K_0 OF PURELY INFINITE SIMPLE REGULAR RINGS

P. ARA, K. R. GOODEARL, AND E. PARDO

Abstract. We extend the notion of a purely infinite simple C*-algebra to the context of unital rings, and we study its basic properties, specially those related to K-Theory. For instance, if R is a purely infinite simple ring, then $K_0(R)^+ = K_0(R)$, the monoid of isomorphism classes of finitely generated projective R-modules is isomorphic to the monoid obtained from $K_0(R)$ by adjoining a new zero element, and $K_1(R)$ is the abelianization of the group of units of R. We develop techniques of construction, obtaining new examples in this class in the case of von Neumann regular rings, and we compute the Grothendieck groups of these examples. In particular, we prove that every countable abelian group is isomorphic to K_0 of some purely infinite simple regular ring. Finally, some known examples are analyzed within this framework.

Introduction

In 1981, Cuntz [12] introduced the concept of a purely infinite simple C*-algebra. This notion has played a central role in the development of the theory of C*-algebras in the last two decades. A large series of contributions, due to Blackadar, Brown, Lin, Pedersen, Phillips, Rørdam and Zhang, among others, reflect the interest in the structure of such algebras. One of the most important advances in the program of classifying separable C*-algebras through K-Theory, proposed by Elliott in the early seventies, was obtained in this context by Kirchberg [16] and Phillips [25], who showed that nuclear separable unital purely infinite simple C*-algebras are classified by K-theoretic invariants.

In this work, we introduce a suitable definition of a purely infinite simple ring. This notion agrees with that of Cuntz in the case of C*-algebras. Moreover, a number of basic results, known in the case of C*-algebras, also hold in the purely algebraic context. In particular the algebraic K_0 and K_1 groups of a purely infinite simple ring follow the same patterns as the corresponding topological K groups of purely infinite simple C*-algebras, found by Cuntz in [12].

Our goal is to use these concepts and ideas in order to advance the knowledge of the K-theory of (von Neumann) regular rings. In particular, we construct examples of

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purely infinite simple regular rings whose K_0 groups are cyclic of arbitrary order, and – in the case when K_0 is finite cyclic – whose finitely generated projective modules are free. By using these constructions, we build examples of purely infinite simple regular rings whose K_0 is any countable abelian group.

A key point of our work lies in the development of new techniques for constructing examples. Our algebras are extensions of the algebras $V_{1,n}$, first considered by Leavitt in [19]. Recall that the Leavitt algebra $V_{1,n}$ over a field k is the k-algebra having a universal right $V_{1,n}$-module isomorphism $V_{1,n}^n \to V_{1,n}$. Our method for getting the extensions of the $V_{1,n}$ works as follows. We consider a local subalgebra of the algebra of noncommutative power series $k\langle\langle X\rangle\rangle$, where $X = \{x_0, \dots, x_n\}$, closed under a certain set of skew derivations, and containing the free algebra $k\langle X\rangle$. Then we construct our extensions of $V_{1,n}$ by universally inverting the right R-module homomorphism $R \to R^{n+1}$ given by left multiplication by the column $(x_0, \dots, x_n)^T$. We prove that K_0 of these algebras is always the cyclic group of order n. Taking R to be the full algebra of noncommutative power series one obtains what could be thought of as a completion of $V_{1,n+1}$. Taking the minimal choice for R, which turns to be the so-called algebra of rational series, one obtains a kind of "algebra of fractions" of $V_{1,n+1}$. The latter algebra is shown to coincide with a construction of Schofield [29], and also with a construction of Rosenmann and Rosset [27]. In particular it enjoys various additional universal properties.

We briefly outline the contents of the paper. In Section 1 we define purely infinite simple rings, and we prove some of their elementary properties. In Section 2 we give the basic patterns for the algebraic K_0 and K_1 of purely infinite simple rings, in analogy with Cuntz's results on the topological K_0 and K_1 of purely infinite simple C*-algebras. Section 3 is devoted to introducing the construction of skew polynomial rings with freely independent indeterminates, which will be a fundamental technical device for the rest of the paper. In Section 4 we outline some known results on Leavitt's algebras, which can be considered as the earliest universal examples in this class, and also as the algebraic relatives of Cuntz's algebras. (The Cuntz algebra $\mathcal{O}_n$ is the C*-completion of the Leavitt algebra $V_{1,n}$ over the field of complex numbers.) Section 5 is the core of the paper: we develop here the construction of our examples, we establish their structure, and we compute their K_0 groups. In Section 6 we present the examples of Schofield [29], as well as those of Rosenmann and Rosset [27], and we show that they are isomorphic. Section 7 is devoted to show that the examples in Section 6 are isomorphic to particular cases of the examples presented in Section 5. As a consequence, we see that these algebras can be obtained in two different ways as universal localizations of free algebras. Finally, we prove in Section 8 that any countable abelian group can be realized as K_0 of a purely infinite simple regular ring.

Aside from a few noted exceptions, all rings and modules in this paper will be assumed to be unital. In fact, most of our rings will be algebras over a base field that we denote k. Ring and algebra homomorphisms, except for embeddings of ideals, will also be assumed to be unital. We will often write nA for the direct sum of n copies of a module A, although in the case of a ring R we prefer the notation R^n for the free right R-module of rank n, whose elements we think of as column vectors. The notations $k\langle X\rangle$ and $k\langle\langle X\rangle\rangle$, respectively, will stand for the free algebra and the noncommutative formal power series algebra over k on a set X.

1. Purely infinite simple rings

We introduce the concept of a purely infinite simple ring, and sketch some basic results on this topic. These results are analogous to those obtained by Cuntz.

First, recall that if R is a ring and $e, f \in R$ are idempotents, we say that e and f are (Murray–von Neumann) *equivalent* (denoted $e \sim f$) provided that there exist elements $x \in eRf$ and $y \in fRe$ such that $xy = e$ and $yx = f$. This is equivalent to demanding that $eR \cong fR$ as right R-modules. Recall also that e and f are said to be *orthogonal* (denoted $e \perp f$) provided that $ef = fe = 0$. In that case, $e+f$ is an idempotent, and $(e+f)R = eR \oplus fR$.

The following useful means of producing orthogonal decompositions of idempotents is old and well known; we sketch the easy proof for convenience.

Lemma 1.1. *Let e be an idempotent in a ring R. If there exist right ideals $A_i \subseteq R$ such that $eR = A_1 \oplus \cdots \oplus A_n$, then there exist pairwise orthogonal idempotents $e_i \in R$ such that $e = e_1 + \cdots + e_n$ and $A_i = e_iR$ for all i.*

Proof. Due to the given direct sum decomposition, the endomorphism ring of the module eR contains orthogonal idempotents $f_1, \dots, f_n$ such that $f_1 + \cdots + f_n$ equals the identity map on eR and each $f_i(eR) = A_i$. The desired idempotents e_i are the images of the f_i under the canonical isomorphism $\mathrm{End}_R(eR) \to eRe$. □

Definitions 1.2. An idempotent e in a ring R is *infinite* if there exist orthogonal idempotents $f, g \in R$ such that $e = f + g$ while $e \sim f$ and $g \neq 0$. In view of Lemma 1.1, e is infinite if and only if eR is isomorphic to a proper direct summand of itself, that is, eR is a *directly infinite* module.

A simple ring R is said to be *purely infinite* if every nonzero right ideal of R contains an infinite idempotent. It will follow from Theorem 1.6 that this concept is left-right symmetric.

Examples 1.3. The class of purely infinite simple rings is rather large; we indicate some subclasses here:

(a) Many purely infinite simple C*-algebras are known; for instance, see [1], [11], [12], [16], [17], [18], [21], [25], [26], [37].
(b) If V is an infinite dimensional vector space over k, then $\mathrm{End}_k(V)$ modulo its unique maximal ideal M is purely infinite. This follows from the fact that if $f \in \mathrm{End}_k(V) \setminus M$, then $\dim_k f(V) = \dim_k V$.
(c) More generally, if R is a regular, right self-injective ring without nonzero directly finite central idempotents, and M is any maximal ideal of R, then R/M is purely

infinite. To see this, consider any element $x \in R \setminus M$. Since R/M is simple, $\sum_{i=1}^{n} a_i x b_i - 1 \in M$ for some $a_i, b_i \in R$, from which we see that R_R embeds in $n(xR) \oplus zR$ for some $z \in M$. By general comparability [14, Corollary 9.15], there is a central idempotent $e \in R$ such that $ezR \lesssim exR$ and $(1-e)xR \lesssim (1-e)zR$; in particular, $(1-e)x \in M$. Since $x \notin M$, we cannot have $e \in M$, and so $1-e \in M$. Now eR is isomorphic to exR. For, since e is a directly infinite central idempotent, we have $(n+1)(eR) \cong eR$ by [14, Theorem 10.16]. Thus, as $eR \lesssim (n+1)(exR) \lesssim (n+1)(eR)$, we have $(n+1)(eR) \cong (n+1)(exR)$ by [14, Theorem 10.14]. Hence, $eR \cong exR$ by [14, Theorem 8.16(b)]. Since the class of e in R/M is 1, we conclude that R/M is isomorphic to $x(R/M)$.

(d) If R is a directly infinite regular ring and all (finitely generated) projective right R-modules are free, then R is a purely infinite simple ring. First, if x is a nonzero element of R, then the projective module xR is free, whence R_R is isomorphic to a direct summand of xR, and consequently $RxR = R$. Thus R is simple. Further, since R_R is directly infinite, so is xR. Hence, all nonzero idempotents in R are infinite, and therefore R is purely infinite.

(e) If A is any directly infinite simple ring, then the direct limit $B = \varinjlim M_{2^n}(A)$ (with block diagonal transition maps) is purely infinite. To show this, let z be a nonzero element of B; then z is the image of a nonzero element $x \in M_{2^n}(A)$ for some n, and we can assume without loss of generality that $n = 0$. For a suitable m we have that $\sum_{i=1}^{2^m} a_i x b_i = 1$ for some a_i, b_i in A. If we use the a_i (respectively, the b_i) as the first row (respectively, column) of $2^m \times 2^m$ matrices whose other entries are zero, we obtain $a, b \in M_{2^m}(A)$ such that $ayb = e_{11}$, where y is the image of x in $M_{2^m}(A)$ and e_{11} is the matrix unit in the top left corner of $M_{2^m}(A)$. Since the idempotent $1 \in A$ is infinite, so is the idempotent ayb in $M_{2^m}(A)$. But yba is an idempotent equivalent to ayb, so it too is infinite. Its image in B is an infinite idempotent contained in the right ideal zB.

Lemma 1.4. *Let R be a simple ring, and let P and Q be finitely generated projective right R-modules. If P is directly infinite, then there exists a nonzero right R-module A such that $P \cong Q \oplus A$.*

Proof. By hypothesis, there exists a nonzero right R-module B such that $P \cong P \oplus B$, whence $P \cong P \oplus mB$ for all $m \in \mathbb{N}$. Since R is simple and B is projective, B is a generator in Mod-R; in particular, there exist $n \in \mathbb{N}$ and a right R-module C such that $nB \cong Q \oplus C$. Therefore

$$P \cong P \oplus nB \cong Q \oplus (P \oplus C),$$

and $P \oplus C$ is nonzero because $P \neq 0$. □

Proposition 1.5. *Suppose that R is a purely infinite simple ring. Then all nonzero finitely generated projective right R-modules are directly infinite; equivalently, all nonzero idempotents in the matrix rings $M_n(R)$ are infinite.*

Consequently, if P and Q are any nonzero finitely generated projective right R-modules, then there exists a nonzero right R-module A such that $P \cong Q \oplus A$.

Proof. By hypothesis, there is at least one infinite idempotent $e \in R$, whence eR is a directly infinite, finitely generated projective right R-module. If Q is any nonzero finitely generated projective right R-module, Lemma 1.4 implies that Q is isomorphic to a direct summand of eR, and so Q is isomorphic to some nonzero right ideal $I \subseteq R$. Now since R is purely infinite, there is an infinite idempotent $f \in I$, whence fR is a directly infinite direct summand of I. It follows that I is directly infinite, whence Q is directly infinite.

The final conclusion of the proposition now follows immediately from Lemma 1.4. □

Theorem 1.6. *Let R be a simple ring. Then R is purely infinite if and only if*

(a) *R is not a division ring, and*
(b) *For every nonzero element $a \in R$, there exist elements $x, y \in R$ such that $xay = 1$.*

Proof. ($\Longrightarrow$): Assume that R is purely infinite. Then R contains an infinite idempotent, and so R cannot be a division ring.

Now consider a nonzero element $a \in R$. By assumption, there exists an infinite idempotent $e \in aR$, and then Proposition 1.5 implies that $eR \cong R \oplus A$ for some A. Now by Lemma 1.1, there exist orthogonal idempotents $f, g \in R$ such that $e = f + g$ and $fR \cong R$. Then $f \sim 1$, and so there are elements $\alpha \in fR$ and $\beta \in Rf$ such that $\alpha\beta = f$ and $\beta\alpha = 1$. Since $f \in eR \subseteq aR$, we also have $f = ar$ for some $r \in R$. Therefore

$$1 = \beta\alpha\beta\alpha = \beta f \alpha = \beta a(r\alpha),$$

and (b) is established.

($\Longleftarrow$): Now assume conditions (a) and (b), and consider a nonzero right ideal $I \subseteq R$. Since R is not a division ring, I must contain a proper nonzero right ideal, say J. Choose a nonzero element $a \in J$, and apply condition (b): there exist $x, y \in R$ such that $xay = 1$. Now $e := ayx$ is an idempotent lying in aR, whence $e \in I$ and $e \neq 1$. Since $(eay)(xe) = e$ and $(xe)(eay) = 1$, we also have $e \sim 1$, and so 1 is infinite. But then e, being equivalent to 1, is infinite too, and we have proved that R is purely infinite. □

Notice that Theorem 1.6 implies that the definition of a purely infinite simple ring given above is left-right symmetric (this can also be shown by a direct argument). Also, we point out that in view of condition (b) of the theorem, our definition agrees with the definition in current use among C*-algebraists. (For the equivalence of this definition with Cuntz's original definition, see [6, Proposition 6.11.5].)

Corollary 1.7. *The class of purely infinite simple rings is closed under Morita equivalence.*

Proof. It suffices to show that if R is a purely infinite simple ring, then $M_2(R)$ and eRe have the same properties, where e is an arbitrary nonzero idempotent in R. It is clear that $M_2(R)$ and eRe are simple.

Because R contains an infinite idempotent, it cannot be artinian. But R is Morita equivalent to eRe, and thus eRe cannot be a division ring. For any nonzero element $a \in eRe$, Theorem 1.6 provides elements $x, y \in R$ such that $xay = 1$, and then exe and eye are elements of eRe such that $(exe)a(eye) = e$. Thus, by the theorem, eRe is purely infinite.

By Proposition 1.5, there is an idempotent $f \in R$ such that $fR \cong 2R$, whence $fRf \cong M_2(R)$. Since fRf is purely infinite by the previous paragraph, we therefore conclude that $M_2(R)$ is purely infinite, as desired. □

Remarks 1.8.

(a) Zhang ([37, Theorem 1]), and also Brown and Pedersen ([7, Proposition 3.9]) have proved that every purely infinite simple C*-algebra has real rank zero, which by [2, Theorem 7.2] is equivalent to the property of being an exchange ring [34]. Thus, the following question imposes itself:

Is every purely infinite simple ring an exchange ring?

(b) Let R be any directly infinite, simple exchange ring which is separative (see [2]). Then R is necessarily purely infinite, as follows. It follows from simplicity and the exchange criterion of Nicholson and Goodearl that any nonzero right ideal of R contains a nonzero idempotent, say e. Then $R_R \lesssim n(eR)$ for some n, whence R is isomorphic to a corner of $M_n(eRe)$, and so $M_n(eRe)$ is directly infinite. By [2, Proposition 2.3], eRe must be directly infinite, that is, the idempotent e is infinite.

(c) If A is a simple exchange ring, then the simple ring $B = \varinjlim M_{2^n}(A)$ (with block diagonal transition maps) will either have stable rank 1 or be purely infinite. For, notice that, if there is some n such that $M_{2^n}(A)$ is directly infinite, then B is purely infinite by Example 1.3(e). Otherwise, A is stably finite, and so it has power-cancellation by [5, Proposition 2.1.8]. This in turn implies that B has cancellation, and thus has stable rank 1 by [36, Theorem 9].

(d) If A is a simple QB-ring (see [3]), then it either has stable rank 1 or is purely infinite [24].

2. K_0 AND K_1 OF PURELY INFINITE SIMPLE RINGS

Cuntz [12] computed the general patterns for K_0 and (topological) K_1 of purely infinite simple C*-algebras. Here we give parallel results for the K_0 and the (algebraic) K_1 groups of purely infinite simple rings. We recall some basics of K-Theory for the convenience of the reader.

Given a ring R, we define $\mathcal{V}(R)$ to be the set of isomorphism classes (denoted $[A]$) of finitely generated projective right R-modules, and we endow $\mathcal{V}(R)$ with the structure of a commutative monoid by imposing the operation

$$[A] + [B] := [A \oplus B]$$

for any isomorphism classes $[A]$ and $[B]$. (The notation $[A]$ will also be used for *stable* isomorphism classes, that is, elements of $K_0(R)^+$, but it should be clear from the context which is meant.) Equivalently [6, Chapter 3], $\mathcal{V}(R)$ can be viewed as the set of equivalence classes of idempotents in $M_\infty(R) = \bigcup_{n=1}^\infty M_n(R)$ with the operation

$$[e] + [f] := \left[\left(\begin{smallmatrix} e & 0 \\ 0 & f \end{smallmatrix}\right)\right]$$

for idempotents $e, f \in M_\infty(R)$. The group $K_0(R)$ is the universal group of $\mathcal{V}(R)$, and the image of the canonical monoid homomorphism $\mathcal{V}(R) \to K_0(R)$ is the *positive cone* of $K_0(R)$, denoted $K_0(R)^+$.

Observe that $\mathcal{V}(R)$ is *conical*, that is, $x + y = 0$ in $\mathcal{V}(R)$ only if $x = y = 0$. Consequently, the subset $\mathcal{V}(R)^* := \mathcal{V}(R) \setminus \{0\}$ is closed under addition. There is a natural pre-order $\leq$ on $\mathcal{V}(R)$, where $x \leq y$ if and only if there exists $z \in \mathcal{V}(R)$ such that $x + z = y$. In terms of isomorphism classes of finitely generated projective modules A and B, we have $[A] \leq [B]$ if and only if A is isomorphic to a direct summand of B. We say that $\mathcal{V}(R)$ is *simple* provided that for each nonzero $x, y \in \mathcal{V}(R)$, there exists $n \in \mathbb{N}$ such that $y \leq nx$. Observe that if R is a simple ring, then $\mathcal{V}(R)$ is a simple monoid (recall that simplicity of R implies that all nonzero projective R-modules are generators in Mod-R). The converse is false (e.g., take $R = \mathbb{Z}$).

Lemma 2.1. *Let R be a purely infinite simple ring. For any $x, y \in \mathcal{V}(R)^*$, there exist $a, b \in \mathcal{V}(R)^*$ such that $x = y + a$ and $y = x + b$.*

Proof. This is just a restatement of the last part of Proposition 1.5. □

Proposition 2.2. *If R is a purely infinite simple ring, then $\mathcal{V}(R)^*$ is a group.*

In particular, any nonzero finitely generated projective R-modules which are stably isomorphic must be isomorphic.

Proof. According to [2, Proposition 2.4], to prove that $\mathcal{V}(R)^*$ is a group it suffices to show that $\mathcal{V}(R)$ is conical and simple, and that for each $x \in \mathcal{V}(R)^*$ there exists $b \in \mathcal{V}(R)^*$ such that $x + b = x$. However, we already know that $\mathcal{V}(R)$ is conical and simple, and the remaining condition follows from Lemma 2.1 (take $y = x$). Therefore $\mathcal{V}(R)^*$ is a group.

The final statement of the proposition now follows. □

An interesting consequence of Proposition 2.2 is that if R is a purely infinite simple ring and $[P]$ is the identity element of $\mathcal{V}(R)^*$, then P is nonzero and $P \oplus Q \cong Q$ for all finitely generated projective right R-modules Q.

Notice that, aside from the fact that the stable rank of a purely infinite simple ring is ∞, Proposition 2.2 shows that the cancellative behavior of finitely generated projective modules is almost the same as in the case of rings of stable rank 1.

If M is an additive monoid, we write $\{0\} \sqcup M$ for the monoid constructed by adjoining a new zero element to M. (To work with this new monoid, one must choose some notation to distinguish between the old and new zero elements.)

Corollary 2.3. *If R is a purely infinite simple ring, then $\mathcal{V}(R) \cong \{0\} \sqcup K_0(R)$, and $K_0(R)^+ = K_0(R)$.*

Proof. Let $\phi : \mathcal{V}(R) \to K_0(R)$ be the natural monoid homomorphism; thus if A is any finitely generated projective right R-module, $\phi([A]) = [A]$ is the stable isomorphism class of A. By Proposition 2.2, $\mathcal{V}(R)^*$ is a group, and so ϕ restricts to a group homomorphism $\phi^* : \mathcal{V}(R)^* \to K_0(R)$. Thus the image of ϕ^* is a subgroup of $K_0(R)$. In particular, this image contains $[0]$, and so $\phi^*(\mathcal{V}(R)^*) = \phi(\mathcal{V}(R)) = K_0(R)^+$. Since $K_0(R)$ is generated by $K_0(R)^+$, we now see that $K_0(R)^+ = K_0(R)$.

If $x, y \in \mathcal{V}(R)^*$ and $\phi^*(x) = \phi^*(y)$, then $x + z = y + z$ for some $z \in \mathcal{V}(R)$. Since either $z = 0$ or z lies in the group $\mathcal{V}(R)^*$, it follows that $x = y$. This shows that ϕ^* is an isomorphism of $\mathcal{V}(R)^*$ onto $K_0(R)^+ = K_0(R)$, and therefore we conclude that $\mathcal{V}(R) \cong \{0\} \sqcup K_0(R)$. □

For completeness, we present the following theorem which shows the parallelism of patterns between algebraic and topological K_1 in the cases of purely infinite simple rings and C*-algebras. However, K_1 will not appear elsewhere in the paper.

Theorem 2.4. *If R is a purely infinite simple ring then $K_1(R) = U(R)^{\mathrm{ab}}$.*

Proof. By [23, Remark after 2.3], R is a GE-ring, so the natural map $\kappa : U(R) \to K_1(R)$ is surjective. In order to show that κ is injective, we will proceed in two steps. The first one is similar to Cuntz's argument in [12].

Step 1. Let v be a unit in the kernel of κ. Assume there is a nonzero idempotent $e \in R$ such that $v = e + (1-e)v(1-e)$. Then $v \in U(R)'$.

Proof of Step 1: Take an idempotent $f < e$ such that $f \sim e$. Set $r_1 = 1-e+f$, and note that $r_1 \sim 1$. Now let $r_2, r_3, \dots$ be orthogonal idempotents such that $r_2 + \dots + r_i \le e - f$ and $r_i \sim 1$ for all $i \ge 2$. Notice that $(r_1 + \dots + r_i)R(r_1 + \dots + r_i) \cong M_i(R)$; this isomorphism may be chosen so that its restriction to r_1Rr_1 sends

$$1 - e \longmapsto \operatorname{diag}(1-e, 0, \dots, 0) \qquad \text{and} \qquad f \longmapsto \operatorname{diag}(e, 0, \dots, 0).$$

Hence, the element $v_i := (1-e)v(1-e) + f + r_2 + \dots + r_i$ corresponds to $\operatorname{diag}(v, 1, \dots, 1)$ under this isomorphism. Since the image of v in $K_1(R)$ is 0, there is some $n \ge 1$ such that $\operatorname{diag}(v, 1, \dots, 1) \in GL_n(R)'$. Consequently, $v_n \in U\big((r_1 + \dots + r_n)R(r_1 + \dots + r_n)\big)'$, and so $v = v_n + e - f - (r_2 + \dots + r_n)$ lies in $U(R)'$, as desired.

Step 2. For any $u \in U(R)$ there is a nonzero idempotent $e \in R$ and a unit of the form $v = e + (1-e)v(1-e)$ such that $u \equiv v \pmod{U(R)'}$.

Proof of Step 2: Any decomposition $1 = e_1 + \dots + e_n$ with $e_1 \sim \dots \sim e_{n-1}$ and $e_n \lesssim e_1$ gives rise to an isomorphism $R \cong S$, where S is the ring of $n \times n$ matrices $A = (a_{ij})$ over $T = e_1Re_1$ of the following form: all the entries a_{in} are in Tf, and all the entries a_{nj} are in fT, where f is the idempotent corresponding to e_n under the subequivalence $e_n \lesssim e_1$. An argument similar to that of Whitehead's Lemma proves that, for $n \ge 4$, the set of elementary matrices relative to the above matrix decomposition of R is contained in $U(R)'$. Now take $n \ge 4$ and a decomposition of 1 of the form indicated. Consider the corresponding matricial representation over $T = e_1Re_1$, which is also a purely infinite simple ring. By the argument in [23, Theorem 2.2], the matrix corresponding to u can be transformed by elementary operations to a matrix of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix},$$

where $*$ is a matrix of size $(n-1) \times (n-1)$. Since $n \geq 4$, the elementary matrices give rise to elements in $U(R)'$, so that u is congruent mod $U(R)'$ to a unit of the form $e_1 + (1-e_1)v(1-e_1)$, as desired. $\square$

3. Skew polynomial rings with freely independent indeterminates

In this section, we introduce a general construction that turns out to be a keystone for our work. Let R be a ring and $(\tau_1, \delta_1), \ldots, (\tau_n, \delta_n)$ some right skew derivations on R. Thus, the τ_i are ring endomorphisms of R, and the δ_i are additive endomorphisms satisfying the rule $\delta(rs) = \delta(r)\tau(s) + r\delta(s)$. Let $Y = \{y_1, \ldots, y_n\}$ be an n-element alphabet and denote by Y^* the free monoid on Y, with identity element denoted 1. Label words in Y^* in the form $y_I = y_{i_1}y_{i_2}\cdots y_{i_t}$ for finite sequences $I = (i_1, \ldots, i_t)$ of indices from $\{1, \ldots, n\}$, and similarly for other quantities like τ_I or δ_I.

We would like to build a ring whose right R-module structure is free with basis Y^*, such that $ry_i = y_i\tau_i(r) + \delta_i(r)$ for all i and all $r \in R$. To define a multiplication on this R-module and verify the ring axioms is tedious; instead, we build a ring of operators on this module, and afterwards carry over the structure.

Let V be the free right R-module with basis Y^*, and let $E = \mathrm{End}_{\mathbb{Z}}(V)$. We let maps in E act on the *right* of their arguments, so that we can identify R with the subring of right multiplication operators in E. Define $z_1, \ldots, z_n \in E$ so that

$$(yr)z_i = (yy_i)\tau_i(r) + y\delta_i(r)$$

for all i, all $r \in R$, and all $y \in Y^*$. Let S be the subring of E generated by R and $z_1, \ldots, z_n$. For all i, all $r, s \in R$, and all $y \in Y^*$, we have

$$\begin{aligned}(ys)(rz_i) &= (ysr)z_i = (yy_i)\tau_i(sr) + y\delta_i(sr) \\ &= (yy_i)\tau_i(s)\tau_i(r) + y\delta_i(s)\tau_i(r) + (ys)\delta_i(r) = (ys)[z_i\tau_i(r) + \delta_i(r)].\end{aligned}$$

Therefore

$$(*) \qquad rz_i = z_i\tau_i(r) + \delta_i(r)$$

for all i and all $r \in R$. In particular, (*) implies that S is generated as a right R-module by the monomials z_I. All $\tau_i(1) = 1$ and $\delta_i(1) = 0$, so $(y)z_i = yy_i$ for all $y \in Y^*$. Consequently, we get by induction that $(1)z_I = y_I$ for all I. Any $s \in S$ can be written as $\sum_I z_Ir_I$ with almost all $r_I = 0$, and $(1)s = \sum_I y_Ir_I$. Since the y_I form a basis for V as a right R-module, the rule $s \mapsto (1)s$ thus gives a right R-module isomorphism from S onto V. We summarize our observations as follows:

Proposition 3.1. *Given a ring R with right skew derivations (τ_i, δ_i) for $i = 1, \dots, n$, there is a ring S containing R as a subring such that:*

(a) *S_R is free with basis Y^*, where Y^* is the free monoid on an n-element set $Y = \{y_1, \dots, y_n\}$.*

(b) *The ring and module multiplications $S \times R \to S$ coincide.*

(c) *The ring and monoid multiplications $Y^* \times Y^* \to S$ coincide.*

(d) *$ry_i = y_i\tau_i(r) + \delta_i(r)$ for all i and all $r \in R$.*

Notation 3.2. We denote the ring S in Proposition 3.1 by $R\langle Y; \tau, \delta\rangle$, where τ and δ are abbreviations for the n-tuples $(\tau_1, \dots, \tau_n)$ and $(\delta_1, \dots, \delta_n)$. To see that $R\langle Y; \tau, \delta\rangle$ is unique in a suitable sense, we establish the following universal property:

Proposition 3.3. *Let R be a ring and $(\tau_1, \delta_1), \dots, (\tau_n, \delta_n)$ right skew derivations on R. Suppose $\phi : R \to T$ is a ring homomorphism and $t_1, \dots, t_n \in T$ are elements such that $\phi(r)t_i = t_i\phi\tau_i(r) + \phi\delta_i(r)$ for all i and all $r \in R$. Then there exists a unique ring homomorphism $\overline{\phi} : R\langle Y; \tau, \delta\rangle \to T$ such that $\overline{\phi}|_R = \phi$ and $\overline{\phi}(y_i) = t_i$ for all i.*

Proof. It suffices to construct a ring with this universal property and show it is isomorphic to $R\langle Y; \tau, \delta\rangle$. More precisely, let $F = \mathbb{Z}\langle z_1, \dots, z_n\rangle$ be the free ring on n letters, let $S_0 = F *_{\mathbb{Z}} R$ be the ring coproduct of F and R, and let S_1 be the factor ring of S_0 by the ideal generated by $(1 * r)(z_i * 1) - z_i * \tau_i(r) - 1 * \delta_i(r)$ for all i and all $r \in R$. Now let $\psi : R \to S_1$ be the map defined by $\psi(r) = \overline{1 * r}$ and set $w_i = \overline{z_i * 1} \in S_1$.

Then for all i and all $r \in R$ we have

$$\psi(r)w_i = w_i\psi\tau_i(r) + \psi\delta_i(r), \tag{†}$$

and $(S_1, \psi, w_1, \dots, w_n)$ is universal with respect to (†). In particular, there is a unique ring homomorphism $\theta : S_1 \to R\langle Y; \tau, \delta\rangle$ such that $\theta\psi$ is the inclusion map $R \to R\langle Y; \tau, \delta\rangle$ and $\theta(w_i) = y_i$ for all i. It is enough to show that θ is an isomorphism.

From (†), we see that $S_1 = \sum_I w_I\psi(R)$. Note that $\theta(\sum_I w_I\psi(r_I)) = \sum_I y_I r_I$ for all finite sums with $r_I \in R$. Since $R\langle Y; \tau, \delta\rangle_R$ is free with basis $\{y_I\}$, it follows that θ is an isomorphism. □

Example 3.4. We give an example of the construction considered above, and show that it coincides with a ring constructed by Tyukavkin ([32], [33]).

Take $R = k\langle\langle X\rangle\rangle$, where $X = \{x_1, \dots, x_n\}$. Let $\tau_1 = \tau_2 = \cdots = \tau_n = \tau$ be the unique k-algebra homomorphism $\tau : R \to R$ sending all x_i to 0. Write elements of R as infinite sums $\sum_{w\in X^*} \lambda_w w$, where $\lambda_w \in k$ and X^* is the free monoid on X. Define k-linear maps $\delta_i : R \to R$ by the rule

$$\delta_i\Big(\sum_{w\in X^*} \lambda_w w\Big) = \sum_{w\in X^*} \lambda_{wx_i} w.$$

We want to check that the δ_i are right τ_i-derivations. Take two elements $r = \sum_w \lambda_w w$ and $r' = \sum_w \mu_w w$ in R, so that $rr' = \sum_w \big(\sum_{uv=w} \lambda_u\mu_v\big)w$. Then we have

$$\delta_i(rr') = \sum_w \Big(\sum_{uv=wx_i} \lambda_u\mu_v\Big)w = \sum_w \Big(\lambda_{wx_i}\mu_1 + \sum_{uv=w} \lambda_u\mu_{vx_i}\Big)w,$$

since $(wx_i)1$ is the only factorization $uv = wx_i$ where x_i is not a right factor of v. Also,

$$\begin{aligned}\delta_i(r)\tau(r') + r\delta_i(r') &= \big(\sum_w \lambda_{wx_i} w\big)\mu_1 + \big(\sum_u \lambda_u u\big)\big(\sum_v \mu_{vx_i} v\big)\\ &= \sum_w \big(\lambda_{wx_i}\mu_1 + \sum_{uv=w} \lambda_u \mu_{vx_i}\big) w,\end{aligned}$$

the same as above. This proves that δ_i is a right τ_i-derivation. It follows that there exists a k-algebra $R\langle Y;\tau,\delta\rangle$. Note that $\tau_i(x_j) = 0$ and $\delta_i(x_j) = \delta_{ij}$ for all i, j. Hence, $x_i y_j = \delta_{ij}$ for all i, j. It follows that the algebra $R\langle Y;\tau,\delta\rangle$ in this case coincides with the algebra constructed by Tyukavkin in [33, page 404].

The examples of Leavitt also arise from the $R\langle Y;\tau,\delta\rangle$ construction, as we observe in the next section.

4. Leavitt's algebras

This section is devoted to showing that some universal examples of non-IBN algebras lie in the class of purely infinite simple rings. Our interest in quoting them here is twofold: on one side, these examples are the algebraic precursors of Cuntz algebras; on the other side, Tyukavkin's example quoted in Example 3.4 are, in some sense, completions of Leavitt's examples.

For any field k, and for any two natural numbers m, n, Leavitt ([19]) introduced the k-algebras $V_{m,n}$, with a universal isomorphism $i : nV_{m,n} \longrightarrow mV_{m,n}$, and $U_{m,n}$, with a universal pair of morphisms $i : nU_{m,n} \longrightarrow mU_{m,n}$ and $j : mU_{m,n} \longrightarrow nU_{m,n}$ such that $ji = 1_{nU_{m,n}}$. Later, Cohn, Skornyakov and Bergman proved some fundamental properties of these algebras. The monoid $\mathcal{V}(R)$ of these examples was computed by Bergman in [4]. For the sake of completeness, we recall here Bergman's result.

Theorem 4.1. ([4, Theorem 6.1]) *For a field k and positive integers m, n, the rings $V_{m,n}$ and $U_{m,n}$ are hereditary. Moreover, $\mathcal{V}(V_{m,n}) = \langle I \mid mI = nI\rangle$, and $\mathcal{V}(U_{m,n}) = \langle I, P \mid mI = nI + P\rangle$.*

We will use the constructions in Section 3 to give some extra information on the algebras $V_{1,n}$ and $U_{1,n}$. First of all, notice that $U_{1,n} = k\langle X\rangle\langle Y;\tau,\delta\rangle$, where X, Y, τ, δ are as in Example 3.4. This is clear once we observe that the row $(y_1, \dots, y_n)$ and the column $(x_1, \dots, x_n)^T$ give a universal pair of morphisms $i : nR_n \longrightarrow R_n$ and $j : R_n \longrightarrow nR_n$ respectively such that $ji = 1_{nR_n}$, where we set $R_n = k\langle X\rangle\langle Y;\tau,\delta\rangle$. The algebra $V_{1,n}$ is thus the algebra obtained by imposing the relation $ij = 1_{R_n}$ in R_n, that is, $V_{1,n} \cong R_n/I_n$, where I_n is the ideal of R_n generated by the idempotent $e_n = 1 - \sum_{i=1}^n y_i x_i$. Following Tyukavkin ([32], [33]) we will call the elements in X^* *monomials* and the elements in Y^* *words*. From our basic relations $x_i y_j = \delta_{ij}$ it follows that to each monomial x_I there corresponds a word y_{I^*} such that $x_I y_{I^*} = 1$, where $I^* = (i_r, \dots, i_1)$ for $I = (i_1, \dots, i_r)$. A *monoword* is any element of the form $y_I x_J$ for some indices I, J. Notice that the monowords $y_I x_J$ form a k-basis of $U_{1,n}$.

Theorem 4.2. *For every natural number $n \geq 2$, $V_{1,n}$ is a purely infinite simple ring, and $K_0(V_{1,n}) \cong \mathbb{Z}/(n-1)\mathbb{Z}$.*

Proof. By Theorem 4.1, $\mathcal{V}(V_{1,n}) = \langle I \mid nI = I \rangle$, from which it follows that $K_0(V_{1,n}) \cong \mathbb{Z}/(n-1)\mathbb{Z}$.

We identify $k\langle X\rangle\langle Y; \tau, \delta\rangle$ with $U_{1,n}$. Let I_n be the ideal of $U_{1,n}$ generated by $e_n := 1 - \sum_{i=1}^n y_i x_i$. As observed before, we have $V_{1,n} \cong U_{1,n}/I_n$. Every element $\alpha \in U_{1,n}$ can be written uniquely as $\alpha = \sum_{I,J} \lambda_{I,J} y_I x_J$ for scalars $\lambda_{I,J}$ which are almost all zero. The *support* of α is the set of pairs (I, J) such that $\lambda_{I,J} \neq 0$; denote by $s(\alpha)$ the cardinality of the support of α. Since $x_i e_n = e_n y_j = 0$ for all i, j, the ideal I_n is spanned (over k) by products of the form $y_I e_n x_J$. It follows that the support of any nonzero element of I_n must contain a pair (I, J) with both I and J nonempty. Consequently, I_n contains no nonzero elements of either $k\langle X\rangle$ or $k\langle Y\rangle$.

We prove that for $\alpha \in U_{1,n} \setminus I_n$, there exist $\beta, \gamma \in U_{1,n}$ such that $\beta\alpha\gamma = 1$. (In fact, β can be taken as a scalar times a monomial, and γ as a word.) If $s(\alpha) = 1$, then $\alpha = \lambda_{I,J} y_I x_J$ with $\lambda_{I,J} \neq 0$, and so

$$(\lambda_{I,J}^{-1} x_{I^*})\alpha y_{J^*} = 1.$$

Now assume that $s(\alpha) = d > 1$. There must be an index i such that $\alpha y_i \notin I_n$, since otherwise we would have $\alpha = \alpha\big(e_n + \sum_{i=1}^n y_i x_i\big) \in I_n$. Note that either $\alpha \in k\langle Y\rangle$ or the total degree in X of αy_i is less than that of α. Hence, by induction on this degree, there is a word y_K such that αy_K is a nonzero polynomial in $k\langle Y\rangle$. Clearly $\alpha y_K \notin I_n$, and so there is a monomial x_I such that $\alpha' = x_I \alpha y_K$ is a polynomial in $k\langle X\rangle$ with nonzero constant term. Clearly, we can choose a word y_J such that $\alpha' y_J \neq 0$ and $s(\alpha' y_J) < d$. Since $x_{J^*}\alpha' y_J$ is a polynomial in $k\langle X\rangle$ with nonzero constant term, we see that $\alpha' y_J \notin I_n$. By induction, there exist $\beta, \gamma \in U_{1,n}$ such that $\beta x_I \alpha y_K y_J \gamma = \beta\alpha' y_J \gamma = 1$. This establishes the claim.

By Theorem 1.6, since $V_{1,n}$ is clearly not a division ring, it is simple and purely infinite. $\square$

Notice that we may identify each $U_{1,n}$ with $k\langle X_n\rangle\langle Y_n; \tau, \delta\rangle$ where $X_n = \{x_1, \dots, x_n\}$ and $Y_n = \{y_1, \dots, y_n\}$ are subsets of fixed infinite sets $\{x_1, x_2, \dots\}$ and $\{y_1, y_2, \dots\}$. In particular, $U_{1,n} \subset U_{1,n+1}$ for all n, and we set U_∞ equal to the union of the $U_{1,n}$.

Theorem 4.3. *The ring U_∞ is simple and purely infinite. Moreover, $K_0(U_\infty) \cong \mathbb{Z}$.*

Proof. Clearly, U_∞ is not a division ring, because it contains at least one infinite idempotent.

Let α be a nonzero element of U_∞. Then $\alpha \in U_{1,n}$ for some n, and $\alpha = \sum_{I,J} \lambda_{I,J} y_I x_J$ where the $\lambda_{I,J} \in k$, the y_I are monomials in $y_1, \dots, y_n$, and the x_J are words in $x_1, \dots, x_n$. We choose an index I' of minimal length with $\lambda_{I',J} \neq 0$ for some J, and then, we choose an index J' of maximal length among the indices J such that (I', J) is in the support of α. Then,

$$(\lambda_{I',J'}{}^{-1}x_{I'^*})\alpha(y_{J'^*}) = 1 + \sum_{I,J} \lambda'_{I,J} y_I x_J \,,$$

and $\lambda'_{\emptyset,J} = 0$ for all $J \neq \emptyset$. Now, since $x_{n+1}, y_{n+1} \in U_\infty$ and $x_{n+1}y_I = 0$ for all nontrivial words y_I in $U_{1,n}$, we have $(x_{n+1}\lambda_{I',J'}{}^{-1}x_{I'^*})\alpha(y_{J'^*}y_{n+1}) = 1$. Thus U_∞ is simple and purely infinite because of Theorem 1.6.

By Theorem 4.1, we have $K_0(U_{1,n}) \cong \mathbb{Z}$ and a generator of $K_0(U_{1,n})$ is provided by the class $[U_{1,n}]$. It follows that the induced homomorphisms $K_0(U_{1,n}) \to K_0(U_{1,n+1})$ are isomorphisms. Since the functor $K_0(-)$ preserves direct limits, we get $K_0(U_\infty) \cong \varinjlim K_0(U_{1,n}) \cong \mathbb{Z}$. $\square$

5. Algebras $R\langle Y;\tau,\delta\rangle$ with other coefficients

Now we proceed to exploit the skew polynomial construction of Section 3, in order to get new examples of purely infinite simple rings. These will be, in some sense, "intermediate" algebras between Leavitt's and Tyukavkin's examples, and, as we will see, they all have the same K-theoretical behavior. We begin as in Example 3.4, except that we now index our variables starting at 0 rather than at 1. For a given natural number n, consider the algebra $k\langle\langle X_n\rangle\rangle$, where $X_n = \{x_0, x_1, \dots, x_n\}$, equipped with the skew derivations (τ_i, δ_i) defined in Example 3.4. Throughout this section, R_n will denote a local subalgebra of $k\langle\langle X_n\rangle\rangle$ containing $k\langle X_n\rangle$ such that R_n is invariant under the δ_i. Of course, one possibility is $R_n = k\langle\langle X_n\rangle\rangle$; another choice of R_n will be important in Section 7. Since the augmentation ideal of $k\langle\langle X_n\rangle\rangle$ has codimension 1, its intersection with R_n must be the maximal ideal of R_n. Thus, all power series in R_n with nonzero constant term are invertible in R_n. Note that by [9, Proposition 2.9.18], R_n is a semifir.

By assumption, the (τ_i, δ_i) restrict to skew derivations on R_n. Thus, we may construct the skew polynomial algebra $S_n = R_n\langle Y_n;\tau,\delta\rangle$, where $Y_n = \{y_0, y_1, \dots, y_n\}$, and our first goal is to determine the structure of this algebra. Except in Proposition 5.8, where we consider the sequence of algebras $S_1, S_2, \dots$, we shall throughout the remainder of the section keep n fixed and write R, S, X, Y for R_n, S_n, X_n, Y_n.

Lemma 5.1. (cf. [33]) *For any nonzero element $r \in R$ there are $w \in Y^*$ and $r' \in R$ such that $rwr' = 1$. In fact, given any nonzero elements $r_1, \dots, r_m \in R$, there exists $w \in Y^*$ such that $r_jw \in R$ for all j and some r_iw is invertible in R.*

Proof. Let $r_1, \dots, r_m$ be nonzero elements of R. For $j = 1, \dots, m$, let l_j denote the minimum length of monomials occurring in r_j. We may assume that $l_1 \leq \cdots \leq l_m$. Pick a monomial x_I of length l_1 occurring in r_1, and set $w = y_{I^*}$. Observe that for any monomial x_J of length at least l_1, either $x_Jw = 0$ or x_Jw is a monomial. Hence, $r_jw \in R$ for all j, and r_1w has nonzero constant term. Consequently, r_1w is invertible in R, by our choice of R. $\square$

Proposition 5.2. (a) *The algebra S is a prime ring, and its socle is the ideal SeS generated by the minimal idempotent $e := 1 - \sum_{i=0}^{n} y_i x_i$. Further,* $\mathrm{Soc}(S)$ *is a regular ideal of S.*

(b) *Suppose that $\phi : R \to V$ is a k-algebra homomorphism and that there exist elements $t_0, \dots, t_n \in V$ such that $\phi(x_i)t_j = \delta_{ij}$ for all i, j. Then ϕ extends uniquely to a k-algebra homomorphism $\overline{\phi} : S \to V$ such that $\overline{\phi}(y_j) = t_j$ for all j.*

Proof. (a) Note that $ey_j = x_j e = 0$ for all j. Consequently, $eS = eR$ and $eSe = ke$.

Given a nonzero element α in S, either $e\alpha = \alpha$ or there is some i such that $x_i\alpha \neq 0$. It follows by induction on the maximum length of words occurring in α that there is a monomial $m \in X^*$ such that $em\alpha \neq 0$. By the observation above, $em\alpha = er$ for some $r \in R$, and r is right invertible in S by Lemma 5.1. In particular, there is some $\gamma \in S$ such that $em\alpha\gamma = e$. It follows at once that S is a prime ring and that SeS is the unique minimal nonzero ideal of S. Then, because $eSe = ke \cong k$ we conclude that eS is a minimal right ideal of S. Therefore $SeS = \mathrm{Soc}(S)$.

Now SeS is a simple ring having a minimal one-sided ideal, and Litoff's Theorem (see [13]) says that it is locally a matrix ring over a division ring (actually the division ring is k in our case). In particular $\mathrm{Soc}(S) = SeS$ is a regular ring.

(b) In view of Proposition 3.3, it suffices to show that

$$\phi(r)t_j = t_j\phi\tau(r) + \phi\delta_j(r) \tag{1}$$

for any $r \in R$ and all $j = 0, \dots, n$. We can write $r = \alpha + r_0x_0 + \cdots + r_nx_n$ for some $\alpha \in k$ and $r_i \in k\langle\langle X\rangle\rangle$. Then $\alpha = \tau(r)$ and $r_i = \delta_i(r)$; in particular, all the $r_i \in R$. Thus

$$\phi(r)t_j = \big(\alpha + \sum_{i=0}^{n} \phi(r_i)\phi(x_i)\big)t_j = \alpha t_j + \phi(r_j) = t_j\phi\tau(r) + \phi\delta_j(r),$$

proving (1). □

The next lemma may be known, but we have not been able to locate any reference. Write $M^* = \mathrm{Hom}_R(M, R)$ for R-modules M.

Lemma 5.3. *Let R be a left semihereditary ring, and let $T = R_\Sigma$ be a universal localization of R. Assume that for all finitely presented right R-modules M such that $M^* = 0$, we have $M \otimes_R T = 0$. Then T is a regular ring and every finitely generated projective right T-module is induced from a finitely generated projective right R-module.*

Proof. We first claim that any finitely presented right R-module M has a decomposition $M = M_1 \oplus M_2$ with M_1 projective and $M_2^* = 0$. In case R is semihereditary on both sides, this follows from [22, Theorem 1.2(3)] (the hypothesis of an involution is not needed in the proof). Let

$$R^m \xrightarrow{f} R^n \xrightarrow{g} M \longrightarrow 0$$

be a resolution of M. Taking duals, we obtain an exact sequence

$$0 \longrightarrow M^* \xrightarrow{g^*} {}^nR \xrightarrow{f^*} {}^mR.$$

Now $f^*({}^nR)$ is a finitely generated submodule of ${}^mR$, so it is projective because R is left semihereditary. Consequently, $g^*(M^*) = ({}^nR)e$ for some idempotent matrix $e \in M_n(R)$. It follows that

$$(1-e)(R^n) = g^{-1}\big(\{x \in M \mid h(x) = 0 \text{ for all } h \in M^*\}\big).$$

In particular, $\ker(g) \subseteq (1-e)(R^n)$. Hence, $M = M_1 \oplus M_2$ where $M_2 = g\big((1-e)(R^n)\big)$ satisfies $M_2^* = 0$ and $M_1 \cong e(R^n)$ is projective.

Now let N be a finitely presented right T-module. By [30, Corollary 4.5], there exists a finitely presented right R-module M such that $M \otimes_R T \cong N$. By the above, we can write $M = M_1 \oplus M_2$ with M_1 projective and $M_2^* = 0$. By hypothesis, we have $M_2 \otimes_R T = 0$, and so $N \cong M_1 \otimes_R T$ is an induced finitely generated projective right T-module. □

Theorem 5.4. *Let R, S, and e be as above, and set $I = SeS$. Let $f : R \to R^{n+1}$ be the homomorphism given by left multiplication by the column $(x_0, \dots, x_n)^T$. Then the universal localization R_f is a purely infinite simple regular ring, and every finitely generated projective R_f-module is free. Moreover, $R_f \cong S/I$, and S is regular.*

Proof. We first prove that $R_f \cong S/I$. For that, it is enough to show that the natural map $R \to S/I$ satisfies the universal property of the universal localization $R \to R_f$. If $\phi : R \to V$ is a k-algebra homomorphism such that $f \otimes 1_V$ is invertible, then there are elements $t_0, \dots, t_n$ in V such that $\phi(x_i)t_j = \delta_{ij}$ and $\sum_{i=0}^n t_i\phi(x_i) = 1$. By Proposition 5.2(b), there exists a unique k-algebra map $\overline{\phi} : S \to V$ such that $\overline{\phi}|_R = \phi$ and $\overline{\phi}(y_i) = t_i$ for all i. Clearly this map factors through S/I and so we get the desired map from S/I to V. This shows that we may identify S/I with R_f. It follows that the localization map $R \to R_f$ is injective (recall from Lemma 5.1 that nonzero elements of R are right invertible in S).

Write $T = S/I = R_f$, and identify R with its image in T. Let us check directly that T is purely infinite simple. (This also follows from the fact that T is regular with every finitely generated projective T-module being free, which we will prove later.) Let α be an element of S which is not in I. There must exist i such that $x_i\alpha \notin I$, since otherwise $\alpha = e\alpha + \sum_{i=0}^n y_i x_i \alpha \in I$. Note that either $\alpha \in R$ or the degree in Y of $x_i\alpha$ is smaller than that of α. We conclude that there is a monomial $m \in X^*$ such that $m\alpha$ is a nonzero element of R. By Lemma 5.1, we get an element $g \in S$ such that $m\alpha g = 1$. This shows that T is a purely infinite simple ring.

Let M be a finitely presented right R-module such that $M^* = 0$. We want to prove that $M \otimes_R T = 0$, in order to apply Lemma 5.3. Take a presentation

$$R^s \to R^t \to M \to 0.$$

Since the functor $(-) \otimes_R T$ is right exact, we get an exact sequence

$$T^s \to T^t \to M \otimes_R T \to 0.$$

Write z_i for the image of y_i in T. The map $R^s \to R^t$ above is given by left multiplication by a $t \times s$ matrix A with coefficients in R, and we have to see that $AT^s = T^t$, that is, the columns of A generate T^t as a right T-module. We proceed by induction on t. If $t = 1$, then there exists a nonzero entry p in A, because $M^* = 0$. By Lemma 5.1, the element p is right invertible in S, so in T, and thus the entries of A generate T as a right T-module. Now assume that $t > 1$. If some entry of A is invertible in R, then by a standard process we can find invertible matrices P and Q over R of appropriate sizes such that $PAQ = \begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}$, where A' is a matrix of size $(t-1)\times(s-1)$. In this case, M can be generated by $t-1$ elements, and induction applies.

In the general case, apply Lemma 5.1 to the entries a_{ij} of A, to obtain a word $w \in Y^*$ such that all $a_{ij}w \in R$ with at least one $a_{ij}w$ invertible. Consequently, there is a vector $v \in T^s$ of the form $(0, \dots, 0, \overline{w}, 0, \dots, 0)^T$ such that the column $q = Av \in AT^s \cap R^s$ has some entry which is invertible in R. Now consider the matrix $C = \begin{pmatrix} q & A \end{pmatrix}$ with coefficients in R, of size $t \times (s+1)$. The finitely presented module $M' := R^t/CR^{s+1}$ is a factor module of M, and consequently $(M')^* = 0$. Moreover, the matrix C has an invertible entry, and so we get as before by induction that $M' \otimes_R T = 0$, or equivalently that $T^t = CT^{s+1}$. But q belongs to the T-submodule of T^t generated by the columns of A, and therefore the columns of A generate T^t as an T-module. We conclude that $M \otimes_R T = 0$, as desired.

By [9, Proposition 2.9.18], R is a semifir. It follows from Lemma 5.3 that T is a regular ring and that every finitely generated projective right T-module is free.

Finally, S/I is a regular ring, and by Proposition 5.2, I is a regular ideal of S. Thus it follows from [14, Lemma 1.3] that S is regular. □

The regularity of S was proved by Tyukavkin ([32], [33]) in the case where $R = k\langle\langle X\rangle\rangle$. In order to compute K_0 groups of both S and R_f, we will need the following technical lemma.

As in the proof of Theorem 5.4, we set $T = S/I = R_f$, and we write T_n in case n requires mention.

Lemma 5.5. *If there is a right R-module map $p : R \to R^s$ which becomes invertible over T, then n divides $s-1$.*

Proof. Write $p = (p_1, \dots, p_s)^T$, where each $p_i \in R$. We will construct, by induction on i, words w_i in Y^* and invertible elements g_i in R such that the following statements hold:

(P_i) There exists an invertible map $p^{(i)} : T \to T^s$ satisfying the following properties:

(1) $p^{(i)}_{i+1}, \dots, p^{(i)}_s \in R$.

(2) The inverse of $p^{(i)}$ is the row $(w_1g_1, \dots, w_ig_i, \alpha_{i+1}, \dots, \alpha_s)$ for some elements $\alpha_{i+1}, \dots, \alpha_s \in T$.

The statement is obvious when $i = 0$. Assume that $0 \le i < s$, and that (P_i) holds. We will prove (P_{i+1}). Note that $p^{(i)}_j w_m = 0$ for $j \ne m$. Without loss of generality, we can assume that the order of the series $p^{(i)}_{i+1}$ is less than or equal to the order of $p^{(i)}_{i+t}$ for all $t \geq 2$. Choose a word $w_{i+1} \in Y^*$ with length equal to the order of $p_{i+1}^{(i)}$ and such that $p_{i+1}^{(i)} w_{i+1}$ is invertible in R. Let $g_{i+1} \in R$ be the inverse of $p_{i+1}^{(i)} w_{i+1}$ and note that

$$1 = p_{i+1}^{(i)} w_{i+1} g_{i+1} = p_{i+1}^{(i)} \alpha_{i+1}.$$

It follows that $u = \alpha_{i+1} p_{i+1}^{(i)} + (1 - w_{i+1} g_{i+1} p_{i+1}^{(i)})$ is invertible in T with inverse $u^{-1} = w_{i+1} g_{i+1} p_{i+1}^{(i)} + (1 - \alpha_{i+1} p_{i+1}^{(i)})$. Therefore $p^{(i+1)} := p^{(i)} u$ is invertible with inverse

$$u^{-1}(w_1 g_1, \dots, w_i g_i, \alpha_{i+1}, \dots, \alpha_s).$$

Note that, for $t > 1$, we have

$$p_{i+t}^{(i)} u = p_{i+t}^{(i)} (1 - w_{i+1} g_{i+1} p_{i+1}^{(i)}).$$

Since the order of $p_{i+t}^{(i)}$ is greater than or equal to the length of w_{i+1}, we conclude that $p_{i+t}^{(i+1)} \in R$, and condition (1) of (P_{i+1}) holds. On the other hand, for $m \leq i$ we have $p_{i+1}^{(i)} w_m = 0$ and so

$$u^{-1} w_m g_m = w_m g_m.$$

We also have

$$u^{-1} \alpha_{i+1} = w_{i+1} g_{i+1} p_{i+1}^{(i)} \alpha_{i+1} + (1 - \alpha_{i+1} p_{i+1}^{(i)}) \alpha_{i+1} = w_{i+1} g_{i+1},$$

and so condition (2) of (P_{i+1}) is also satisfied. Therefore the induction works.

Take $q_i = g_i p_i^{(s)} \in T$ for $i = 1, \dots, s$. Then

$$\sum_{i=1}^{s} w_i q_i = 1, \tag{1}$$

$q_i w_i \neq 0$ for all i, and $q_i w_j = 0$ for $i \neq j$. We claim that these conditions imply $s \equiv 1 \pmod{n}$. We proceed by induction on the maximum of the lengths of the w_i. This maximum is 0 if and only if $s = 1$ (and then $q_1 = 1$). So assume that $s > 1$. In this case all w_i are different from 1. Fix $\ell \in \{0, \dots, n\}$. Left multiplying (1) by x_ℓ and right multiplying it by y_ℓ, and letting $A_\ell = \{i : x_\ell w_i \neq 0\}$, we have

$$\sum_{i \in A_\ell} (x_\ell w_i)(q_i y_\ell) = 1.$$

Note that $\{1, \dots, s\}$ is the disjoint union of the family $\{A_\ell \mid \ell = 0, \dots, n\}$. Observe also that for $i, j \in A_\ell$ we have $(q_i y_\ell)(x_\ell w_j) = q_i w_j$. So this term is 0 if $i \neq j$ and nonzero if $i = j$. By induction, $|A_\ell| \equiv 1 \pmod{n}$. Therefore

$$s = \sum_{\ell=0}^{n} |A_\ell| \equiv n + 1 \equiv 1 \pmod{n},$$

as desired. $\square$

Theorem 5.6. *Let R, T, and $f : R \to R^{n+1}$ be as above. Then $K_0(T)$ is a cyclic group of order n, generated by $[T]$.*

Proof. Consider the homomorphism $\iota^* : K_0(R) \to K_0(T)$, where $\iota : R \to T$ denotes the localization map. Since our ring R is a semifir, we have $K_0(R)$ infinite cyclic, generated by $[R]$. Also, by Lemma 5.3 and the proof of Theorem 5.4, the map ι^* is a group epimorphism. Thus, it is enough to show that the kernel of ι^* is generated by $n[R]$ to get the desired result.

Suppose that $[R^r] - [R^s]$ lies in $\ker(\iota^*)$, for some nonnegative integers r, s, that is, $[T^r] - [T^s] = 0$ in $K_0(T)$. Then $T^{r+t} \cong T^{s+t}$ for some $t \geq 0$. Since $[R^r] - [R^s] = [R^{r+t}] - [R^{s+t}]$, we may thus assume that $T^r \cong T^s$. By [30, Corollary 4.4], there is an isomorphism $T^{r+u} \to T^{s+u}$, for some $u \geq 0$, which is induced from a map $R^{r+u} \to R^{s+u}$. We can again reduce to the case that $u = 0$. Thus, it suffices to prove that if there exists a map $g : R^r \to R^s$ that becomes invertible over T, then n divides $r - s$. After taking the direct sum of g with an identity map $R^m \to R^m$ for a suitable m, we may assume that $r = \ell n + 1$ for some positive integer ℓ. Since the map $f : R \to R^{n+1}$ becomes an isomorphism over T, there exists a homomorphism $h : R \to R^r$ which becomes invertible over T, and the same holds for $gh : R \to R^s$. By Lemma 5.5, n divides $s - 1$, and hence also $r - s$, as desired. $\square$

Corollary 5.7. *Let R and S be as above. For $\ell, m \in \mathbb{N}$, we have $S^\ell \cong S^m$ if and only if $\ell = m$. Moreover, $K_0(S)$ is an infinite cyclic group, with generator $[S]$.*

Proof. Suppose that $S^\ell \cong S^m$ for some $\ell \geq m > 0$. Consider $p \in \mathbb{N}$ such that $p \geq n$ and $p > \ell - m$. If we take $R_p = k\langle\langle X_p \rangle\rangle$, then S embeds in S_p, whence $S_p^\ell \cong S_p^m$ and so $T_p^\ell \cong T_p^m$. Thus, by Theorem 5.6 (applied to T_p), we get that p divides $\ell - m$, which implies that $\ell = m$, as desired.

Now, by the above argument, we have that the class $[S] \in K_0(S)$ generates an infinite cyclic subgroup. By Proposition 5.2, I is the socle of S, and I is generated by the minimal right ideal eS. Hence, $K_0(I)$ is infinite cyclic, with generator $[eS]$. Since $y_0x_0, \dots, y_nx_n$ are orthogonal idempotents equivalent to 1, we have $eS \oplus S^{n+1} \cong S$, and so $[eS] = -n[S]$ in $K_0(S)$. In particular, it follows that the natural map $K_0(I) \to K_0(S)$ is injective. Using that, we get the following commutative diagram, whose bottom row is exact because S is regular.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{-n} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle 1\mapsto[eS]} & & \big\downarrow{\scriptstyle 1\mapsto[S]} & & \big\downarrow{\scriptstyle [1]_n\mapsto[T]} & & \\ 0 & \longrightarrow & K_0(I) & \longrightarrow & K_0(S) & \longrightarrow & K_0(T) & \longrightarrow & 0 \end{array}$$

Moreover, the map $\mathbb{Z}/n\mathbb{Z} \to K_0(T)$ is an isomorphism by Theorem 5.6. Thus, by the Five Lemma we conclude that $K_0(S)$ is infinite cyclic with generator $[S]$, as desired. $\square$

Now consider a sequence of algebras $S_1, S_2, \dots$ of the type discussed in this section. Assume the sequence has been chosen so that $R_1 \subset R_2 \subset \dots$; we could, of course, take $R_n = k\langle\langle X_n \rangle\rangle$ for all n, but different choices will be needed in Section 8. We have natural inclusions $S_n \subset S_{n+1}$ for all n, and we set $S_\infty = \bigcup_{n=1}^\infty S_n$. Note that S_∞ is a regular ring.

Proposition 5.8. *Let $S_1, S_2, \dots, S_\infty$ be as above. The ring S_∞ is simple, regular, and purely infinite, and $K_0(S_\infty)$ is an infinite cyclic group, with generator $[S_\infty]$.*

Proof. Since each S_n is regular (Theorem 5.4), so is S_∞. Next, notice that S_∞ is not a division ring, because it contains at least one infinite idempotent. Let α be a nonzero element of S_∞. Then $\alpha \in S_n$ for some n, and the proof of Proposition 5.2(a) shows that there exist $s, t \in S_n$ such that $s\alpha t = 1 - \sum_{i=0}^{n} y_i x_i$. Consequently, $(x_{n+1}s)\alpha(ty_{n+1}) = 1$. Thus, S_∞ is simple and purely infinite because of Theorem 1.6.

We finally compute $K_0(S_\infty)$. By Corollary 5.7, $K_0(S_n) \cong \mathbb{Z}$ for all n, with $[S_n] \mapsto 1$. Proceeding as in Theorem 4.3, we obtain $K_0(S_\infty) \cong \varinjlim K_0(S_n) = \mathbb{Z}$, with $[S_\infty] \mapsto 1$. $\square$

Theorem 5.6 and Proposition 5.8 provide us with examples of purely infinite simple regular rings whose K_0 groups are cyclic of arbitrary order. Using these rings as basic building blocks, we can construct purely infinite simple regular rings whose K_0's are arbitrary countable abelian groups – see Section 8.

6. The Rosenmann-Rosset and Schofield constructions

In this section we consider two constructions associated to the free algebra over a field k, and we will prove that they are isomorphic purely infinite simple regular k-algebras. In Section 7, we will relate these algebras to the ones constructed in Section 5. Throughout this section, R will denote a free k-algebra on the finite alphabet $\{x_0, \dots, x_n\}$ with $n > 0$.

First, we briefly quote an example of Rosenmann and Rosset [27, Section 3]. For the general theory of rings of quotients, we refer the reader to [31]. Let $\mathcal{F}_{fc}$ be the Gabriel topology whose basic neighborhoods of 0 are the right ideals of finite codimension; see [27, Theorem 1.1]. Consider the $\mathcal{F}_{fc}$-localization of R, denoted by R_{fc}. Then R_{fc} is a ring and we can associate to each right R-module a right R_{fc}-module M_{fc} by the rule $M_{fc} = \varinjlim_{I \in \mathcal{F}_{fc}} \mathrm{Hom}(I, M)$. (Note that since every right ideal in R is projective we have $M_{fc} = M_{\mathcal{F}_{fc}}$ by [27, Lemma 2.3].) Recall that a module M is *torsion* (with respect to $\mathcal{F}_{fc}$) if for every $x \in M$, its annihilator, $\mathrm{ann}_R(x)$ is in $\mathcal{F}_{fc}$. If M is torsion then $M_{fc} = 0$. Clearly, M is torsion if and only if all its cyclic submodules are finite dimensional.

We recall one of the main results in [27]:

Theorem 6.1. [27, Theorem 5.1] *Let R and R_{fc} be as above. Then, for $\ell, m \in \mathbb{N}$, we have $R_{fc}^{\ell} \cong R_{fc}^{m}$ if and only if $\ell \equiv m \pmod{n}$.*

Another construction was quoted by A. H. Schofield in a private communication [29]. We thank him for allowing us to present this construction here.

We denote by $\mathcal{P}$ the category of finitely generated projective right R-modules. Let $\Phi = \mathrm{Mor}(\mathcal{P})$ denote the class of all homomorphisms between finitely generated projectives.

Proposition 6.2. [29] *Let R be as above. Let Σ be the family of all <u>mono</u>morphisms in Φ whose images have finite codimension.*

(a) Σ *is composition-closed: If* $f, g \in \Sigma$ *and the composition* fg *is defined, then* $fg \in \Sigma$.

(b) *Every* $f \in \Sigma$ *is a non-zero-divisor in* Φ*: If* $g \in \Phi$ *and* fg *is defined (respectively,* gf *is defined), then* $fg = 0$ *(respectively,* $gf = 0$*) implies* $g = 0$.

(c) Σ *satisfies the right Ore condition relative to* Φ*: If* $f \in \Sigma$ *and* $g \in \Phi$ *with the same codomain, then there exist* $f' \in \Sigma$ *and* $g' \in \Phi$ *such that* $gf' = fg'$ *(with appropriate conditions on domains and codomains).*

Proof. Part (a) is clear.

(b) Let $f \in \Sigma$ and $g \in \Phi$. Since f is injective it is clear that $fg = 0$ implies $g = 0$. Now assume that $gf = 0$. Let Q be the codomain of f. Then Q is a finitely generated free R-module, and since $\mathrm{Im}(f) \subseteq \ker(g)$, we see that $\ker(g)$ has finite codimension. Hence, $\mathrm{Im}(g)$ is finite-dimensional and a free R-module, so it is zero.

(c) Now assume that $f : X \to Y$ is in Σ and that $g : Z \to Y$ is in Φ. Form the pullback

$$\begin{array}{ccc} P & \xrightarrow{f'} & Z \\ {\scriptstyle g'}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

Then $gf' = fg'$. We will show that $P \in \mathcal{P}$ and $f' \in \Sigma$, which will finish the proof. As is well known, since f is injective it follows that f' is injective (e.g., [28, Exercise 2.47]). On the other hand $f'(P) = g^{-1}(f(X))$. Thus $Z/f'(P)$ embeds in $Y/f(X)$, which is finite dimensional, and so $f'(P)$ has finite codimension in Z. Since f' is injective, P is free of finite rank by [20, Theorem 4], and therefore $f' \in \Sigma$. $\square$

Continue with R and Σ as above. Let $\mathcal{C}$ be the set of all pairs (f, s) with $f \in \Phi$ and $s \in \Sigma$ such that f and s have the same domain. We define a relation on $\mathcal{C}$. Let (f, s) and (f', s') be in $\mathcal{C}$, with

$$Q \xleftarrow{f} P \xrightarrow{s} U \qquad \text{and} \qquad Q' \xleftarrow{f'} P' \xrightarrow{s'} U'.$$

Then $(f, s) \sim (f', s')$ if and only if $Q = Q'$ and $U = U'$ and there are maps $h : U'' \to P$ and $h' : U'' \to P'$ such that $sh = s'h' \in \Sigma$ and $fh = f'h'$. By [35, Section 10.3], $\sim$ is an equivalence relation, and we get a quotient category $\mathcal{P}\Sigma^{-1}$. Now let Q be the ring consisting of all equivalence classes $[(f, s)]$ with $(f, s) \in C$ and $f, s : P \to R$. Notice that $Q = \mathrm{End}_{\mathcal{P}\Sigma^{-1}}(R)$. There is a canonical map $\phi : R \to Q$ defined by $\phi(a) = [(a, 1)]$. The full subcategory of Mod-Q consisting of all the induced finitely generated projective modules is equivalent to the category with the same objects as $\mathcal{P}$ and morphisms $\mathcal{P}\Sigma^{-1}$. The map $\phi : R \to Q$ gives the universal localization of R with respect to Σ. Note that since Σ is a non-zero-divisor class in Φ, the map ϕ is injective.

Since ${}_RQ$ is the direct limit of $\mathrm{Hom}_R(eR^m, R)s^{-1}$, where $s : eR^m \to R$ is in Σ and e ranges through all the idempotent matrices over R, and since $\mathrm{Hom}_R(eR^m, R)$ is projective as a left R-module, it follows that ${}_RQ$ is flat.

Theorem 6.3. [29] *Let R and Σ be as above. Then the universal localization $Q = R_\Sigma$ is a purely infinite simple regular ring.*

Proof. Let N be a finitely presented right R-module. By [20, Theorem 2], there is a finitely generated free submodule L of N of finite codimension. There is an exact sequence

$$0 \to R^\ell \to R^m \to N/L \to 0,$$

and so the map $R^\ell \to R^m$ must be in Σ, which gives $(N/L) \otimes_R Q = 0$. Now since ${}_RQ$ is flat, we have an exact sequence

$$0 \to L \otimes_R Q \to N \otimes_R Q \to (N/L) \otimes_R Q \to 0.$$

Since $(N/L) \otimes_R Q = 0$ we conclude that $N \otimes_R Q$ is a free Q-module. Since every finitely presented Q-module is induced from a finitely presented R-module, we conclude that Q is a regular ring such that every finitely generated projective module is free. Thus Q is simple and purely infinite by Example 1.3(d). □

Theorem 6.4. *Let R, R_{fc} and Q be as above. Then, $R_{fc} \cong Q$ as k-algebras.*

Proof. Let $s \in \Sigma$. Since finitely generated projective right R-modules are free, we can assume without loss of generality that $s : R^\ell \to R^m$ for some $\ell, m \in \mathbb{N}$, with $N := \operatorname{coker}(s)$ finite-dimensional. We have therefore a short exact sequence

$$0 \longrightarrow R^\ell \xrightarrow{\ s\ } R^m \longrightarrow N \longrightarrow 0.$$

Proceeding as in [27, page 368] and taking into account that N is a torsion module with respect to $\mathcal{F}_{fc}$ and thus $N_{fc} = 0$, we get an exact sequence

$$0 \longrightarrow R^\ell_{fc} \xrightarrow{\ s\ } R^m_{fc} \longrightarrow 0.$$

We conclude that every map in Σ becomes invertible over R_{fc}. Thus, using the universal property of Q with respect to R and Σ, we conclude that there exists a unique k-algebra homomorphism $\rho : Q \to R_{fc}$ which restricts to the identity map on R. Since Q is simple by Theorem 6.3, ρ is an injective homomorphism. Now, if $I_R \leq R_R$ is a right ideal of finite codimension, $i : I \to R$ is the natural inclusion map, and $f : I \to R$ is any right R-module homomorphism, then $\rho([(f, i)]) = [f]$, whence ρ is an isomorphism, as desired. □

Schofield also proved that $K_0(Q) \cong \mathbb{Z}/n\mathbb{Z}$. Because of Theorem 6.4 and Proposition 2.2, this implies the result in Theorem 6.1 ([27, Theorem 5.1]). We will give an alternate proof of this fact in Theorem 7.6, by using the techniques developed in Section 5.

7. Isomorphism of Q with T

In this section we will prove that the algebra Q (and so also the algebra R_{fc}) constructed in Section 6 is isomorphic to a particular instance of our construction in Section 5. For this, we need to introduce the algebra of rational series, denoted by $k_{\mathrm{rat}}\langle X \rangle$, associated to a finite alphabet X. This algebra plays an important role in other parts of mathematics, particularly formal language theory and the theory of codes, see [8].

We fix some notation for the rest of this section. Let $X = X_n = \{x_0, \dots, x_n\}$ be a finite alphabet, with $n > 0$. The *division closure* of $k\langle X\rangle$ in $k\langle\langle X\rangle\rangle$ is called the *algebra of rational series* and it is denoted by $k_{\text{rat}}\langle X\rangle$. By definition, $k_{\text{rat}}\langle X\rangle$ is the smallest subalgebra of $k\langle\langle X\rangle\rangle$ containing $k\langle X\rangle$ and closed under inversion, in the sense that if an element of the subalgebra is invertible in $k\langle\langle X\rangle\rangle$, then it is already invertible in the subalgebra. The algebra $k_{\text{rat}}\langle X\rangle$ is local, with maximal ideal consisting of the elements with constant term 0. By [9, Exercise 7.1.10], a square matrix over $k\langle\langle X\rangle\rangle$ (respectively $k_{\text{rat}}\langle X\rangle$) is invertible if and only if its image under the augmentation map is invertible over k (see also [10, pp. 408-409]). It follows from this that $k_{\text{rat}}\langle X\rangle$ coincides with the *rational closure* of $k\langle X\rangle$ in $k\langle\langle X\rangle\rangle$, that is, $k_{\text{rat}}\langle X\rangle$ is the smallest subalgebra of $k\langle\langle X\rangle\rangle$ containing $k\langle X\rangle$ and such that every square matrix with coefficients in the subalgebra which is invertible over $k\langle\langle X\rangle\rangle$ is already invertible over the subalgebra. The right τ_i-derivations δ_i on $k\langle\langle X\rangle\rangle$ defined in Example 3.4 coincide with the left transductions associated to the monomials x_i. These left transductions are defined in [9, page 105] as follows. For a fixed monomial $x_{i_1} \cdots x_{i_r}$ of degree r define the left transduction for this monomial as the k-linear map $a \mapsto a^*$ of $k\langle X\rangle$ into itself which sends any monomial of the form $bx_{i_1} \cdots x_{i_r}$ to b and all other monomials to 0. This map extends in the obvious way to the power series algebra $k\langle\langle X\rangle\rangle$. By [9, page 135], the algebra of rational series $k_{\text{rat}}\langle X\rangle$ is invariant under all transductions. We conclude that $k_{\text{rat}}\langle X\rangle$ is a local subalgebra of $k\langle\langle X\rangle\rangle$ containing $k\langle X\rangle$ and invariant under the δ_i. Therefore we can build the algebras $S = S_n = k_{\text{rat}}\langle X\rangle\langle Y; \tau, \delta\rangle$ and $T = T_n = S/I = k_{\text{rat}}\langle X\rangle_f$ based on $k_{\text{rat}}\langle X\rangle$; see Section 5.

An important property of $k_{\text{rat}}\langle X\rangle$ is that it is a universal localization of $k\langle X\rangle$. Although this fact seems to be well known, we do not have an explicit reference, and so we sketch a proof as follows.

Proposition 7.1. *Let Σ' be the set of those square matrices over $k\langle X\rangle$ which become invertible over $k\langle\langle X\rangle\rangle$. Then $k_{\text{rat}}\langle X\rangle$ is the universal localization of $k\langle X\rangle$ with respect to Σ'.*

Proof. Note that Σ' is the set of matrices of the form $A_0 + B$, where A_0 is an invertible matrix over k and all the entries of B have constant term 0. Since $k_{\text{rat}}\langle X\rangle$ is the rational closure of $k\langle X\rangle$ in $k\langle\langle X\rangle\rangle$, we have a unique algebra homomorphism $\phi : k\langle X\rangle_{\Sigma'} \to k_{\text{rat}}\langle X\rangle$ which is the identity on $k\langle X\rangle$. It is easy to see from [9, Theorem 7.1.2] that ϕ is surjective. (This holds in general when we consider the rational closure of an inclusion of rings.)

By [9, Lemma 5.9.4], the inclusion map $k\langle X\rangle \to k\langle\langle X\rangle\rangle$ is honest (i.e., it sends full matrices to full matrices), and hence so is the inclusion map $k\langle X\rangle \to k_{\text{rat}}\langle X\rangle$. Since Σ' is a factor-closed, multiplicative set of matrices over $k\langle X\rangle$, it now follows from [9, Proposition 7.5.7(ii)] that ϕ is injective. Therefore ϕ is an isomorphism. $\square$

By a result of Bergman and Dicks ([30, Theorem 4.9]), every universal localization of a hereditary ring is also hereditary. In particular, $k_{\text{rat}}\langle X\rangle$ is a hereditary ring, and indeed it is a fir by [9, Theorem 7.10.7]. The algebra $k\langle\langle X\rangle\rangle$ is just a semifir, but not a fir; see [9, Proposition 2.9.18 and Proposition 5.10.9]. We summarize the results obtained in Section 5 for the algebras S and T based on the algebra of rational series $k_{\text{rat}}\langle X\rangle$, along with the facts just mentioned.

The following notation will be helpful. If $a_0, \dots, a_n$ are elements in a k-algebra V, let $\Sigma'_k(a_0, \dots, a_n)$ denote the set of square matrices over V of the form $A_0 + B$ where A_0 is an invertible matrix over k and the entries of B consist of noncommutative polynomials with zero constant term evaluated at $(a_0, \dots, a_n)$. (Note that to check whether all such matrices are invertible over V, it suffices to consider those for which A_0 is an identity matrix.) In particular, if $V = k\langle X\rangle$ and the $a_i = x_i$, then $\Sigma'_k(x_0, \dots, x_n)$ equals the set Σ' in Proposition 7.1.

Theorem 7.2. *Let $\Sigma_1 = \Sigma'_k(x_0, \dots, x_n) \cup \{(x_0, \dots, x_n)^T\}$. Set $S = k_{\text{rat}}\langle X\rangle\langle Y; \tau, \delta\rangle$, let I be the ideal generated by the idempotent $e = 1 - \sum_{i=0}^n y_i x_i$, and write $T = S/I$. Then T is the universal localization of $k\langle X\rangle$ with respect to Σ_1. In particular, T is a hereditary algebra.*

To write the universal property of T in more elementary form, suppose that V is a k-algebra containing elements $a_0, \dots, a_n$ and $b_0, \dots, b_n$ such that

(1) *$a_i b_j = \delta_{ij}$ for all i, j.*
(2) *$b_0 a_0 + b_1 a_1 + \cdots + b_n a_n = 1$.*
(3) *All matrices in $\Sigma'_k(a_0, \dots, a_n)$ are invertible over V.*

Then there exists a unique k-algebra homomorphism $\psi : T \to V$ such that $\psi(x_i) = a_i$ for all i and $\psi(y_j) = b_j$ for all j.

Proof. That $T = k\langle X\rangle_{\Sigma_1}$ follows from Proposition 7.1 and Theorem 5.4. The fact that T is hereditary follows from [30, Theorem 4.9]. □

Now we bring the construction of Theorem 6.3 into play, but with a slight change of notation. Let $Y = \{y_0, \dots, y_n\}$ and set $R = k\langle Y\rangle$. We may identify R with the k-subalgebra of S generated by Y. If r is any nonzero element of R and y_J is a monomial of maximum length occurring in r, then $x_{J^*} r$ is an element of $k\langle X\rangle$ with nonzero constant term. Hence, $x_{J^*} r$ is invertible in S, and so $r \notin I$. Therefore $R \cap I = 0$, and thus we can identify R with its image in T under the quotient map.

Let Q be the universal localization of R with respect to the set Σ of monomorphisms between finitely generated projective right R-modules with finite dimensional cokernels. We want to prove that all maps in Σ are invertible over T.

We need the following fact:

Lemma 7.3. *T is flat as a left R-module.*

Proof. We first check that ${}_RS$ is free. To see this, let $\{p_\alpha\}$ be a k-basis of $k_{\text{rat}}\langle X\rangle$. Then the elements of S can be uniquely written as $\sum \lambda_{I,\alpha} y_I p_\alpha$. We have

$$S = \bigoplus_I y_I k_{\text{rat}}\langle X\rangle = \bigoplus_{I,\alpha} k y_I p_\alpha = \bigoplus_\alpha (\bigoplus_I k y_I) p_\alpha = \bigoplus_\alpha R p_\alpha,$$

so ${}_RS$ is free. Since S is regular, ${}_ST$ is flat, hence ${}_RT$ is flat. □

Proposition 7.4. *Every map in Σ is invertible over T, and so there is a unique k-algebra homomorphism $Q \to T$ which is the identity on R.*

Proof. Notice that every finitely generated projective right R-module is free. Consider a short exact sequence

$$0 \to R^i \xrightarrow{\ g\ } R^j \longrightarrow N \longrightarrow 0$$

with $g \in \Sigma$. Since ${}_RT$ is flat, we have an exact sequence

$$(*) \qquad 0 \to T^i \xrightarrow{\ g\ } T^j \longrightarrow N \otimes_R T \longrightarrow 0,$$

and it is enough to see that $N \otimes_R T = 0$ to get the desired result.

Since $g \in \Sigma$ we have $\dim_k(N) < \infty$, and so $J := \operatorname{ann}_R(N)$ is an ideal of R of finite codimension. Since T is regular, the sequence (*) splits, so $N \otimes_R T$ is a projective right T-module; in particular, it embeds in T_T. For $\gamma \in N$, $(\gamma \otimes 1)R$ has finite k-dimension since $\gamma J = 0$ and $\dim_k(R/J) < \infty$. So it is enough to show that T does not contain nonzero finite-dimensional right R-submodules. We will prove this in the following form: If $\alpha \in S \setminus I$, then αR contains an infinite family of elements which are linearly independent modulo I.

If $\alpha \in S \setminus I$, then, as shown in the proof of Theorem 5.4, we can find $m \in X^*$ such that $m\alpha \in k_{\text{rat}}\langle X\rangle$ and $m\alpha \neq 0$. By Lemma 5.1, there is a word $w \in Y^*$ such that the product $p = m\alpha w$ is an invertible element of $k_{\text{rat}}\langle X\rangle$. As noted earlier in the section, $R \cap I = \{0\}$, whence $1, y_1, y_1^2, \dots$ are k-linearly independent modulo I. Thus, the sequence $p^{-1}m\alpha w, p^{-1}m\alpha w y_1, p^{-1}m\alpha w y_1^2, \dots$ is linearly independent modulo I. It follows immediately that $\alpha w, \alpha w y_1, \alpha w y_1^2, \dots$ is linearly independent modulo I, as desired.

Therefore all maps in Σ are indeed invertible over T, so we get a k-algebra homomorphism $\psi : Q \to T$ which is the identity on R. □

We are now ready to prove the main result of this section.

Theorem 7.5. *Let T be the universal localization of $k\langle X\rangle$ with respect to the set Σ_1 described in Theorem 7.2. Let Q be the universal localization of $R = k\langle Y\rangle$ with respect to the set Σ described in Section 6. Then there is a unique k-algebra isomorphism $\psi : Q \to T$ which is the identity on R.*

Proof. By Proposition 7.4, there exists a unique k-algebra homomorphism $\psi : Q \to T$ which is the identity on R. Since Q is simple, the map ψ must be injective. Now ψ sends the row matrix $(y_0, \dots, y_n)$ over R to the row matrix $(y_0, \dots, y_n)$ over T, which is the inverse of the column $(x_0,\dots,x_n)^T$ over T. The row $(y_0,\dots,y_n)$ is invertible over Q, so there exists a column $(x'_0,\dots,x'_n)^T$ over Q which is the inverse of $(y_0,\dots,y_n)$, and obviously $\psi(x'_i)=x_i$ for all i. Let A be an $m\times m$ matrix in $\Sigma'_k(x'_0,\dots,x'_n)$. Then $\psi(A)$ is invertible in $M_m(T)$, and so, ψ being injective, A must be a non-zero-divisor in $M_m(Q)$. Since Q is regular by Theorem 6.3, we get that A is invertible in $M_m(Q)$. By the universal property of $T=k\langle X\rangle_{\Sigma_1}$, there exists a unique k-algebra homomorphism $\varphi: T\to Q$ sending $(x_0,\dots,x_n)^T$ to $(x'_0,\dots,x'_n)^T$. The row matrix $(y_0,\dots,y_n)$, being the inverse of $(x_0,\dots,x_n)^T$ in T as well as the inverse of $(x'_0,\dots,x'_n)^T$ in Q, must be sent to itself by φ. Therefore we conclude that ψ and φ are mutually inverse isomorphisms. $\square$

As a consequence of our results in Section 5, we can derive an alternate proof of Schofield's result that $K_0(Q)\cong\mathbb{Z}/n\mathbb{Z}$. As we observed at the end of Section 6, this implies in turn the Rosenmann-Rosset result [27, Theorem 5.1].

Theorem 7.6. *Let Q be as above. Then $K_0(Q)\cong\mathbb{Z}/n\mathbb{Z}$, with $[Q]\mapsto[1]_n$.*

Proof. This follows immediately from Theorems 7.5 and 5.6. $\square$

8. Realizing groups as K_0 of purely infinite simple regular rings

Rørdam has proved that all countable abelian groups appear as K_0's of purely infinite simple C*-algebras [26, Theorem 8.1]. In this section we prove a similar result for purely infinite simple regular rings. Our construction follows the same pattern as Rørdam's but requires much more care with the technical details, since we have to work with more complicated universal properties.

For a field K and an integer $n\geq 2$, we will write $S_{n,K}$ and $T_{n,K}$ for the K-algebra versions of the algebras constructed in Section 7. Namely,

$$S_{n,K}=K_{\mathrm{rat}}\langle X_n\rangle\langle Y_n;\tau,\delta\rangle\,,$$

where $X_n=\{x_0,x_1,\dots,x_n\}$ and $Y_n=\{y_0,y_1,\dots,y_n\}$, and $T_{n,K}=S_{n,K}/I_{n,K}$, where $I_{n,K}$ is the ideal of $S_{n,K}$ generated by the idempotent $e_n:=1-\sum_{p=0}^n y_px_p$. By Theorems 5.4 and 5.6, $T_{n,K}$ is a purely infinite simple regular ring, and $K_0(T_{n,K})\cong\mathbb{Z}/n\mathbb{Z}$ with $[T_{n,K}]\mapsto[1]_n$. We denote by $T_{0,K}$ the inductive limit of the sequence $(S_{n,K})_{n\geq2}$ along the natural inclusions $S_{n,K}\hookrightarrow S_{n+1,K}$. Note that $T_{0,K}$ contains the idempotents $e_0,e_1,\dots$ and that

$$x_ie_n=\begin{cases}0 & (i\leq n)\\ x_i & (i>n)\,,\end{cases}\qquad\qquad e_ny_j=\begin{cases}0 & (j\leq n)\\ y_j & (j>n)\,.\end{cases}$$

By Proposition 5.8, $T_{0,K}$ is simple, regular, and purely infinite. Moreover, $K_0(T_{0,K})\cong\mathbb{Z}$ with $[T_{0,K}]\mapsto 1$.

We shall require the universal property for $T_{n,K}$, $(n\geq 2)$, given in Theorem 7.2. Analogously, $T_{0,K}$ possesses the following universal property:

Lemma 8.1. *Let V be a K-algebra containing elements $a_0,a_1,\dots$ and $b_0,b_1,\dots$ such that $a_ib_j=\delta_{i,j}$ for all i,j, and such that all the matrices in $\Sigma'_K(a_0,a_1,\dots)$ are invertible* *over V. Then there exists a unique K-algebra homomorphism $\psi : T_{0,K} \to V$ such that $\psi(x_i) = a_i$ for all i and $\psi(y_j) = b_j$ for all j.*

Proof. This follows from Lemmas 7.1 and 5.2(b). $\square$

We will take as canonical representatives of the nonzero cyclic groups the groups $\mathbb{Z}_m$, the integers mod m, where m is either 0 or an integer larger than 1. For uniformity of notation, we write $[a]_0 = a$ for $a \in \mathbb{Z}$. Every group homomorphism $\varphi : \mathbb{Z}_n \to \mathbb{Z}_m$ is given by multiplication by some integer ℓ, and we choose ℓ such that $1 \le \ell \le m$ in case $m \ge 2$. Recall that for any ring R and $\ell > 0$, there is a canonical group isomorphism $K_0(M_\ell(R)) \cong K_0(R)$ with $[M_\ell(R)] \mapsto \ell[R]$. Hence, we identify

$$\big(K_0(M_\ell(T_{m,K})), [M_\ell(T_{m,K})]\big) = (\mathbb{Z}_m, [\ell]_m)$$

for all $\ell > 0$ and $m \in \{0\} \cup \{2, 3, \dots\}$.

It will be convenient to set $e_{-\ell} = e_\ell = 1 - y_0x_0 - y_1x_1 - \cdots - y_\ell x_\ell$ for all $\ell \ge 0$. If $\ell \le 0$, then by $M_\ell(T_{0,K})$ we will understand the corner ring $e_\ell T_{0,K} e_\ell$. In this case $K_0(M_\ell(T_{0,K})) \cong K_0(T_{0,K})$ with $[M_\ell(T_{0,K})] \mapsto [e_\ell T_{0,K}]$. Note that $y_0x_0, \dots, y_{-\ell}x_{-\ell}$ are pairwise orthogonal idempotents equivalent to 1, and also orthogonal to e_ℓ. Since $e_\ell + y_0x_0 + y_1x_1 + \cdots + y_{-\ell}x_{-\ell} = 1$, we have $[e_\ell T_{0,K}] + (-\ell+1)[T_{0,K}] = [T_{0,K}]$ in $K_0(T_{0,K})$, whence $[e_\ell T_{0,K}] = \ell[T_{0,K}]$. Hence, we can make the identification

$$\big(K_0(M_\ell(T_{0,K})), [M_\ell(T_{0,K})]\big) = (\mathbb{Z}_0, [\ell]_0),$$

in parallel with the previous identifications.

The following lemma and corollary give the key for our construction.

Lemma 8.2. *Let $\varphi : \mathbb{Z}_n \to \mathbb{Z}_m$ be a group homomorphism given by multiplication by ℓ, where we adopt the above conventions. Let K be a field and t an indeterminate. Then there exists a K-algebra homomorphism $\psi : T_{n,K} \to M_\ell(T_{m,K(t)})$ such that $K_0(\psi) = \varphi$.*

Proof. With the canonical identifications we have made for the K_0 groups, it is automatic that $K_0(\psi) = \varphi$ will hold for any K-algebra homomorphism $\psi : T_{n,K} \to M_\ell(T_{m,K(t)})$. Thus, only the existence of such homomorphisms needs to be established. We will consider several cases, in the first two of which $\ell > 0$.

Case 1. Assume that $n, m \ge 2$.

We can write $\ell n = hm$ for some positive integer h. There are elements $x'_0, \dots, x'_{hm}$ and $y'_0, \dots, y'_{hm}$ in $T_{m,K(t)}$ implementing an isomorphism $T_{m,K(t)} \cong T^{hm+1}_{m,K(t)}$ and with each x'_i being a nontrivial product of the standard elements x_i of $T_{m,K(t)}$. For example, take $x'_0, \dots, x'_{hm}$ and $y'_0, \dots, y'_{hm}$ to be the sequences

$$x_0^h, x_1x_0^{h-1}, \dots, x_mx_0^{h-1}, x_1x_0^{h-2}, \dots, x_mx_0^{h-2}, \dots, x_1, \dots, x_m \qquad \text{and}$$
$$y_0^h, y_0^{h-1}y_1, \dots, y_0^{h-1}y_m, y_0^{h-2}y_1, \dots, y_0^{h-2}y_m, \dots, y_1, \dots, y_m,$$

respectively. Define matrices $A_i, B_j \in M_\ell(T_{m,K(t)})$ for $i,j = 0,1,\dots,n$ as follows:

$$A_0 = \operatorname{diag}(t,\dots,t,x_0')\,, \qquad (A_i)_{\alpha,\beta} = \delta_{\beta,\ell} x'_{(i-1)\ell+\alpha} \quad (1 \le i \le n)\,,$$
$$B_0 = \operatorname{diag}(t^{-1},\dots,t^{-1},y_0')\,, \qquad (B_j)_{\alpha,\beta} = \delta_{\alpha,\ell} y'_{(i-1)\ell+\beta} \quad (1 \le j \le n)\,.$$

It is easy to check that $A_iB_j = \delta_{i,j}I$ for all i,j and $\sum_{i=0}^n B_iA_i = I$.

Observe that if $p(Z_0,\dots,Z_n)$ is a noncommutative polynomial over K with zero constant term, then $p(A_0,\dots,A_n)$ is a matrix in $M_\ell(T_{m,K(t)})$ whose entries come from either $tK[t]$ or $K[t]\langle X_m\rangle X_m$ (where the latter notation refers to the left ideal of $K[t]\langle X_m\rangle$ generated by X_m). Consequently, any $r\times r$ matrix in $\Sigma'_K(A_0,\dots,A_n)$, when viewed as a block form of an $\ell r\times \ell r$ matrix over $T_{m,K(t)}$, consists of a sum $C_0+C_1+C_2$ where C_0 is an invertible matrix over K, all entries of C_1 lie in $tK[t]$, and all entries of C_2 lie in $K[t]\langle X_m\rangle X_m$. Now C_0+C_1 is an invertible matrix over $K(t)$ (note that we need the variable t to achieve this statement), whence $C_0+C_1+C_2$ lies in $\Sigma'_{K(t)}(x_0,\dots,x_m)$. It follows that every matrix in $\Sigma'_K(A_0,\dots,A_n)$ is invertible over $M_\ell(T_{m,K(t)})$. By the universal property of the algebras $T_{n,K}$ (Theorem 7.2), there is a unique K-algebra homomorphism $T_{n,K}\to M_\ell(T_{m,K(t)})$ sending (x_i) to (A_i) and (y_j) to (B_j).

Case 2. Assume that $n=0$ and $\ell>0$ (with m arbitrary).

This case is easier than the previous one. Just take an infinite sequence of diagonal matrices A_i, B_j in $M_\ell(T_{m,K})$, where the A_i's consist of nontrivial products of the x's in the diagonal, the B_j's consist of nontrivial products of the y's in the diagonal, and they satisfy the rules $A_iB_j = \delta_{ij}$. For example, take

$$A_i = \operatorname{diag}(x_0x_1^i,\dots,x_0x_1^i) \qquad \text{and} \qquad B_j = \operatorname{diag}(y_1^jy_0,\dots,y_1^jy_0).$$

for $i,j = 0,1,\dots$. Then we can identify $\Sigma'_K(A_0,A_1,\dots)$ with a subset of $\Sigma'_K(x_0,x_1)$, and so every matrix in $\Sigma'_K(A_0,A_1,\dots)$ is invertible over $M_\ell(T_{m,K})$. By the universal property of $T_{0,K}$ (Lemma 8.1), there is a K-algebra homomorphism $\psi: T_{0,K}\to M_\ell(T_{m,K})$. (Here the use of the new variable t is not necessary.)

Case 3. Now assume that $n=0$ and $\ell\le 0$. Necessarily, $m=0$.

Recall that in this case our convention is to set $M_\ell(T_{0,K}) = e_\ell T_{0,K} e_\ell$. Define the following elements A_i, B_j in $M_\ell(T_{0,K})$ for $i,j\ge 0$:

$$A_i = e_\ell x_{-\ell+1+i} \qquad \text{and} \qquad B_j = y_{-\ell+1+j}e_\ell.$$

Then one has $A_iB_j = \delta_{ij}e_\ell$ for all i,j. If C is a matrix in $\Sigma'_K(A_0,A_1,\dots)$, then $C = \tilde e_\ell\Gamma = \tilde e_\ell\Gamma\tilde e_\ell$ for some matrix Γ in $\Sigma'_K(x_{-\ell+1},x_{-\ell+2},\dots)$, where $\tilde e_\ell = \operatorname{diag}(e_\ell,\dots,e_\ell)$. Note that Γ is invertible over $K_{\mathrm{rat}}\langle x_{-\ell+1},x_{-\ell+2},\dots\rangle$, with $\tilde e_\ell\Gamma^{-1} = \tilde e_\ell\Gamma^{-1}\tilde e_\ell$, and so C is invertible over $M_\ell(T_{0,K})$ with inverse $\tilde e_\ell\Gamma^{-1}$. Thus in this case too we get our desired homomorphism $T_{0,K}\to M_\ell(T_{0,K})$.

Case 4. Finally, suppose that $m=0$ and $n\ge 2$. Then necessarily $\ell=0$.

Set $E = \operatorname{diag}(e_n,1,1,\dots,1)$ in $M_{n+1}(T_{0,K(t)})$, and observe that $E\sim e_0$. Hence, $M_0(T_{0,K(t)})$ is isomorphic to the $K(t)$-algebra $M = EM_{n+1}(T_{0,K(t)})E$, and so it suffices to produce a K-algebra homomorphism $T_{n,K}\to M$. Let us index rows and columns of matrices in $M_{n+1}(T_{0,K(t)})$ by $0,1,\dots,n$.

Now consider matrices A_i, B_j in M for $i, j = 0, 1, \dots, n$ where

$$A_0 = \operatorname{diag}(te_n, x_0, \dots, x_0)\,, \qquad (A_i)_{\alpha,\beta} = \delta_{\beta,i} \begin{cases} te_n & (\alpha = 0) \\ x_\alpha & (\alpha \geq 1) \end{cases} \quad (1 \leq i \leq n)\,,$$

$$B_0 = \operatorname{diag}(t^{-1}e_n, y_0, \dots, y_0)\,, \qquad (B_j)_{\alpha,\beta} = \delta_{\alpha,j} \begin{cases} t^{-1}e_n & (\beta = 0) \\ y_\beta & (\beta \geq 1) \end{cases} \quad (1 \leq j \leq n)\,.$$

Then we have $A_iB_j = \delta_{i,j}E$ for all i, j and $\sum_{i=0}^n B_iA_i = E$. If $p(Z_0, \dots, Z_n)$ is a non-commutative polynomial over K with zero constant term, then $p(A_0, \dots, A_n) = EC = ECE$ for some matrix $C \in M_{n+1}(T_{0,K(t)})$ whose entries come from either $tK[t]\langle X_n\rangle$ or $K\langle X_n\rangle X_n$, and where $C_{i0} = 0$ for $i = 1, \dots, n$. Consequently, any matrix D in $\Sigma'_K(A_0, \dots, A_n)$ can be written as $D = \widetilde{E}\Delta = \widetilde{E}\Delta\widetilde{E}$ for some Δ in $\Sigma'_{K(t)}(x_0, \dots, x_n)$, where $\widetilde{E} = \operatorname{diag}(E, \dots, E)$. Now Δ is invertible over $K(t)_{\text{rat}}\langle X_n\rangle$, and we compute that $\widetilde{E}\Delta^{-1} = \widetilde{E}\Delta^{-1}\widetilde{E}$. Thus D is invertible over M, with inverse $\widetilde{E}\Delta^{-1}$. This shows that every matrix in $\Sigma'_K(A_0, \dots, A_n)$ is invertible over M. Therefore by the universal property of $T_{n,K}$, there is a K-algebra homomorphism $T_{n,K} \to M$, and we are done. □

Corollary 8.3. *Let K be a field and t an indeterminate. Let R be a K-algebra Morita equivalent to some $T_{n,K}$ and S a $K(t)$-algebra Morita equivalent to some $T_{m,K(t)}$, where $n, m \in \{0\} \cup \{2, 3, \dots\}$. Let $\varphi : K_0(R) \to K_0(S)$ be a group homomorphism such that $\varphi([R]) = [S]$. Then there exists a K-algebra homomorphism $\psi : R \to S$ such that $K_0(\psi) = \varphi$.*

Proof. There exists a finitely generated projective right $T_{n,K}$-module A such that $R \cong \operatorname{End}(A)$, and $[A] = \ell[T_{n,K}]$ for some $\ell \in \mathbb{Z}$, where we may assume $1 \leq \ell \leq n$ in case $n \geq 2$. If $\ell > 0$, then $A \cong T_{n,K}^\ell$ by Proposition 2.2, whence $R \cong M_\ell(T_{n,K})$. If $\ell \leq 0$, then $n = 0$ and $[A] = [e_\ell T_{0,K}]$, in which case $R \cong e_\ell T_{0,K} e_\ell = M_\ell(T_{0,K})$. Hence, there is no loss of generality in assuming that $R = M_\ell(T_{n,K})$. Likewise, we may assume that $S = M_{\ell'}(T_{m,K(t)})$ for some $\ell' \in \mathbb{Z}$, where $1 \leq \ell' \leq m$ in case $m \geq 2$.

As above, we can make the identifications

$$\begin{gathered}
\big(K_0(T_{n,K}), [T_{n,K}]\big) = (\mathbb{Z}_n, [1]_n) \\
(K_0(R), [R]) = \big(K_0(T_{n,K}), \ell[T_{n,K}]\big) = (\mathbb{Z}_n, [\ell]_n) \\
\big(K_0(T_{m,K(t)}), [T_{m,K(t)}]\big) = (\mathbb{Z}_m, [1]_m) \\
(K_0(S), [S]) = \big(K_0(T_{m,K(t)}), \ell'[T_{m,K(t)}]\big) = (\mathbb{Z}_m, [\ell']_m).
\end{gathered}$$

Then φ is identified with a homomorphism $\mathbb{Z}_n \to \mathbb{Z}_m$ given by multiplication by an integer h, where we can assume that $1 \leq h \leq m$ in case $m \geq 2$, and $[h\ell]_m = [\ell']_m$. Finally, identify

$$\big(K_0(M_h(T_{m,K(t)})), [M_h(T_{m,K(t)})]\big) = \big(K_0(T_{m,K(t)}), h[T_{m,K(t)}]\big) = (\mathbb{Z}_m, [h]_m).$$

By Lemma 8.2, there exists a K-algebra homomorphism $\psi_0 : T_{n,K} \to M_h(T_{m,K(t)})$ such that $K_0(\psi_0) = \varphi$. If $\ell > 0$, then ψ_0 induces a K-algebra homomorphism $\psi_1 : R \to$

$M_\ell(M_h(T_{m,K(t)}))$ such that $K_0(\psi_1) = \varphi$. There is a nonzero finitely generated projective right S-module B such that $[B] = [h]_m$ in $K_0(S)$ and $\mathrm{End}_S(B) \cong M_h(T_{m,K(t)})$. Since $\ell[B] = [\ell h]_m = [\ell']_m = [S]$, it follows from Proposition 2.2 that $\ell B \cong S_S$, whence $M_\ell(M_h(T_{m,K(t)})) \cong S$. On composing the latter isomorphism with ψ_1, we obtain a K-algebra homomorphism $\psi : R \to S$ such that $K_0(\psi) = \varphi$.

Now suppose that $\ell \le 0$, so that $n = 0$ and $R = e_\ell T_{0,K} e_\ell$. In this case, ψ_0 restricts to a K-algebra homomorphism $\psi_1 : R \to f M_h(T_{m,K(t)}) f$, where $f = \psi_0(e_\ell)$. Since $[e_\ell T_{0,K}] = \ell[T_{0,K}]$ in $K_0(T_{0,K})$, we have

$$[f M_h(T_{m,K(t)})] = \ell[M_h(T_{m,K(t)})] = \ell[h]_m = [\ell']_m$$

in $K_0(M_h(T_{m,K(t)}))$. There is a nonzero finitely generated projective right $M_h(T_{m,K(t)})$-module C such that $[C] = [\ell']_m$ in $K_0(M_h(T_{m,K(t)}))$ and $\mathrm{End}(C) \cong S$. By Proposition 2.2, $f M_h(T_{m,K(t)}) \cong C$, whence $f M_h(T_{m,K(t)}) f \cong S$. As in the previous paragraph, on composing the latter isomorphism with ψ_1, we obtain the desired K-algebra homomorphism $\psi : R \to S$. $\square$

We are now ready to prove our realization result.

Theorem 8.4. *Let G be a countable abelian group, $u \in G$, and k any field. Then there exists a purely infinite simple regular k-algebra R such that $K_0(R) \cong G$ with $[R] \mapsto u$.*

Proof. We can write (G, u) as the inductive limit of a sequence

$$(G_0, u_0) \xrightarrow{\varphi_0} (G_1, u_1) \xrightarrow{\varphi_1} \cdots,$$

where each G_n is a nonzero finitely generated abelian group, $u_n \in G_n$, and $\varphi_n : G_n \to G_{n+1}$ is a group homomorphism such that $\varphi_n(u_n) = u_{n+1}$. For each n, we may assume that

$$(G_n, u_n) = (G_{n,1}, u_{n,1}) \times \cdots \times (G_{n,r(n)}, u_{n,r(n)})$$

where each $G_{n,i}$ is nonzero and cyclic.

Let $t_1, t_2, \dots$ be independent indeterminates over k, and set $K_n = k(t_1, \dots, t_n)$ for $n = 0, 1, \dots$ (thus $K_0 = k$). For each n, let $\mathcal{M}_n$ denote the class of K_n-algebras Morita equivalent to ones of the form T_{*,K_n}. Choose $R_{n,1}, \dots, R_{n,r(n)}$ in $\mathcal{M}_n$ together with identifications $(K_0(R_{n,i}), [R_{n,i}]) = (G_{n,i}, u_{n,i})$ for all i. Set $R_n = R_{n,1} \times \cdots \times R_{n,r(n)}$, so that $(K_0(R_n), [R_n]) = (G_n, u_n)$. We shall construct K_n-algebra homomorphisms $\psi_n : R_n \to R_{n+1}$ such that $K_0(\psi_n) = \varphi_n$ and each of the component maps $\psi_{n,j} : R_n \to R_{n+1,j}$ is an embedding. Then the inductive limit of the sequence

$$R_0 \xrightarrow{\psi_0} R_1 \xrightarrow{\psi_1} \cdots$$

will be a regular k-algebra with $(K_0(R), [R]) \cong (G, u)$. Because the $\psi_{n,j}$ are embeddings, we see that for any nonzero element $a \in R_n$, there exist elements $x, y \in R_{n+1}$ such that $x\psi_n(a)y = 1$. Therefore R will be a purely infinite simple ring.

It only remains to build the homomorphisms ψ_n. Fix n, write φ_n as a matrix of group homomorphisms

$$\eta_{i,j} : G_{n,i} \to G_{n+1,j},$$

and note that $\eta_{1,j}(u_{n,1})+\cdots+\eta_{r(n),j}(u_{n,r(n)}) = u_{n+1,j}$ for all j. Hence, there exist finitely generated projective right $R_{n+1,j}$-modules $P_1,\dots,P_{r(n)}$ such that $[P_i] = \eta_{i,j}(u_{n,i})$ for all i and $P_1\oplus\cdots\oplus P_{r(n)} \cong R_{n+1,j}$. Since $R_{n+1,j}$ is purely infinite simple, there exists a nonzero finitely generated projective right $R_{n+1,j}$-module P such that $[P]$ is the identity element of the group $\mathcal{V}(R_{n+1,j})^*$ (see Proposition 2.2). We can replace each P_i by $P\oplus P_i$, and so we may assume that all $P_i \neq 0$. Consequently, there exist nonzero pairwise orthogonal idempotents $f_{1,j},\dots,f_{r(n),j}$ in $R_{n+1,j}$ such that $f_{1,j}+\cdots+f_{r(n),j} = 1$ and $[f_{i,j}R_{n+1,j}] = \eta_{i,j}(u_{n,i})$ for all i.

Each corner $f_{i,j}R_{n+1,j}f_{i,j}$ belongs to $\mathcal{M}_{n+1}$, and we can make the identification

$$\big(K_0(f_{i,j}R_{n+1,j}f_{i,j}), [f_{i,j}R_{n+1,j}f_{i,j}]\big) = \big(G_{n+1,j}, \eta_{i,j}(u_{n,i})\big).$$

By Corollary 8.3, there exist K_n-algebra homomorphisms $\theta_{i,j} : R_{n,i} \to f_{i,j}R_{n+1,j}f_{i,j}$ such that $K_0(\theta_{i,j}) = \eta_{i,j}$. Since the $R_{n,i}$ are simple algebras, the $\theta_{i,j}$ are embeddings. Consequently, the rule $\psi_{n,j}(a) = \theta_{1,j}(a)+\cdots+\theta_{r(n),j}(a)$ defines a K_n-algebra embedding $\psi_{n,j} : R_n \to R_{n+1,j}$ such that $K_0(\psi_{n,j}) = (\eta_{1,j},\dots,\eta_{r(n),j})$. These $\psi_{n,j}$, finally, are the components for the desired K_n-algebra homomorphism $\psi_n : R_n \to R_{n+1}$. $\square$

Remark 8.5. Note that we get a countable k-algebra R in Theorem 8.4 in case we start with a countable field k.

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193, Bellaterra (Barcelona), Spain
E-mail address: para@mat.uab.es

Department of Mathematics, University of California, Santa Barbara, CA 93106, USA
E-mail address: goodearl@math.ucsb.edu

Departamento de Matemáticas, Universidad de Cádiz, Aptdo. 40, 11510 Puerto Real (Cádiz), Spain
E-mail address: enrique.pardo@uca.es