i = \left(a_is_ia_{i+1}^2s_{i+1}\cdots s_{n-1}a_n^2s_n\right)\left(a_1^2s_1\cdots s_{i-2}a_{i-1}^2s_{i-1}a_i\right)$$

for $i = 1, \dots, n$. Note that the d_i are subwords of c^2 and so they are non-zero. Therefore, Lemma 3 implies that $\text{r-ann}(d_i) = \text{r-ann}(a_i)$, for each i. Moreover, the sum $\sum_{i=1}^n d_iR$ is direct, because $d_iR \subseteq a_iR$. Hence,

$$\text{r-ann}(d_1 + \cdots + d_n) = \bigcap_{i=1}^n \text{r-ann}(d_i) = \bigcap_{i=1}^n \text{r-ann}(a_i) = 0.$$

Note that each d_i is a reordering of the letters of c and so, as G is abelian, $\deg(d_i) = \deg(c)$. Therefore $d_1 + \cdots + d_n$ is a homogeneous regular element lying in I. □

Proof of Theorem 1. The proof of Theorem 1 from Theorem 4 is essentially the same as that in the ungraded case and is left to the reader (see [**5**, Proof of Theorem C.I.1.6]). □

An Application

Recent motivation to prove Theorem 1 has come from the study of quantized coordinate rings, where group-gradings can be used to partition prime spectra in useful ways (see [**2**, Section 6] and [**1**, Part II] for a full discussion). The cited work required an extra hypothesis, which can now be removed.

First consider an arbitrary group G and a G-graded ring R, and let $\text{gr-spec}\,R$ denote the set of gr-prime ideals of R. For any ideal P of R, let $(P:\text{gr})$ denote the largest graded ideal contained in P. Note that if P is prime, then $(P:\text{gr})$ is gr-prime. Hence, there is a partition

$$\operatorname{spec} R = \bigsqcup_{J \in \text{gr-spec}\, R} \operatorname{spec}_J R,$$

where $\operatorname{spec}_J R = \{P \in \operatorname{spec} R \mid (P:\text{gr}) = J\}$. When R is noetherian and G is free abelian of finite rank, each $\operatorname{spec}_J R$ is homeomorphic to the scheme of irreducible subvarieties of an affine algebraic variety, as follows.

Theorem 5. *Let R be a right noetherian ring graded by an abelian group G, and let J be a gr-prime ideal of R.*

(a) *The set $\mathcal{E}_J$ of homogeneous regular elements of R/J is a right denominator set, and the localization $R_J = (R/J)[\mathcal{E}_J^{-1}]$ is a gr-simple, gr-artinian ring.*
(b) *The localization map $R \to R/J \to R_J$ induces a Zariski-homeomorphism of* $\operatorname{spec}_J R$ *onto* $\operatorname{spec} R_J$.
(c) *Contraction and extension induce mutually inverse Zariski-homeomorphisms between* $\operatorname{spec} R_J$ *and* $\operatorname{spec} Z(R_J)$.
(d) *If G is free abelian of rank $r < \infty$, then $Z(R_J)$ is a commutative Laurent polynomial ring over the field $Z(R_J)_1$ in r or fewer indeterminates.*

Proof. Without loss of generality, $J = 0$. Part (a) follows from Theorem 1. In view of Theorem 4, all nonzero graded ideals of R meet $\mathcal{E}_0$, and so $\operatorname{spec}_0 R$ consists of the prime ideals disjoint from $\mathcal{E}_0$. Hence, part (b) follows from standard localization theory. Parts (c) and (d) follow from [**1**, Corollary 4.3 and Lemma 4.1(d)]. $\square$

In the applications to quantized coordinate rings, the grading arises from a torus action, as follows. Let R be a right noetherian algebra over a field k, let $H = (k^\times)^r$ be an algebraic torus over k, and suppose that we have a rational action of H on R by k-algebra automorphisms. This implies that R is graded by the character group $G = \widehat{H}$, where the homogeneous component R_g corresponding to a character g is just the g-eigenspace for the H-action on R. (See [**2**, §6.1] or [**1**, §5.1] for details.) In this setting, $\text{gr-spec}\,R$ is just the set of H-prime ideals of R, and $(P:\text{gr}) = \bigcap_{h\in H} h(P)$ for any ideal P. Theorem 5 applies, and the set $\mathcal{E}_J$ is just the set of regular H-eigenvectors in R/J. The field $Z(R_J)_1$ coincides with $Z(R_J)^H$, the fixed subfield of $Z(R_J)$ under the induced H-action. In fact, $Z(R_J)^H = Z(\operatorname{Fract} R/J)^H$, by the argument of [**1**, Proof of Theorem 5.3(c)].

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