QUANTUM n-SPACE AS A QUOTIENT OF CLASSICAL n-SPACE

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Abstract. The prime and primitive spectra of $\mathcal{O}_{\mathbf{q}}(k^n)$, the multiparameter quantized coordinate ring of affine n-space over an algebraically closed field k, are shown to be topological quotients of the corresponding classical spectra, $\operatorname{spec} \mathcal{O}(k^n)$ and $\max \mathcal{O}(k^n) \approx k^n$, provided the multiplicative group generated by the entries of $\mathbf{q}$ avoids -1.

Introduction

In the representation theory of a noncommutative ring A, the natural analog of the maximal spectrum of a commutative ring is $\operatorname{prim} A$, the set of primitive ideals, equipped with the Zariski (Jacobson) topology. Thus, if A is a quantization of a classical coordinate ring $\mathcal{O}(V)$ over an algebraically closed field, one can view $\operatorname{prim} A$ as a quantization of the variety V. The question then naturally arises, how are $\operatorname{prim} A$ and V related? Some "piecewise" relations are known in many cases. For instance, if G is a connected, semisimple, complex algebraic group and H is a maximal torus of G, then various (generic) quantizations A of $\mathcal{O}(G)$ exhibit the following properties: H acts on A via automorphisms; there are only finitely many H-orbits in $\operatorname{prim} A$, and they are locally closed; each H-orbit in $\operatorname{prim} A$ is homeomorphic to a torus; and each H-orbit in $\operatorname{prim} A$ is a set-theoretic quotient of a locally closed subset of G that is stable under translation by H (see [**8**, **9**, **11**, **12**, **10**]). Similar pictures have been observed to hold for quantized coordinate rings of affine spaces (see [**4**, **15**, **3**, **6**]). One is therefore led to conjecture that in the above situations, $\operatorname{prim} A$ as a whole is a topological quotient of V. Similarly, $\operatorname{spec} A$ should be a topological quotient of $\operatorname{spec} \mathcal{O}(V)$. In the "smallest" cases, such as $A = \mathcal{O}_q(SL_2(k))$ and $A = \mathcal{O}_q(k^2)$, these conjectures can easily be verified by direct calculation. In this paper we establish these conjectures for multiparameter quantum affine spaces, when -1 cannot be written as a product of the defining parameters.

To provide some detail, first let k be an algebraically closed field. By a quantum k-affine n-space (i.e., a quantization of the coordinate ring of k^n), we mean an algebra with generators $x_1, x_2, \dots, x_n$ subject only to the commutator relations $x_i x_j = q_{ij} x_j x_i$, for a chosen set of nonzero scalars $q_{ij} \in k$. (To avoid degeneracies, one assumes that $\mathbf{q} = (q_{ij})$ is a multiplicatively antisymmetric $n \times n$ matrix.) We denote this algebra as $\mathcal{O}_{\mathbf{q}}(k^n)$. Letting $H := (k^{\times})^n$ be an algebraic n-torus, it is not hard to verify that the natural action of H on the vector space spanned by $x_1, \dots, x_n$ extends to an action by k-algebra automorphisms on $\mathcal{O}_{\mathbf{q}}(k^n)$. Consequently, there are induced H-actions on the prime and primitive spectra of $\mathcal{O}_{\mathbf{q}}(k^n)$.

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Our methods now require us to make the following technical assumption: Either the subgroup of $k^\times$ generated by the q_{ij} does not contain -1, or the characteristic of k is 2. (While our present techniques do not appear to work in the absence of this hypothesis, we do not know if our main results will remain valid without it.) Now set $A = \mathcal{O}_{\mathbf{q}}(k^n)$, and let $R := \mathcal{O}(k^n)$ be a commutative polynomial ring in n variables, also equipped with the natural action of H by automorphisms. In the main result (4.11), we construct H-equivariant topological quotient maps

$$\operatorname{spec} R \to \operatorname{spec} A \quad \text{and} \quad k^n \approx \max R \to \operatorname{prim} A.$$

Moreover, a description of the fibers of the second map is provided. Roughly speaking, we show that $\operatorname{prim} A$ is equal to k^n modulo the actions of a compatible system of groups acting separately on each H-orbit, and that $\operatorname{spec} A$ can be obtained from $\operatorname{spec} R$ in an analogous fashion.

Our analysis involves two separate steps:

(I) The spaces $\operatorname{spec} A$ and $\operatorname{prim} A$ are homeomorphic to certain sets of ideals of R, denoted $\mathcal{G}$-$\operatorname{spec} R$ and $\mathcal{G}$-$\max R$, which are equipped with Zariski topologies.

(II) We obtain topological quotient maps $\operatorname{spec} R \to \mathcal{G}$-$\operatorname{spec} R$ and $\max R \to \mathcal{G}$-$\max R$, by sending each prime (respectively, maximal) ideal P of R to the largest member of $\mathcal{G}$-$\operatorname{spec} R$ contained within P.

Step II works very generally, and so we begin with that. Section 1 contains an axiomatic treatment that produces topological quotients of prime and primitive spectra in a noetherian ring R, using certain families of ideals of R. In Section 2, we restrict to the case of a commutative polynomial ring R over k and a family of ideals stable under a suitably compatible family of groups of automorphisms of R. This last setup yields the topological quotients of $\operatorname{spec} R$ and $\max R$ that will appear in our main theorem. Step I is developed in Sections 3 and 4. Section 3 is devoted to the case of quantum tori, which yields the necessary information about individual "strata" of prime and primitive ideals in A; in Section 4, we glue these results together over the whole prime and primitive spectra of A, and obtain the main theorem. Section 5 contains explicit calculations of the resulting topological quotient maps $k^n \to \operatorname{prim} A$ for the case where A is a single parameter quantum affine space, and Section 6 applies the main theorem to twists of more general graded commutative algebras.

We remind the reader that a map $\phi : X \to Y$ between topological spaces is a *topological quotient map* provided ϕ is surjective and the topology on Y coincides with the quotient topology induced by ϕ, that is, a subset $W \subseteq Y$ is closed in Y precisely when $\phi^{-1}(W)$ is closed in X. When such a map exists, Y is completely determined (up to homeomorphism) by X and the fibers of ϕ.

1. Quotients of prime spectra

We begin the work of this paper by specifying sufficient conditions under which a collection of ideals in a noetherian ring R, equipped with a Zariski topology, will form a topological quotient of the prime spectrum of R. These conditions are modelled on standard properties of the collection of those ideals of R which are prime relative to the ideals invariant under some set of operators, such as the collection of G-prime ideals where G is a group of automorphisms of R, or the collection of Δ-prime ideals where Δ is a set of derivations on R. Conditions for a subcollection to be a topological quotient of the primitive or the maximal spectrum of R are also given. At the end of the section, we specialize to the case of G-prime ideals, this being the case we shall need for application to quantum tori.

1.1. Let R be a noetherian ring, and let $X\text{-spec}\, R$ be a nonempty set of ideals of R. The elements of $X\text{-spec}\, R$ will be referred to as *X-prime* ideals, and intersections of collections of X-prime ideals will be termed *X-semiprime* ideals. We will further assume, throughout this section, that

(a) For each prime ideal P of R there exists a (unique) X-prime ideal, denoted $(P : X)$, that is maximum among X-semiprime ideals contained in P. In other words, $(P : X)$ is an X-prime ideal contained in P and containing all X-semiprime ideals which are contained in P.

(b) If P is a prime ideal of R minimal over an X-prime ideal Q, then $Q = (P : X)$. In particular, for each X-prime ideal Q there exists a prime ideal P such that $Q = (P : X)$.

1.2. If I is an ideal of R then $V(I)$ will denote the set of prime ideals containing I. The set of prime ideals of R will be denoted $\text{spec}\, R$ and will be equipped with the standard Zariski topology: The closed subsets are precisely those of the form $V(I)$, for ideals I of R. The sets of (left) primitive and maximal ideals of R will be denoted $\text{prim}\, R$ and $\max R$ respectively, and each of these sets will be given the relative topology from $\text{spec}\, R$.

1.3 Lemma. *Let Q be an X-prime ideal in R, and let I and J be X-semiprime ideals of R such that $IJ \subseteq Q$. Then I or J is contained in Q.*

Proof. Choose a prime ideal P such that $Q = (P : X)$. Then $IJ \subseteq P$, and so one of I or J is contained in P, say $I \subseteq P$. But Q is the unique maximum X-semiprime ideal contained in P, and so $I \subseteq Q$. $\square$

1.4. If I is an ideal of R, define $V_X(I)$ to be the set of X-prime ideals of R containing I; note that $V_X(I) = V_X(J)$, where J is the intersection of the X-prime ideals of R containing I (with the convention that the intersection of an empty collection of ideals is equal to R itself). By (1.3), if $J_1, \dots, J_t$ are X-semiprime ideals of R, then $V_X(J_1) \cup \cdots \cup V_X(J_t) = V_X(J_1 \cap \cdots \cap J_t)$. Therefore, the standard Zariski topology generalizes to a topology on $X\text{-spec}\, R$: The closed sets are those subsets of the form $V_X(I)$, for ideals I of R.

1.5. By our assumptions in (1.1), there is a surjection

$$\pi : \text{spec}\, R \to X\text{-spec}\, R$$

sending each prime ideal P to $(P : X)$. We first record two trivial observations concerning the behavior of π with respect to closed sets.

Lemma. *Let J be an X-semiprime ideal of R.*

(a) $\pi(V(J)) = V_X(J)$.

(b) $\pi^{-1}(V_X(J)) = V(J)$. $\square$

1.6. A subset U of $\operatorname{spec} R$ will be termed *π-stable* provided U is a union of fibers of π, or equivalently, provided $U = \pi^{-1}(\pi(U))$.

Lemma. *Let U be a closed π-stable subset of* $\operatorname{spec} R$. *Then there exists an X-semiprime ideal J of R such that $U = V(J)$.*

Proof. Choose an ideal I of R such that $U = V(I)$, and let $P_1, \dots, P_t$ be the prime ideals minimal over I. Then $P_1, \dots, P_t$ are the minimal elements of U. For $1 \le i \le t$, set $Q_i = (P_i : X)$, and set $J = Q_1 \cap \cdots \cap Q_t$. Observe that $U \subseteq V(J)$. Next, let P be a prime ideal minimal over J. Note that P is minimal over Q_i, for some $1 \le i \le t$, and so $(P : X) = Q_i = (P_i : X)$. Therefore, since U is π-stable, $P \in U$. Consequently, $V(J) \subseteq U$, and hence $U = V(J)$. $\square$

1.7 Proposition. (a) *π is continuous.*

(b) *π maps π-stable closed subsets of* $\operatorname{spec} R$ *to closed subsets of* $X\text{-}\operatorname{spec} R$.

(c) *π is a topological quotient map.*

Proof. (a) Let V be a closed subset of $X\text{-}\operatorname{spec} R$. As seen in (1.4), there exists an X-semiprime ideal J in R such that $V = V_X(J)$. By (1.5b), $\pi^{-1}(V) = V(J)$.

(b) Let U be a π-stable closed subset of $\operatorname{spec} R$. It follows from (1.6) that there exists an X-semiprime ideal J in R such that $U = V(J)$. By (1.5a), $\pi(U) = V_X(J)$.

(c) Assume that V is a subset of $X\text{-}\operatorname{spec} R$ whose inverse image $U = \pi^{-1}(V)$ is closed in $\operatorname{spec} R$. It only remains to show that V is closed. However, U is clearly π-stable, and so V is closed by (b). $\square$

1.8. Let us say that a subset U of $\operatorname{prim} R$ (or of $\max R$) is *relatively π-stable* provided $U = V \cap \operatorname{prim} R$ (or $U = V \cap \max R$) for some π-stable subset V of $\operatorname{spec} R$.

Proposition. (a) *Suppose that $\bigcap U$ is X-semiprime for every relatively π-stable subset U of* $\operatorname{prim} R$. *Then the restriction of π to the surjection*

$$\operatorname{prim} R \longrightarrow \pi(\operatorname{prim} R)$$

is a topological quotient map (with respect to the relative topologies).

(b) *Similarly, if the intersection of any relatively π-stable subset of* $\max R$ *is X-semiprime, then π restricts to a topological quotient map*

$$\max R \longrightarrow \pi(\max R).$$

Proof. (a) Let $\pi_{\mathrm{prim}} : \operatorname{prim} R \to \pi(\operatorname{prim} R)$ denote the restriction of π. It is immediate from the continuity of π that π_{prim} is continuous. Now let W be a subset of $\pi(\operatorname{prim} R)$ whose inverse image $U = \pi_{\mathrm{prim}}^{-1}(W)$ is closed in $\operatorname{prim} R$. Since U is relatively π-stable, its intersection is by hypothesis an X-semiprime ideal J. Moreover, $U = V(J) \cap \operatorname{prim} R$ because U is closed in $\operatorname{prim} R$. Then $W = \pi(U) \subseteq \pi(V(J)) = V_X(J)$. If $I \in V_X(J) \cap \pi(\operatorname{prim} R)$, there exists $P \in \operatorname{prim} R$ such that $\pi(P) = I \supseteq J$. Since $P \supseteq \pi(P)$, we have $P \in V(J) \cap \operatorname{prim} R = U$, and so $I \in W$. Thus $W = V_X(J) \cap \pi(\operatorname{prim} R)$, a relatively closed subset of $\pi(\operatorname{prim} R)$.

(b) Use the proof above, with "prim" replaced by "max" everywhere. $\square$

The preceding observations apply easily to group actions, as follows.

1.9. For the remainder of this section, let G be a group acting by automorphisms on R. An ideal I of R, such that $g(I) = I$ for all $g \in G$, is termed a *G-ideal*. Note that the set of G-ideals of R is closed under products, sums, and intersections. A (set-theoretically) proper G-ideal Q of R is *G-prime* provided $IJ \not\subseteq Q$ for all G-ideals I and J not contained in Q. An intersection of G-prime ideals will be termed *G-semiprime*. For any ideal I of R, set $(I : G) = \bigcap_{g\in G} g(I)$. It is easy to see that $(I : G)$ is the unique maximum G-ideal contained in I.

1.10 Lemma. (cf. [**5**, Remarks 4*, 5*]) (a) *Let P be a prime ideal of R. Then $(P : G)$ is a G-prime ideal.*

(b) *Let Q be a G-prime ideal of R, and let P be a prime ideal of R minimal over Q. Then $Q = (P : G)$, and every prime ideal minimal over Q is in the G-orbit of P. In particular, Q is semiprime.*

Proof. Part (a) follows from an argument mimicking the proof of (1.3).

(b) Let $P_1, \dots, P_t$ be the prime ideals of R minimal over Q, and set $N_i = (P_i : G)$ for each $1 \le i \le t$. Note that $N_1, \dots, N_t$ are G-ideals, all containing Q, and that some product of them is contained within Q. Therefore, $Q = N_i$ for some i, because Q is G-prime. Now consider an arbitrary prime ideal P_j minimal over Q. Since $P_j \supseteq Q = N_i$, we see that P_j contains a product of prime ideals in the G-orbit of P_i, and so $P_j \supseteq g(P_i) \supseteq Q$ for some $g \in G$. Therefore, $P_j = g(P_i)$, and $Q = (P_i : G) = (P_j : G)$. Part (b) follows. $\square$

1.11. Let $G\text{-spec}\, R$ denote the set of G-prime ideals of R. It follows from (1.9) and (1.10) that $G\text{-spec}\, R$ satisfies the axioms for $X\text{-spec}\, R$ specified in (1.1). In particular, if $V_G(I)$ denotes the set of G-prime ideals of R that contain a given ideal I of R, then by (1.4) there is a Zariski topology on $G\text{-spec}\, R$: The closed sets are those of the form $V_G(I)$, for ideals I of R. With respect to this topology, we can use (1.7c) to deduce that the assignment

$$P \mapsto (P : G)$$

is a topological quotient map from $\operatorname{spec} R$ onto $G\text{-spec}\, R$.

Let $G\text{-max}\, R$ denote the set of maximal proper G-ideals. Observe that $G\text{-max}\, R$ is a subset of $G\text{-spec}\, R$ and that $G\text{-max}\, R$ is comprised of the maximal members of $G\text{-spec}\, R$. Equip $G\text{-max}\, R$ with the relative topology. While each member of $G\text{-max}\, R$ must be equal to $(M : G)$ for some $M \in \max R$, the converse fails in general. For example, let R be a polynomial ring $k[x]$ over an infinite field k, and let $G = k^\times$ act on R by the rule $g.f(x) = f(gx)$. Then $(\langle x - \alpha\rangle : G) = 0$ for all nonzero $\alpha \in k$, whereas $G\text{-max}\, R$ consists of the single maximal ideal $\langle x\rangle$.

In the present setting, the hypothesis of (1.8b) is easily verified, as follows. Suppose U is a relatively π-stable subset of $\max R$, and set $J = \bigcap U$. Since $\max R$ is stable under the action of G (within $\operatorname{spec} R$), so is U, and hence $J = \bigcap_{M\in U} (M : G)$. Thus by (1.10a), J is G-semiprime, as desired. We therefore conclude from (1.8b) that the assignment

$$M \mapsto (M : G)$$

is a topological quotient map from $\max R$ onto $\pi(\max R)$.

1.12. Suppose that G has no proper subgroups of finite index. Then all finite G-orbits of ideals of R are singletons, and so all G-prime ideals of R are prime, by (1.10b). We see, in this case, that G-spec R is a subset of spec R and that the topology on G-spec R described in (1.11) is the relative topology.

1.13. We conclude this section by considering a particular case of the group action setting that will be needed later.

Let k be an algebraically closed field, $R = k[y_1^{\pm 1}, \dots, y_n^{\pm 1}]$ a Laurent polynomial ring over k, and $H = (k^\times)^n$ the standard algebraic k-torus of rank n. Let H act on R by k-algebra automorphisms in the natural manner; namely,

$$(h_1, \dots, h_n).f(y_1, \dots, y_n) = f(h_1 y_1, \dots, h_n y_n),$$

for $(h_1, \dots, h_n) \in H$ and $f(y_1, \dots, y_n) \in R$. Since k is algebraically closed, the induced action of H on $\max R$ is transitive.

Proposition. *Let G be a subgroup of H.*

(a) *The set $\{(M : G) \mid M \in \max R\}$ coincides with G-$\max R$.*

(b) *If G is a closed subgroup of H, the fibers of the map $M \mapsto (M : G)$, from $\max R$ to G-$\max R$, are precisely the G-orbits in $\max R$.*

Proof. (a) Given $Q \in G$-$\max R$, choose a maximal ideal $N \supseteq Q$. Then $(N : G) \supseteq Q$, and so $Q = (N : G)$ by the maximality of Q.

Conversely, consider $Q = (M : G)$ where $M \in \max R$. Choose $Q' \in G$-$\max R$ containing Q. By the preceding paragraph, there exists a maximal ideal M' in $\max R$ for which $Q' = (M' : G)$. Since H acts transitively on $\max R$, there exists an $h \in H$ such that $M = h(M')$. Since H is abelian, $Q = (M : G) = (h(M') : G) = h(Q')$. Hence $Q' \supseteq h(Q')$, and so $Q' = h(Q') = Q$. Thus $Q \in G$-$\max R$.

(b) Let $M, M' \in \max R$. If M and M' are in the same G-orbit, then $(M : G) = (M' : G)$. Conversely, assume that $(M : G) = (M' : G)$. On intersecting with the ring of G-invariants R^G, we thus obtain

$$M \cap R^G = (M : G) \cap R^G = (M' : G) \cap R^G = M' \cap R^G.$$

Next, we may view R as the coordinate algebra of H and identify H with $\max R$. By [**14**, Theorem 48, p. 220], for example, there is an identification of H/G with $\max R^G$ that produces the following commutative diagram:

$$\begin{array}{ccc} H & \xrightarrow{\text{quotient}} & H/G \\ \| & & \| \\ \max R & \xrightarrow{\text{restriction}} & \max R^G \end{array}$$

Letting H act on itself and on H/G by right translation (i.e., $g \in H$ acts on $h \in H$ by $g.h = hg^{-1}$), the diagram above is H-equivariant.

Since the cosets of G in H are the G-orbits with respect to right translation, it follows that the fibers of the restriction map $\max R \to \max R^G$ are the G-orbits in $\max R$. Therefore M and M' lie in the same G-orbit, as desired. $\square$

2. Quotients of affine space

Throughout this section, k denotes a field, and R denotes a commutative polynomial ring $k[y_1, \dots, y_n]$. We study a "piecewise" action on R by a compatible system of groups, together with a collection of ideals which we show satisfies the axioms of the previous section. The resulting quotients of the prime and maximal spectra of R will play a crucial role in our main theorem. Namely, as we shall prove in Section 4, the quotients of $\operatorname{spec} R$ and $\max R$ with respect to a suitable system of groups turn out to be homeomorphic to the prime and primitive spectra, respectively, of a quantum affine n-space.

The setup we develop here amounts to patching together finitely many quotients by group actions, as follows. We first partition $\operatorname{spec} R$ into the 2^n locally closed sets determined by which subsets of $\{y_1, \dots, y_n\}$ are contained in given prime ideals. These subsets are homeomorphic, via localization, to the prime spectra of the Laurent polynomial rings obtained by factoring some of the y_i out of R and inverting the remainder. Our "piecewise group action", finally, amounts to compatible choices of groups acting as automorphisms on the above localizations.

2.1. Set $H = (k^\times)^n$, the algebraic k-torus of rank n, and equip H with the Zariski topology. As in (1.13), let H act on R by the automorphisms

$$(h_1, \dots, h_n).f(y_1, \dots, y_n) = f(h_1 y_1, \dots, h_n y_n).$$

We will also consider the induced actions of H on $\operatorname{spec} R$ and $\max R$. Next, let W denote the set of subsets of $\{1, \dots, n\}$. For each $w \in W$, let $\operatorname{spec}_w R$, equipped with the relative topology, denote the set of prime ideals P of R such that

$$P \cap \{y_1, \dots, y_n\} = \{y_i \mid i \in w\}.$$

Note that $\operatorname{spec} R = \bigsqcup_{w \in W} \operatorname{spec}_w R$. Also note that $\operatorname{spec}_w R$ is stable under H, and that there is an obvious H-equivariant homeomorphism

$$\operatorname{spec}_w R \xrightarrow{\cong} \operatorname{spec} k[y_i^{\pm 1} \mid i \notin w].$$

2.2. Let $\mathcal{G} = \{G_w \mid w \in W\}$ be a family of subgroups of H, indexed by W, subject to the following compatibility hypothesis: $(P : G_v) \subseteq (P : G_w)$ for all $v \subseteq w$ and every $P \in \operatorname{spec}_w R$. We do not exclude the possibility that the G_w, for $w \in W$, are identically the same. Set

$$\mathcal{G}\text{-spec}_w R \;=\; \{(P : G_w) \mid P \in \operatorname{spec}_w R\},$$

for $w \in W$. Note, since the y_i are G_w-eigenvectors, that $Q \cap \{y_1, \dots, y_n\} = \{y_i \mid i \in w\}$ for all $Q \in \mathcal{G}\text{-spec}_w R$. Consequently, the sets $\mathcal{G}\text{-spec}_w R$ for $w \in W$ are pairwise disjoint, and we set

$$\mathcal{G}\text{-spec} R = \bigsqcup_{w \in W} \mathcal{G}\text{-spec}_w R.$$

The members of $\mathcal{G}\text{-spec} R$ will be termed $\mathcal{G}$*-prime* ideals, the intersections of collections of $\mathcal{G}$-prime ideals will be called $\mathcal{G}$*-semiprime* ideals, and the set of $\mathcal{G}$-prime ideals containing an ideal I of R will be denoted $V_{\mathcal{G}}(I)$. Note from (1.10b) that all $\mathcal{G}$-semiprime ideals are semiprime.

2.3 Lemma. *$\mathcal{G}$-spec R satisfies the conditions in* (1.1)*, with* $(P : \mathcal{G}) = (P : G_w)$ *for* $P \in \operatorname{spec}_w R$.

Proof. To verify (1.1a), let $P \in \operatorname{spec}_w R$, let $Q = (P : G_w)$, and let Q' be an arbitrary $\mathcal{G}$-semiprime ideal of R contained in P. We will show that $Q' \subseteq Q$. To start, write

$$Q' = \bigcap_{v \in W} N_v,$$

where each N_v is an intersection of ideals from $\mathcal{G}\text{-spec}_v R$, and where each N_v contains Q'. Since $Q' \subseteq P$, some $N_v \subseteq P$. Then since $y_i \in N_v$ for all $i \in v$, we must have $v \subseteq w$. Note that N_v is G_v-stable and so is contained in $(P : G_v)$. By the compatibility hypothesis, $N_v \subseteq Q$, whence $Q' \subseteq Q$, and (1.1a) is established.

To check (1.1b), let $Q \in \mathcal{G}\text{-spec}_w R$ and let P be a prime ideal of R minimal over Q. Observe first that $P \in \operatorname{spec}_v R$ for some $v \supseteq w$. Next, it follows from (1.10a) that Q is G_w-prime, and then from (1.10b) that $Q = (P : G_w)$. Moreover, since each of the variables y_i is a G_w-eigenvector, it now follows that $v = w$, whence $Q = (P : \mathcal{G})$ and (1.1b) is satisfied. □

2.4. Following (1.4), we equip $\mathcal{G}$-spec R with the Zariski topology where the closed subsets are the $V_{\mathcal{G}}(I)$, for ideals I of R. In case none of the G_w has proper subgroups of finite index, it follows from (1.12) that $\mathcal{G}\text{-spec}\, R \subseteq \operatorname{spec} R$, and then the topology on $\mathcal{G}$-spec R coincides with the relative topology.

Note, for each $w \in W$, that $\operatorname{spec}_w R$ is a union of H-orbits. Therefore, since H is abelian, we see that $h(P : \mathcal{G}) = (h(P) : \mathcal{G})$ for all $h \in H$ and $P \in \operatorname{spec} R$. From (1.7) we now obtain:

Proposition. *The assignment $P \mapsto (P : \mathcal{G})$ produces an H-equivariant topological quotient map from* spec R *onto* $\mathcal{G}$-spec R. □

2.5. For $w \in W$, set $\max_w R = \max R \cap \operatorname{spec}_w R$. Note that $\max R = \bigsqcup_{w \in W} \max_w R$. Moreover, $\max_w R$ is equal to the set of maximal members of $\operatorname{spec}_w R$. Next, set

$$\begin{aligned} \mathcal{G}\text{-max}_w R \;&=\; \{(M : G_w) \mid M \in \max_w R\} \qquad (\text{for } w \in W); \\ \mathcal{G}\text{-max}\, R \;&=\; \bigsqcup_{w \in W} \mathcal{G}\text{-max}_w R = \{(M : \mathcal{G}) \mid M \in \max R\}. \end{aligned}$$

Note that the definition of $\mathcal{G}\text{-max}_w R$ above and the definition of $G_w\text{-max}\, R$ in (1.11) follow different patterns. However, we shall see in (2.6) that when k is algebraically closed, $\mathcal{G}\text{-max}_w R$ coincides with $\big(G_w\text{-max}\, R\big) \cap \big(\mathcal{G}\text{-spec}_w R\big)$.

Proposition. *The assignment $M \mapsto (M : \mathcal{G})$ produces an H-equivariant topological quotient map from* max R *onto* $\mathcal{G}$-max R.

Proof. By (1.8b), it suffices to show that if U is any relatively π-stable subset of max R and $J = \bigcap U$, then J is $\mathcal{G}$-semiprime.

To start, set $U_w = U \cap \max_w R$ for $w \in W$, and note that U_w is stable under G_w since $(g(M) : \mathcal{G}) = (M : \mathcal{G})$ for all $M \in \max_w R$ and $g \in G_w$. Hence,

$$\bigcap U_w = \bigcap_{M \in U_w} (M : G_w) = \bigcap_{M \in U_w} (M : \mathcal{G}),$$

which is $\mathcal{G}$-semiprime. Therefore $J = \bigcap_{w\in W}\big(\bigcap U_w\big)$ is a $\mathcal{G}$-semiprime ideal, as required. □

2.6 Proposition. *Assume that k is algebraically closed. For $w \in W$, the set $\mathcal{G}\text{-max}_w R$ coincides with the set of maximal members of $\mathcal{G}\text{-spec}_w R$.*

Proof. Set $m = n - |w|$ and $R_w = \big(R/\langle y_i \mid i \in w\rangle\big)[y_j^{-1} \mid j \notin w]$. Note that R_w is a Laurent polynomial ring in the (images of the) m indeterminates y_j, for $j \notin w$. We thereby obtain an action of the m-torus $(k^\times)^m$ on R_w, following (1.13). Note that the actions on R_w by G_w and H, induced from their actions on R, factor through the $(k^\times)^m$-action. Moreover, localization provides H-equivariant bijections

$$\text{max}_w R \to \text{max}\, R_w \qquad \text{and} \qquad \mathcal{G}\text{-spec}_w R \to G_w\text{-spec}\, R_w.$$

The proposition therefore follows from (1.13a). □

2.7. We close this section by considering the fibers of the map in (2.5) more closely. First note that each $\text{max}_w R$ is a disjoint union of fibers. Next, on $\text{max}_w R$ the map is given by the rule $M \mapsto (M : G_w)$, and so the fibers within $\text{max}_w R$ are unions of G_w-orbits. Our interest is in conditions under which the fibers within $\text{max}_w R$ coincide with the G_w-orbits.

The maximal ideals in $\text{max}_w R$ have the form

$$\langle y_i \mid i \in w\rangle + \langle f_1, \dots, f_t\rangle$$

where $f_1, \dots, f_t \in k[y_j \mid j \notin w]$. These ideals are all fixed by the group

$$H_w = \{(h_1, \dots, h_n) \in H \mid h_j = 1 \text{ for all } j \notin w\};$$

thus $(M : G_w) = (M : G_w H_w)$ for all $M \in \text{max}_w R$. Hence, there is no loss of generality in assuming that $G_w \supseteq H_w$.

Proposition. *Fix $w \in W$, and assume that k is algebraically closed. Assume that $G_w \supseteq H_w$ and that G_w/H_w is a closed subgroup of H/H_w. Then the fibers of the map $M \mapsto (M : \mathcal{G})$, from $\text{max}_w R$ to $\mathcal{G}\text{-max}_w R$, are precisely the G_w-orbits in $\text{max}_w R$.*

Proof. We shift everything to the localization R_w, as in the proof of (2.6), and we recall the $(k^\times)^m$-action on R_w described therein, where $m = n - |w|$. Now observe that the induced action of H/H_w on R_w is identical to the $(k^\times)^m$-action on R_w. Therefore, the proposition follows from (1.13b). □

3. Quantum tori

As in [**6**], much of our analysis of quantum affine spaces can be reduced, via localization, to quantum tori. This portion of our analysis is carried out in the present section. It leads, in particular, to a version of our main theorem for an arbitrary multiparameter quantum torus $\mathcal{O}_{\mathbf{q}}((k^\times)^n)$ over an algebraically closed field k, which establishes that $\text{spec}\, \mathcal{O}_{\mathbf{q}}((k^\times)^n)$ and $\text{prim}\, \mathcal{O}_{\mathbf{q}}((k^\times)^n)$ can be presented as topological quotients of $\text{spec}\, \mathcal{O}((k^\times)^n)$ and $\text{max}\, \mathcal{O}((k^\times)^n)$, respectively. No restriction on the parameter matrix $\mathbf{q}$ is needed here; it is only in patching quantum tori together to cover a quantum affine space, as in the following section, that we will need to avoid -1.

Let k be a field. From (3.7) onward, we will assume that k is algebraically closed.

3.1. Let $\mathbf{q} = (q_{ij})$ be a multiplicatively antisymmetric $n\times n$ matrix over k; that is, $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$, for all i, j. Let $A = \mathcal{O}_{\mathbf{q}}((k^\times)^n)$ be the corresponding multiparameter quantized coordinate ring of the torus $(k^\times)^n$, that is, the k-algebra generated by elements $x_1^{\pm 1}, \dots, x_n^{\pm 1}$ subject only to the relations $x_i x_j = q_{ij} x_j x_i$ for all i, j. This algebra is also known as a *McConnell-Pettit algebra*, after [**13**]; in the notation of that paper, $A = P(\mathbf{q})$.

Let us first express A in terms of ordered monomials, using standard multi-index notation. Thus A has a k-basis of monomials, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, for n-tuples $\alpha = (\alpha_1, \dots, \alpha_n)$ from the group $\Gamma := \mathbb{Z}^n$. Define $\sigma : \Gamma \times \Gamma \to k^\times$ by

$$\sigma(\alpha, \beta) = \prod_{i,j=1}^{n} q_{ij}^{\alpha_i \beta_j}.$$

Then σ determines the commutation rules in A; namely, $x^\alpha x^\beta = \sigma(\alpha,\beta) x^\beta x^\alpha$ for $\alpha, \beta \in \Gamma$. Moreover, σ is an alternating bicharacter on Γ, because

$$\sigma(\alpha,\alpha) = 1, \qquad \sigma(\beta,\alpha) = \sigma(\alpha,\beta)^{-1}, \qquad \text{and} \qquad \sigma(\alpha, \beta+\beta') = \sigma(\alpha,\beta)\sigma(\alpha,\beta'),$$

for $\alpha, \beta, \beta' \in \Gamma$.

3.2. It is convenient to view A as a twisted group algebra of Γ. One way to do this is to use the function $d : \Gamma \times \Gamma \to k^\times$ such that $x^\alpha x^\beta = d(\alpha,\beta) x^{\alpha+\beta}$ for all $\alpha, \beta \in \Gamma$. It follows from the associativity of multiplication in A that d must be a 2-cocycle on Γ. Thus by writing A in terms of the k-basis $\{x^\alpha \mid \alpha \in \Gamma\}$, we have expressed A as a twisted group algebra $k^d\Gamma$.

Note that $\sigma(\alpha,\beta) = d(\alpha,\beta) d(\beta,\alpha)^{-1}$ for $\alpha, \beta \in \Gamma$. Conversely, if c is any 2-cocycle on Γ such that $c(\alpha,\beta)c(\beta,\alpha)^{-1} = \sigma(\alpha,\beta)$ for $\alpha, \beta \in \Gamma$, then $k^c\Gamma \cong A$. Namely, $k^c\Gamma$ has a basis $\{x_\alpha \mid \alpha \in \Gamma\}$ such that $x_\alpha x_\beta = c(\alpha,\beta) x_{\alpha+\beta}$ for $\alpha, \beta \in \Gamma$. If $\epsilon_1, \dots, \epsilon_n$ denotes the standard basis for Γ, then $k^c\Gamma$ is generated by the elements $x_{\epsilon_1}^{\pm 1}, \dots, x_{\epsilon_n}^{\pm 1}$, which satisfy the same commutation rules as $x_1, \dots, x_n$, namely

$$x_{\epsilon_i} x_{\epsilon_j} = c(\epsilon_i,\epsilon_j) x_{\epsilon_i+\epsilon_j} = c(\epsilon_i,\epsilon_j) c(\epsilon_j,\epsilon_i)^{-1} x_{\epsilon_j} x_{\epsilon_i} = \sigma(\epsilon_i,\epsilon_j) x_{\epsilon_j} x_{\epsilon_i} = q_{ij} x_{\epsilon_j} x_{\epsilon_i}$$

for all i, j. Therefore there exists a k-algebra isomorphism $k^c\Gamma \to A$ such that $x_{\epsilon_i} \mapsto x_i$ for all i. Note that this isomorphism sends each x_α to a scalar multiple of x^α.

While the cocycle d is useful for some purposes, it has some drawbacks; for example, it is neither symmetric nor antisymmetric in general. We therefore leave the choice of a particular cocycle until later (see (3.5) and (3.12)).

3.3. Let c be any 2-cocycle on Γ such that $c(\alpha,\beta)c(\beta,\alpha)^{-1} = \sigma(\alpha,\beta)$ for $\alpha, \beta \in \Gamma$, and identify A with $k^c\Gamma$. This allows us to write A in terms of a k-basis $\{x_\alpha \mid \alpha \in \Gamma\}$, with multiplication given by

$$x_\alpha x_\beta = c(\alpha,\beta) x_{\alpha+\beta}$$

for $\alpha, \beta \in \Gamma$. We shall also need the ordinary group algebra $k\Gamma$. Let us express this algebra in terms of a k-basis $\{y_\alpha \mid \alpha \in \Gamma\}$, with $y_\alpha y_\beta = y_{\alpha+\beta}$ for $\alpha, \beta \in \Gamma$.

3.4. Set $H = \mathrm{Hom}(\Gamma, k^\times)$, an algebraic torus of rank n (which will henceforth be equipped with the Zariski topology). Write the application of H to Γ in terms of a pairing $\langle -, - \rangle : H \times \Gamma \to k^\times$. Define actions of H on both A and $k\Gamma$ via k-algebra automorphisms such that

$$h.x_\alpha = \langle h, \alpha \rangle x_\alpha \quad \text{and} \quad h.y_\alpha = \langle h, \alpha \rangle y_\alpha,$$

for $h \in H$ and $\alpha \in \Gamma$.

3.5. Let $\Phi_c : A \to k\Gamma$ be the k-linear isomorphism such that $\Phi_c(x_\alpha) = y_\alpha$ for $\alpha \in \Gamma$. Observe that Φ_c is H-equivariant. The extent to which Φ_c fails to preserve products can be expressed in terms of c: since $x_\alpha x_\beta = c(\alpha, \beta) x_{\alpha+\beta}$, we have

$$\Phi_c(x_\alpha x_\beta) = c(\alpha, \beta) y_{\alpha+\beta} = c(\alpha, \beta) \Phi_c(x_\alpha) \Phi_c(x_\beta)$$

for $\alpha, \beta \in \Gamma$.

As in [**6**, 1.2], set $S = \mathrm{rad}(\sigma) = \{\alpha \in \Gamma \mid \sigma(\alpha, -) \equiv 1\}$, a subgroup of Γ; then

$$Z(A) = k[x_\alpha \mid \alpha \in S].$$

If we identify the group algebra kS with the k-linear span of $\{y_\alpha \mid \alpha \in S\}$ inside $k\Gamma$, then $\Phi_c(Z(A)) = kS$. In fact:

Lemma. *Assume that $c \equiv 1$ on $S \times \Gamma \subseteq \Gamma \times \Gamma$. Then the map Φ_c restricts to a k-algebra isomorphism of $Z(A)$ onto kS. With respect to this isomorphism, Φ_c itself gives a semilinear isomorphism of A as $Z(A)$-module onto $k\Gamma$ as kS-module: $\Phi_c(za) = \Phi_c(z)\Phi_c(a)$ for all $z \in Z(A)$ and $a \in A$.*

Proof. By assumption, $\Phi_c(x_\alpha x_\beta) = \Phi_c(x_\alpha)\Phi_c(x_\beta)$ for $\alpha \in S$ and $\beta \in \Gamma$. Consequently, $\Phi_c(za) = \Phi_c(z)\Phi_c(a)$ for all $z \in Z(A)$ and $a \in A$, and the lemma follows. $\square$

We shall need the following fact to show that S equals the intersection of the kernels of some homomorphisms from H (see (3.7)).

3.6 Lemma. *The order of the torsion subgroup of Γ/S is not divisible by* $\mathrm{char}\, k$.

Proof. The torsion subgroup of Γ/S is a finite abelian group, hence a finite direct sum of finite cyclic groups, and so the order of this subgroup equals the product of the orders of certain of its elements. Thus it suffices to show that the order of each torsion element of Γ/S is not divisible by $\mathrm{char}\, k$.

Let $\alpha + S$ be a torsion element of Γ/S, and choose a basis $\epsilon_1, \dots, \epsilon_n$ for Γ. For each positive integer m, we have

$$m\alpha \in S \iff \sigma(m\alpha, \gamma_i) = 1 \text{ for all } i \iff \sigma(\alpha, \epsilon_i)^m = 1 \text{ for all } i.$$

Hence, the order of $\alpha + S$ equals the order of the element $(\sigma(\alpha, \epsilon_1), \dots, \sigma(\alpha, \epsilon_n))$ in $(k^\times)^n$, and the latter order is clearly not divisible by $\mathrm{char}\, k$. $\square$

3.7. From now until the end of the section, assume that k is algebraically closed.

We use $^\perp$ to denote orthogonals of subsets of H and Γ, relative to the pairing $\langle -, - \rangle$. In particular,

$$S^\perp = \{h \in H \mid \langle h, \alpha \rangle = 1 \text{ for } \alpha \in S\} = \{h \in H \mid \ker h \supseteq S\},$$

a closed subgroup of H, and

$$S^{\perp\perp} = \{\alpha \in \Gamma \mid \langle h, \alpha \rangle = 1 \text{ for } h \in S^\perp\} = \bigcap_{h \in S^\perp} \ker h,$$

a subgroup of Γ.

Lemma. *$S^{\perp\perp} = S$.*

Proof. Consider $\alpha \in \Gamma \backslash S$. The order of the coset $\alpha + S$ in Γ/S is either infinite, or – if finite – not divisible by $\operatorname{char} k$, in view of (3.6). In either case, the subgroup $\mathbb{Z}(\alpha + S) \subseteq \Gamma/S$ supports a nontrivial homomorphism to $k^\times$. Since $k^\times$ is divisible, any such homomorphism extends to a homomorphism $\Gamma/S \to k^\times$. Thus, there exists a homomorphism $h : \Gamma \to k^\times$ (i.e., $h \in H$) such that $S \subseteq \ker h$ and $h(\alpha) \neq 1$, which verifies that $\alpha \notin S^{\perp\perp}$. □

3.8 Proposition. (a) *$A^{S^\perp} = Z(A)$ and $(k\Gamma)^{S^\perp} = kS$.*

(b) *Every $S^\perp$-stable ideal of $k\Gamma$ is generated by its contraction to kS.*

(c) *Every ideal of A is generated by its contraction to $Z(A)$.*

(d) *Assume that $c \equiv 1$ on $S \times \Gamma$. The map $I \mapsto \Phi_c(I)$, for ideals I of A, provides a product preserving, H-equivariant lattice isomorphism from the lattice of ideals of A onto the lattice of $S^\perp$-ideals of $k\Gamma$.*

Proof. (a) It follows from (3.7) that $A^{S^\perp}$ is spanned by the x_α for $\alpha \in S$, and that $(k\Gamma)^{S^\perp}$ is spanned by the y_α for $\alpha \in S$. The second part of (a) is thus clear, and the first follows from (3.5).

(b) Consider an $S^\perp$-stable ideal I in $k\Gamma$. Since $S^\perp$ acts semisimply on $k\Gamma$, this ideal must be spanned by $S^\perp$-eigenvectors. Let $a \in I$ be an $S^\perp$-eigenvector, and choose an element γ from the support of a. Then $x_\gamma^{-1} a$ is also an $S^\perp$-eigenvector and contains 1 in its support. Hence $S^\perp$ fixes $x_\gamma^{-1} a$. Thus $x_\gamma^{-1} a \in (k\Gamma)^{S^\perp} = kS$, and so $a \in (k\Gamma)(I \cap kS)$. Therefore $I = (k\Gamma)(I \cap kS)$.

(c) See, for example, [**6**, 1.4].

(d) To start, note that $U \mapsto \Phi_c(U)$ produces an H-equivariant, bijective lattice isomorphism from the lattice of k-subspaces U of A onto the lattice of k-subspaces of $k\Gamma$.

Next, let I be an ideal of A. It follows from (c) that $I = I'A$ for some ideal I' of $Z(A)$. By (3.5), $\Phi_c(I')$ is an ideal of kS, and $\Phi_c(I) = \Phi_c(I')\Phi_c(A) = \Phi_c(I')k\Gamma$ is an ideal of $k\Gamma$. Furthermore, since $\Phi_c(I')$ is an ideal of kS, and since kS is fixed pointwise by $S^\perp$, we see that $\Phi_c(I')k\Gamma$ is stable under $S^\perp$. Similarly, it follows from (b) and (3.5) that Φ_c^{-1} maps each $S^\perp$-ideal of $k\Gamma$ to an ideal of A. Therefore the set maps Φ_c and Φ_c^{-1} provide mutually inverse bijections between the lattices of ideals of A and $S^\perp$-ideals of $k\Gamma$. It follows easily from the preceding paragraph that these bijections are H-equivariant lattice isomorphisms.

Now let I and J be ideals of A. It only remains to show that $\Phi_c(IJ) = \Phi_c(I)\Phi_c(J)$. As above, write $I = I'A$ and $J = J'A$, for ideals I' and J' of $Z(A)$. Then $IJ = I'J'A$, and so by (3.5), $\Phi_c(IJ) = \Phi_c(I')\Phi_c(J')k\Gamma = \Phi_c(I)\Phi_c(J)$. □

3.9. Let $S^\perp\text{-}\operatorname{spec} k\Gamma$ and $S^\perp\text{-}\max k\Gamma$ denote the sets of $S^\perp$-prime and maximal proper $S^\perp$-ideals of $k\Gamma$, respectively. As noted in (1.11), $S^\perp\text{-}\max k\Gamma \subseteq S^\perp\text{-}\operatorname{spec} k\Gamma$. Equip these sets with their respective Zariski topologies, again following (1.11).

Proposition. *Assume that $c \equiv 1$ on $S \times \Gamma$. The set maps Φ_c and Φ_c^{-1} provide pairs of mutually inverse, H-equivariant, homeomorphisms*

$$\operatorname{spec} A \longleftrightarrow S^\perp\text{-}\operatorname{spec} k\Gamma \qquad \textit{and} \qquad \max A \longleftrightarrow S^\perp\text{-}\max k\Gamma.$$

Proof. This follows from (3.8d). $\square$

3.10 Proposition. *The assignment $P \mapsto (P : S^\perp)$, for prime ideals P of $k\Gamma$, produces H-equivariant topological quotient maps*

$$\operatorname{spec} k\Gamma \to S^\perp\text{-}\operatorname{spec} k\Gamma \qquad \textit{and} \qquad \max k\Gamma \to S^\perp\text{-}\max k\Gamma.$$

Proof. By (1.13a), $S^\perp\text{-}\max k\Gamma = \{(M : S^\perp) \mid M \in \max k\Gamma\}$, and so the given assignment does map $\max k\Gamma$ onto $S^\perp\text{-}\max k\Gamma$. The given maps are H-equivariant because H is abelian. The remainder of the proposition follows from (1.11). $\square$

The two propositions above provide the information necessary to exhibit $\operatorname{spec} A$ and $\operatorname{prim} A$ as topological quotients of their classical analogs, as in the following theorem. A cocycle c with the required condition is constructed in (3.12).

3.11 Theorem. *Assume that $c \equiv 1$ on $S \times \Gamma$. Then the rule $P \mapsto \Phi_c^{-1}(P : S^\perp)$, for prime ideals P of $k\Gamma$, defines H-equivariant topological quotient maps*

$$\operatorname{spec} k\Gamma \to \operatorname{spec} A \qquad \textit{and} \qquad \max k\Gamma \to \max A.$$

The fibers of the second map are exactly the $S^\perp$-orbits in $\max k\Gamma$.

Proof. The first part follows from (3.9) and (3.10). The final part is a consequence of (1.13b). $\square$

3.12. In order to apply (3.11), we must be able to choose the cocycle c to be trivial on $S \times \Gamma$. This can be done as follows:

Lemma. *There exists a 2-cocycle c on Γ such that $c \equiv 1$ on $S \times \Gamma$ and $c(\alpha,\beta)c(\beta,\alpha)^{-1} = \sigma(\alpha,\beta)$ for $\alpha,\beta \in \Gamma$.*

Proof. As in [**6**, 1.6], Γ has a basis $\gamma_1,\dots,\gamma_n$ such that S is generated by $m_1\gamma_1,\dots,m_t\gamma_t$, for some $t \le n$ and some positive integers m_i. Set $p_{ij} = \sigma(\gamma_i,\gamma_j)$ for all i,j, and note that the matrix (p_{ij}) is multiplicatively antisymmetric. Given $\alpha,\beta \in \Gamma$, write $\alpha = \sum_i a_i\gamma_i$ and $\beta = \sum_j b_j\gamma_j$ for some $a_i, b_j \in \mathbb{Z}$; then

$$\sigma(\alpha,\beta) = \prod_{i,j} p_{ij}^{a_ib_j}.$$

Now define

$$c(\alpha,\beta)=\prod_{i<j}p_{ij}^{a_ib_j}$$

for $\alpha=\sum_i a_i\gamma_i$ and $\beta=\sum_j b_j\gamma_j$ in Γ. Then c is a bicharacter on Γ, and hence a 2-cocycle. It is easily checked that $c(\alpha,\beta)c(\beta,\alpha)^{-1}=\sigma(\alpha,\beta)$ for $\alpha,\beta\in\Gamma$. Since $m_i\gamma_i\in S$, we obtain

$$1=\sigma(m_i\gamma_i,\gamma_j)=p_{ij}^{m_i}$$

for all i,j. Hence, if $\alpha=\sum_i a_i\gamma_i\in S$, then $p_{ij}^{a_i}=1$ for all i,j, and therefore $c(\alpha,\beta)=1$ for all $\beta\in\Gamma$. $\square$

3.13 Remark. It would result in a stronger theorem if we could conclude in (3.11) that the given maps were closed. However, this conclusion is not possible in general. For instance, take $n=3$, choose $p\in k^\times$ not a root of unity, and take

$$\mathbf{q}=\begin{pmatrix}1&1&1\\1&1&p^2\\1&p^{-2}&1\end{pmatrix}.$$

Then σ is given by the rule $\sigma(s,t)=p^{2(s_2t_3-s_3t_2)}$, and we may choose c to be given by

$$c(s,t)=p^{s_2t_3-s_3t_2}.$$

We compute that $S=\mathbb{Z}(1,0,0)$ and $S^\perp=\{(h_1,h_2,h_3)\in H\mid h_1=1\}$.

Now write $k\Gamma=k[y_1^{\pm1},y_2^{\pm1},y_3^{\pm1}]$, where y_1,y_2,y_3 correspond to the standard basis for Γ, and let

$$V=\{\langle y_1-\alpha_1,\ y_2-\alpha_2,\ y_3-\alpha_3\rangle\mid \alpha_i\in k^\times \text{ and } (\alpha_1-1)\alpha_2=1\},$$

a closed subset of $\max k\Gamma$. The map $M\mapsto(M:S^\perp)$ sends V to the set

$$\{\langle y_1-\alpha_1\rangle\mid \alpha_1\in k^\times \text{ and } \alpha_1\neq 1\},$$

a non-closed subset of $S^\perp\text{-}\max k\Gamma$, and thus $M\mapsto\Phi_c^{-1}(M:S^\perp)$ sends V to a non-closed subset of $\max A$.

4. Quantum affine spaces

We now present and prove the main results of this paper. In essence, we partition the prime and primitive spectra of a quantum affine space into corresponding spectra of quantum tori, apply the results of the previous section to these spectra, and then patch everything together. It is the patching process that requires a more careful choice of cocycle than previously.

We continue to assume that the base field k is algebraically closed.

4.1. Again let $\mathbf{q} = (q_{ij})$ be a multiplicatively antisymmetric $n \times n$ matrix over k. Now, however, let $A = \mathcal{O}_{\mathbf{q}}(k^n)$ be the corresponding multiparameter quantized coordinate ring of affine n-space. In other words, A is the k-algebra generated by elements $x_1, \dots, x_n$, subject only to the relations $x_i x_j = q_{ij} x_j x_i$. We will treat A as a twisted semigroup algebra, as follows.

Set $\Gamma = \mathbb{Z}^n$ and $\Gamma^+ = (\mathbb{Z}^+)^n$, and define an alternating bicharacter σ on Γ by the rule

$$\sigma(\alpha, \beta) = \prod_{i,j=1}^{n} q_{ij}^{\alpha_i \beta_j}$$

as in (3.1). If c is any 2-cocycle on Γ such that $c(\alpha,\beta)c(\beta,\alpha)^{-1} = \sigma(\alpha,\beta)$ for $\alpha, \beta \in \Gamma$, we can write A as the twisted semigroup algebra $k^c\Gamma^+$, viewed in the obvious way as a subalgebra of the twisted group algebra $k^c\Gamma$.

We will need to apply the results of Section 3 to the localizations

$$A_w = \big(A/\langle x_i \mid i \in w\rangle\big)[x_j^{-1} \mid j \notin w]$$

as w ranges over the subsets of $\{1, \dots, n\}$. Thus, we require a cocycle c whose restrictions to the subgroups of Γ generated by subsets of the standard basis all satisfy the hypothesis of (3.11). The cocycle obtained in (3.12) is not sufficient for this purpose. With a minor technical assumption, we can construct a suitable cocycle as follows.

4.2. Let $\langle q_{ij}\rangle$ denote the subgroup of $k^\times$ generated by the q_{ij}. Assume from now on that either $-1 \notin \langle q_{ij}\rangle$ or $\operatorname{char} k = 2$.

Lemma. *There exists an alternating bicharacter c on Γ such that $c^2 = \sigma$ and such that $c(\alpha,\beta) = 1$ whenever $\sigma(\alpha,\beta) = 1$.*

Proof. Since k is algebraically closed, the abelian group $k^\times$ is divisible, and so it contains a divisible hull D for the subgroup $\langle q_{ij}\rangle$. In view of our hypotheses, $\langle q_{ij}\rangle$ contains no elements of order 2, whence the same holds for D. Now each q_{ij} has a unique square root in D, say p_{ij}, and it follows from the uniqueness of the choices that (p_{ij}) is a multiplicatively antisymmetric matrix over k. Thus, the rule

$$c(\alpha, \beta) = \prod_{i,j=1}^{n} p_{ij}^{\alpha_i \beta_j}$$

defines an alternating bicharacter c on Γ such that $c^2 = \sigma$. For any $\alpha, \beta \in \Gamma$ such that $\sigma(\alpha,\beta) = 1$, we have $c(\alpha,\beta)^2 = 1$ and therefore $c(\alpha,\beta) = 1$, because D has no elements of order 2. $\square$

4.3. Fix an alternating bicharacter c as in (4.2). Then c is a 2-cocycle on Γ and

$$c(\alpha,\beta)c(\beta,\alpha)^{-1} = c(\alpha,\beta)^2 = \sigma(\alpha,\beta)$$

for $\alpha, \beta \in \Gamma$. As in (3.3), we write $k^c\Gamma$ in terms of a basis $\{x_\alpha \mid \alpha \in \Gamma\}$ such that $x_\alpha x_\beta = c(\alpha,\beta)x_{\alpha+\beta}$ for $\alpha, \beta \in \Gamma$. Thus $A = k^c\Gamma^+$ equals the subspace of $k^c\Gamma$ spanned by the x_α for $\alpha \in \Gamma^+$.

Similarly, we write the group algebra $k\Gamma$ in terms of a basis $\{y_\alpha \mid \alpha \in \Gamma\}$, with $y_\alpha y_\beta = y_{\alpha+\beta}$ for $\alpha, \beta \in \Gamma$, and we identify the semigroup algebra $R := k\Gamma^+$ with the subspace of $k\Gamma$ spanned by the y_α for $\alpha \in \Gamma^+$. We also view R as a polynomial ring $k[y_1, \dots, y_n]$, as in Section 2, where $y_i = y_{\epsilon_i}$ and ϵ_i is the i-th standard basis element of Γ.

4.4. Following (3.4), set $H = \mathrm{Hom}(\Gamma, k^\times)$ and write the application of H to Γ in terms of a pairing $\langle -, - \rangle : H \times \Gamma \to k^\times$. We again have actions of H on $k^c\Gamma$ and $k\Gamma$ via k-algebra automorphisms such that $h.x_\alpha = \langle h, \alpha \rangle x_\alpha$ and $h.y_\alpha = \langle h, \alpha \rangle y_\alpha$, for $h \in H$ and $\alpha \in \Gamma$. The subalgebras A and R are stable under these actions.

4.5. Let W denote the set of subsets of $\{1, \dots, n\}$. For $w \in W$, let $\mathrm{spec}_w A$ be the set, equipped with the relative Zariski topology, of those prime ideals P in A such that

$$P \cap \{x_1, \dots, x_n\} = \{x_i \mid i \in w\}.$$

Then $\mathrm{spec}\, A$ is the disjoint union of the sets $\mathrm{spec}_w A$. Likewise, $\mathrm{prim}\, A$ is the disjoint union of the sets $\mathrm{prim}_w A = \mathrm{prim}\, A \cap \mathrm{spec}_w A$, each of which is also endowed with the relative topology. Note that the sets $\mathrm{spec}_w A$ and $\mathrm{prim}_w A$ are invariant under the action of H. Also, it was proved in [**6**, 2.3], for $w \in W$, that the set $\mathrm{prim}_w A$ coincides with the set of maximal members of $\mathrm{spec}_w A$. Define $\mathrm{spec}_w R$ and $\max_w R$ similarly, as in (2.1) and (2.5). Recall that $\max_w R$ equals the set of maximal elements of $\mathrm{spec}_w R$.

It is convenient to label the localizations

$$A_w = \big(A/\langle x_i \mid i \in w \rangle\big)[x_j^{-1} \mid j \notin w] \qquad \text{and} \qquad R_w = \big(R/\langle y_i \mid i \in w \rangle\big)[y_j^{-1} \mid j \notin w],$$

and to identify A_w (respectively, R_w) with the k-subalgebra of $k^c\Gamma$ (respectively, $k\Gamma$) spanned by the x_α (respectively, y_α) for those $\alpha \in \Gamma$ such that $\alpha_i = 0$ for $i \in w$. In view of the preceding paragraph and [**7**, 9.20], for example, we see that localization induces H-equivariant homeomorphisms

$$\begin{aligned} \mathrm{spec}_w A &\longrightarrow \mathrm{spec}\, A_w & \qquad \mathrm{spec}_w R &\longrightarrow \mathrm{spec}\, R_w \\ \mathrm{prim}_w A &\longrightarrow \max A_w & \qquad \max_w R &\longrightarrow \max R_w. \end{aligned}$$

4.6. As in (3.5), we have an H-equivariant k-linear isomorphism $\Phi = \Phi_c : k^c\Gamma \to k\Gamma$ such that $\Phi(x_\alpha) = y_\alpha$ for $\alpha \in \Gamma$. This map restricts to an H-equivariant k-linear isomorphism from A onto R. Note that Φ also restricts to H-equivariant k-linear isomorphisms $\Phi_w : A_w \to R_w$ for $w \in W$.

4.7. Retain $\epsilon_1, \dots, \epsilon_n$ as the standard basis for Γ, and fix $w \in W$. Let Γ_w be the subgroup of Γ generated by $\{\epsilon_i \mid i \notin w\}$, and let σ_w and c_w denote the restrictions of σ and c to Γ_w. With this notation, the identifications made in (4.5) are $A_w = k^{c_w}\Gamma_w$ and $R_w = k\Gamma_w$.

4.8. Set $S_w = \mathrm{rad}(\sigma_w) = \{\alpha \in \Gamma_w \mid \sigma(\alpha, -) \equiv 1 \text{ on } \Gamma_w\}$, for $w \in W$. As in (3.7), set

$$S_w^\perp = \{h \in H \mid \langle h, \alpha \rangle = 1 \text{ for } \alpha \in S_w\} \quad \text{and} \quad \Gamma_w^\perp = \{h \in H \mid \langle h, \alpha \rangle = 1 \text{ for } \alpha \in \Gamma_w\}.$$

Because $\sigma \equiv 1$ on $S_w \times \Gamma_w$, it follows from (4.2) that $c \equiv 1$ on $S_w \times \Gamma_w$.

Lemma. *Let $v \subseteq w$ in W.*
(a) $S_w^\perp \subseteq S_v^\perp \Gamma_w^\perp$.
(b) $(P : S_v^\perp) \subseteq (P : S_w^\perp)$ *for all* $P \in \operatorname{spec}_w R$.

Proof. (a) Observe that $\Gamma_w \subseteq \Gamma_v$ and that $S_v \cap \Gamma_w \subseteq S_w$. Any $h \in S_w^\perp$ is a homomorphism $\Gamma \to k^\times$ such that $\ker h \supseteq S_w \supseteq S_v \cap \Gamma_w$. Hence, $h|_{\Gamma_w}$ extends to a homomorphism $h_1 : \Gamma_w + S_v \to k^\times$ such that $S_v \subseteq \ker h_1$. Since $k^\times$ is divisible, h_1 extends to a homomorphism $h_2 : \Gamma \to k^\times$. Now h_2 is a homomorphism in H such that $h_2 = h$ on Γ_w and $h_2 = 1$ on S_v. Thus $h_2 \in S_v^\perp$ and $h_2^{-1}h \in \Gamma_w^\perp$, whence $h \in S_v^\perp \Gamma_w^\perp$ as desired.

(b) Observe that $\Gamma_w^\perp$ acts trivially on $R/\langle y_i \mid i \in w\rangle$. Hence, P is stable under $\Gamma_w^\perp$, and so $(P : S_v^\perp) = (P : S_v^\perp \Gamma_w^\perp)$. In view of part (a), $(P : S_v^\perp \Gamma_w^\perp) \subseteq (P : S_w^\perp)$, and we are done. $\square$

4.9. Set $\mathcal{G} = \{S_w^\perp \mid w \in W\}$, and note by (4.8) that $\mathcal{G}$ satisfies the hypotheses in (2.2), as applied to R. Define $\mathcal{G}\text{-spec}\, R$ as in (2.2), and define $(P : \mathcal{G})$ to be $(P : S_w^\perp)$ for $P \in \operatorname{spec}_w R$, as in (2.3). Apply the Zariski topology to $\mathcal{G}\text{-spec}\, R$, following (2.4). We may also use (2.4) to conclude that the assignment $P \mapsto (P : \mathcal{G})$ is a topological quotient map from $\operatorname{spec} R$ onto $\mathcal{G}\text{-spec}\, R$.

Next, define $\mathcal{G}\text{-max}\, R$ as in (2.5). It follows from (2.6), for each $w \in W$, that $\mathcal{G}\text{-max}_w\, R$ is the set of maximal members of $\mathcal{G}\text{-spec}_w\, R$. From (2.5) it follows that the assignment $M \mapsto (M : \mathcal{G})$, for $M \in \max R$, produces an H-equivariant topological quotient map from $\max R$ onto $\mathcal{G}\text{-max}\, R$. Moreover, by (2.7), the fibers in $\max R$ over points in $\mathcal{G}\text{-max}_w\, R$ are precisely the $S_w^\perp$-orbits in $\max_w R$.

It remains to relate $\operatorname{spec} A$ to $\mathcal{G}\text{-spec}\, R$ and $\operatorname{prim} A$ to $\mathcal{G}\text{-max}\, R$.

4.10 Lemma. *The set function $I \mapsto \Phi(I)$, for ideals I of R, produces an H-equivariant homeomorphism from* $\operatorname{spec} A$ *onto* $\mathcal{G}\text{-spec}\, R$ *that restricts to a homeomorphism from* $\operatorname{prim} A$ *onto* $\mathcal{G}\text{-max}\, R$.

Proof. Given $w \in W$, there exists an H-equivariant commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\Phi} & R \\ {\scriptstyle\lambda_w}\big\downarrow & & \big\downarrow{\scriptstyle\mu_w} \\ A_w & \xrightarrow{\Phi_w} & R_w \end{array}$$

where λ_w and μ_w are the respective localization maps. Let $Q \in \mathcal{G}\text{-spec}_w\, R$, and write $Q = (P : S_w^\perp)$ for some $P \in \operatorname{spec}_w R$. Then $P = \mu_w^{-1}(P')$ for some $P' \in \operatorname{spec} R_w$; the ideal $Q' = (P' : S_w^\perp)$ lies in $S_w^\perp\text{-}\operatorname{spec} R_w$; and $Q = \mu_w^{-1}(Q')$. By (3.9), $\Phi_w^{-1}(Q') \in \operatorname{spec} A_w$, and so $\lambda_w^{-1}\Phi_w^{-1}(Q') \in \operatorname{spec}_w A$. Since $\lambda_w^{-1}\Phi_w^{-1}(Q') = \Phi^{-1}\mu_w^{-1}(Q') = \Phi^{-1}(Q)$, we thus see that the set $\Phi^{-1}(Q)$ is a prime ideal of A, lying in $\operatorname{spec}_w A$. Similarly, $\Phi^{-1}(M) \in \operatorname{prim}_w A$ for all $M \in \mathcal{G}\text{-max}_w\, R$.

Since Φ and Φ_w are bijections, analogous arguments show that $\Phi(Q) \in \mathcal{G}\text{-spec}_w\, R$, for all $Q \in \operatorname{spec}_w A$, and $\Phi(P) \in \mathcal{G}\text{-max}_w\, R$, for all $P \in \operatorname{prim}_w A$.

The results above, taken over all $w \in W$, show that the set maps $I \mapsto \Phi(I)$ and $J \mapsto \Phi^{-1}(J)$ yield pairs of mutually inverse bijections

$$\operatorname{spec} A \longleftrightarrow \mathcal{G}\text{-spec}\, R \qquad \text{and} \qquad \operatorname{prim} A \longleftrightarrow \mathcal{G}\text{-max}\, R.$$

These maps are H-equivariant because Φ and Φ^{-1} are H-equivariant. Finally, note that the topologies on all of the spaces considered in this section can be defined using subsets of A and R instead of ideals – for instance, the closed subsets of $\operatorname{spec} A$ are precisely those of the form $\{Q \in \operatorname{spec} A \mid Q \supseteq X\}$, for arbitrary subsets $X \subseteq A$. It therefore follows that the above maps are homeomorphisms, as desired. $\square$

Combining (4.9) and (4.10), we obtain the main theorem of the paper, which we state as follows, using the notation developed in (4.2)–(4.8).

4.11 Theorem. *Suppose that k is an algebraically closed field, that $\mathbf{q} = (q_{ij})$ is a multiplicatively antisymmetric $n \times n$ matrix over k, that $A = \mathcal{O}_{\mathbf{q}}(k^n)$, and that $R = \mathcal{O}(k^n)$. Further assume that $-1 \notin \langle q_{ij} \rangle$ or that $\operatorname{char} k = 2$. Let $H = (k^\times)^n$ act by k-algebra automorphisms on A and on R in the standard manner. Then there exist H-equivariant topological quotient maps*

$$\operatorname{spec} R \to \operatorname{spec} A \qquad \textit{and} \qquad \max R \to \operatorname{prim} A$$

given by $P \mapsto \Phi^{-1}(P : S_w^\perp)$, for $P \in \operatorname{spec}_w R$. The fibers of the second map, over points in $\operatorname{prim}_w A$, consist precisely of the $S_w^\perp$-orbits within $\max_w R$. $\square$

5. Some examples

We illustrate (4.11) by calculating the explicit form of the maps from $\max R$ to $\operatorname{prim} A$ in the cases of the standard single parameter quantum affine spaces, and we comment on some more general cases related to bilinear forms and Poisson brackets. Let k, $\mathbf{q}$, A, and R be as in (4.11), and let

$$\Psi : \max R \longrightarrow \operatorname{prim} A$$

denote the topological quotient map given by the theorem therein. Composing Ψ with the natural isomorphism of affine n-space onto $\max R$, we obtain a similar topological quotient map

$$\psi : k^n \longrightarrow \operatorname{prim} A.$$

For simplicity of notation, we describe ψ rather than Ψ.

5.1. First, let A be the standard one-parameter quantization $\mathcal{O}_q(k^n)$, for some $q \in k^\times$. Then A is generated by $x_1, \dots, x_n$ such that $x_i x_j = q x_j x_i$ for all $i < j$. The alternating bicharacter σ as in (4.1) can be expressed by

$$\sigma(\alpha, \beta) = q^{b(\alpha,\beta)},$$

where $b : \Gamma \times \Gamma \to \mathbb{Z}$ is the alternating bilinear form

$$b(\alpha, \beta) = \sum_{i<j} \alpha_i \beta_j - \sum_{i>j} \alpha_i \beta_j.$$

To complete the hypotheses of (4.11), we must assume that q is either not a root of unity or an odd root of unity. In either case, q has a square root $p \in k^\times$ such that $-1 \notin \langle p \rangle$ (if $\operatorname{char} k \neq 2$), and the rule

$$c(\alpha, \beta) = p^{b(\alpha,\beta)}$$

defines an alternating bicharacter c on Γ satisfying the conclusions of (4.2). Hence, we can identify A with $k^c\Gamma^+$ for this c.

5.2. Continue with (5.1), and suppose that q is not a root of unity. We calculate $\psi(\boldsymbol{\lambda})$ for points $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ in k^n. This depends on the position of $\boldsymbol{\lambda}$, with respect to the stratification of k^n corresponding to the subsets $\max_w R$. Set

$$(k^n)_w = \{\boldsymbol{\lambda} \in k^n \mid \lambda_i = 0 \text{ for } i \in w \text{ and } \lambda_j \neq 0 \text{ for } j \notin w\}$$

for $w \in W$.

If $n - |w|$ is even, we calculate that $S_w = 0$. In this case,

$$\psi(\boldsymbol{\lambda}) = \langle x_i \mid i \in w \rangle$$

for all $\boldsymbol{\lambda} \in (k^n)_w$.

Now suppose that $n - |w|$ is odd. List the elements of the complement of w in ascending order, say

$$\{1, \dots, n\} \setminus w = \{w_1 < w_2 < \cdots < w_{2m+1}\},$$

and then set

$$w_+ = \{w_1, w_3, \dots, w_{2m+1}\} \qquad \text{and} \qquad w_- = \{w_2, w_4, \dots, w_{2m}\}.$$

Finally, write $\gamma_\pm(w) = \sum_{i \in w_\pm} \epsilon_i \in \Gamma$, where $\epsilon_1, \dots, \epsilon_n$ is, as usual, the standard basis for Γ. We calculate that $S_w = \mathbb{Z}\big(\gamma_+(w) - \gamma_-(w)\big)$, and that

$$\psi(\boldsymbol{\lambda}) = \langle x_i \mid i \in w \rangle + \big\langle \big(\prod_{j \in w_-} \lambda_j\big) x_{\gamma_+(w)} - \big(\prod_{j \in w_+} \lambda_j\big) x_{\gamma_-(w)} \big\rangle$$

for $\boldsymbol{\lambda} \in (k^n)_w$.

If one wishes to write the last formula in terms of ordinary monomials in the x_i, additional scalar factors are introduced. For instance, suppose that $n = 3$ and $w = \varnothing$, so that $w_+ = \{1, 3\}$ and $w_- = \{2\}$. Since $x_1 x_3 = c(\epsilon_1, \epsilon_3) x_{(1,0,1)} = p x_{\gamma_+(w)}$, we have

$$\psi(\lambda_1, \lambda_2, \lambda_3) = \langle p^{-1} \lambda_2 x_1 x_3 - \lambda_1 \lambda_3 x_2 \rangle$$

for $(\lambda_1, \lambda_2, \lambda_3) \in (k^3)_w = (k^\times)^3$.

5.3. Continue with (5.1), but suppose now that q is a primitive t-th root of unity, for some odd $t > 1$. In this case, the correct choice for p is $q^{(t+1)/2}$. The form of $\psi(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in (k^n)_w$ again depends on the parity of $n - |w|$.

If $n - |w|$ is even, we calculate that $S_w = t\Gamma_w$, and that

$$\psi(\boldsymbol{\lambda}) = \langle x_i \mid i \in w \rangle + \langle x_j^t - \lambda_j^t \mid j \notin w \rangle$$

for $\boldsymbol{\lambda} \in (k^n)_w$.

If $n - |w|$ is odd, we calculate that $S_w = t\Gamma_w + \mathbb{Z}\big(\gamma_+(w) - \gamma_-(w)\big)$, and that

$$\psi(\boldsymbol{\lambda}) = \langle x_i \mid i \in w \rangle + \langle x_j^t - \lambda_j^t \mid j \notin w \rangle + \big\langle \big(\prod_{j \in w_-} \lambda_j\big) x_{\gamma_+(w)} - \big(\prod_{j \in w_+} \lambda_j\big) x_{\gamma_-(w)} \big\rangle$$

for $\boldsymbol{\lambda} \in (k^n)_w$.

5.4. Let $b : \Gamma \times \Gamma \to \mathbb{Z}$ be an arbitrary alternating bilinear form, let $q \in k^\times$ be a non-root of unity, and set $q_{ij} = q^{b(\epsilon_i, \epsilon_j)}$ for all i, j. The alternating bicharacter σ associated with A as in (4.1) then takes the form $\sigma(\alpha, \beta) = q^{b(\alpha,\beta)}$. To obtain a bicharacter c as in (4.2), take a square root p for q, and define $c(\alpha, \beta) = p^{b(\alpha,\beta)}$. Observe that since q is not a root of unity, $S_w = \operatorname{rad}(\sigma|_{\Gamma_w}) = \operatorname{rad}(b|_{\Gamma_w})$ for $w \in W$.

Now suppose that $\operatorname{char} k = 0$. There is a Poisson bracket on R such that $\{y_\alpha, y_\beta\} = b(\alpha, \beta) y_\alpha y_\beta$ for $\alpha, \beta \in \Gamma$. The analysis in [**15**, 1.4] (with $\mathbb{C}$ replaced by k) shows that the $S_w^\perp$-orbits in $(k^n)_w$ coincide with the minimal Poisson subvarieties relative to the Poisson structure just defined. Therefore the fibers of ψ in this case coincide with the minimal Poisson subvarieties of k^n.

When $k = \mathbb{C}$, these minimal Poisson subvarieties are not necessarily the same as the symplectic leaves in k^n. See [**15**, 1.4] for necessary and sufficient conditions.

5.5. The relation $\sigma = q^b$ in (5.4) is stronger than necessary for the analysis given – it would suffice to have an alternating bilinear form $b : \Gamma \times \Gamma \to k$ such that $\operatorname{rad}(\sigma|_{\Gamma_w}) = \operatorname{rad}(b|_{\Gamma_w})$ for all $w \in W$. To find further examples where such b exist, and some situations in which the fibers of a map analogous to ψ coincide with symplectic leaves, see [**15**, Section 3].

5.6. To test whether the hypothesis $-1 \notin \langle q_{ij} \rangle$ in (4.11) is necessary, the natural examples to investigate are the algebras $\mathcal{O}_{-1}(k^n)$ in characteristic different from 2. Our method works precisely when $\mathcal{O}_{-1}(k^n)$ can be expressed in the form $k^c\Gamma^+$ for some 2-cocycle c such that $c \equiv 1$ on $S_w \times \Gamma_w$ for all $w \in W$. It is easy to find such a c in case $n = 2$, and one also exists in case $n = 3$. However, it can be shown that no such c exists when $n \geq 4$. Thus, our method cannot handle $\mathcal{O}_{\mathbf{q}}(k^n)$ for arbitrary $\mathbf{q}$ and n. We leave it as an open question whether $\operatorname{prim} \mathcal{O}_{\mathbf{q}}(k^n)$ is always an H-equivariant topological quotient of $\max \mathcal{O}(k^n)$.

6. Twists of graded commutative algebras

To conclude the paper, we apply the main theorem (4.11) to a class of twists of finitely generated commutative graded algebras. Assume throughout that k is a field, that R is a k-algebra graded by an abelian group G (which we write additively), and that $c : G \times G \to k^\times$ is a 2-cocycle.

6.1. There exists a G-graded k-algebra R' equipped with a G-graded k-linear isomorphism

$$\begin{array}{ccc} R & \longrightarrow & R' \\ r & \longmapsto & r' \end{array}$$

such that, for all $\alpha, \beta \in G$, the multiplication of arbitrary homogeneous elements $r' \in R'_\alpha$ and $s' \in R'_\beta$ is given by the rule $r's' = c(\alpha, \beta)(rs)'$. Up to a G-graded k-algebra isomorphism, R' depends only on the cohomology class of c [**2**, p. 888]. We call R' the *twist of R by c* (cf. [**2**, Section 3]), and we refer to the above function $R \to R'$ as the *twist map (associated to c)*. If U is a subset of R, we will use U' to denote the image of U under the twist map.

Note, for computational purposes, that if I and J are (G-)homogeneous ideals of R then $I'J' = (IJ)'$.

6.2. Recall that a proper homogeneous ideal K, in R or R', is *graded-prime* provided it contains no product IJ of homogeneous ideals I and J not contained in K. Every ideal of R or R' contains a unique maximum homogeneous ideal, and an argument similar to (1.3) demonstrates that the maximum homogeneous ideal within a prime ideal is graded-prime.

Lemma. *Suppose that G is torsionfree and that R is noetherian. If $P_1, \dots, P_m$ are the minimal prime ideals of R, then $P_1, \dots, P_m$ are homogeneous, and $P'_1, \dots, P'_m$ are the minimal prime ideals of R'.*

Proof. Let P_i° denote the largest homogeneous ideal contained in P_i. Since G is torsionfree, the graded-prime ideal P_i° must actually be a prime ideal, by [**1**, Corollary 3.3]. Hence, $P_i = P_i^\circ$ by minimality.

Since the twist map preserves homogeneous ideals and their products, it follows that P'_i is a graded-prime ideal of R'. A second application of [**1**, Corollary 3.3] then shows that P'_i is a prime ideal of R'. Now some product of the P_i is equal to 0 in R, and by homogeneity the corresponding product of the P'_i must be 0 in R'. It follows that every minimal prime ideal of R' must occur among the P'_i.

Let $Q'_1, \dots, Q'_n$ be the minimal prime ideals of R', and denote their respective preimages under the twist map by $Q_1, \dots, Q_n$. Arguing as above, we see that the Q_j are prime ideals of R, and that all minimal prime ideals of R occur among the Q_j. The lemma follows. □

6.3 Theorem. *Let k be an algebraically closed field, let R be a commutative affine k-algebra graded by a torsionfree abelian group G, and let A be the twist of R by a 2-cocycle $c : G \times G \to k^\times$. Assume either that -1 is not in the subgroup of $k^\times$ generated by the image of c, or that* $\operatorname{char} k = 2$. *Then there exist topological quotient maps*

$$\operatorname{spec} R \to \operatorname{spec} A \qquad \textit{and} \qquad \max R \to \operatorname{prim} A.$$

Proof. Since R is affine, its support is contained in a finitely generated subgroup $G^{\mathrm{fg}} \subseteq G$. Then R is also graded by G^{fg}, and A is equal to the twist of R by $c|_{G^{\mathrm{fg}}}$. Hence, there is no loss of generality in assuming that G is finitely generated. Thus G is now a free abelian group of finite rank.

Set $\sigma(\alpha, \beta) = c(\alpha, \beta)c(\beta, \alpha)^{-1}$, for $\alpha, \beta \in G$. By [**2**, Proposition 1(ii), p. 888], σ is an alternating bicharacter on G. Since either $-1 \notin \langle \operatorname{im} \sigma \rangle$ or $\operatorname{char} k = 2$, it follows from (4.2) that there exists an alternating bicharacter d on G with the following properties: $d^2 = \sigma$, and $d(\alpha, \beta) = 1$ whenever $\sigma(\alpha, \beta) = 1$. By [**2**, Proposition 1(i), p. 888], d and c are in the same cohomology class, and hence A can also be written as the twist of R by d. Thus, there is no loss of generality in assuming that c is an alternating bicharacter and that $c(\alpha, \beta) = 1$ whenever $c(\alpha, \beta)^2 = 1$.

The inverse of the twist map is a G-graded k-linear isomorphism $\Phi : A \to R$ such that $\Phi(ab) = c(\alpha, \beta)\Phi(a)\Phi(b)$ for $a \in A_\alpha$ and $b \in A_\beta$. Choose homogeneous elements $r_1, \dots, r_n$ generating R as a k-algebra, and let $\delta_1, \dots, \delta_n \in G$ be the degrees of these elements.

Set $\Gamma = \mathbb{Z}^n$ and $\Gamma^+ = (\mathbb{Z}^+)^n$, and let $\rho : \Gamma \to G$ be the group homomorphism given by

$$\rho(\alpha_1, \dots, \alpha_n) = \alpha_1\delta_1 + \cdots + \alpha_n\delta_n.$$

Set $\widetilde{c}(\alpha,\beta) = c(\rho(\alpha),\rho(\beta))$ for $\alpha,\beta \in \Gamma$. Then $\widetilde{c}$ is an alternating bicharacter on Γ such that $\widetilde{c}(\alpha,\beta) = 1$ whenever $\widetilde{c}(\alpha,\beta)^2 = 1$. Now set $r^\alpha = r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_n^{\alpha_n} \in R_{\rho(\alpha)}$ and $a_\alpha = \Phi^{-1}(r^\alpha) \in A_{\rho(\alpha)}$ for $\alpha \in \Gamma^+$. Since R is commutative, $r^\alpha r^\beta = r^{\alpha+\beta}$ for $\alpha,\beta \in \Gamma^+$, whence

$$\Phi(a_\alpha a_\beta) = c(\rho(\alpha),\rho(\beta)) r^\alpha r^\beta = \widetilde{c}(\alpha,\beta) r^{\alpha+\beta} = \widetilde{c}(\alpha,\beta)\Phi(a_{\alpha+\beta}),$$

and so $a_\alpha a_\beta = \widetilde{c}(\alpha,\beta) a_{\alpha+\beta}$ for $\alpha,\beta \in \Gamma^+$.

Now set $\widetilde{R} = k\Gamma^+$ and $\widetilde{A} = k^{\widetilde{c}}\Gamma^+$. Write $\widetilde{R}$ in terms of a k-basis $\{y_\alpha \mid \alpha \in \Gamma^+\}$ such that $y_\alpha y_\beta = y_{\alpha+\beta}$ for $\alpha,\beta \in \Gamma^+$, and write $\widetilde{A}$ in terms of a k-basis $\{x_\alpha \mid \alpha \in \Gamma^+\}$ such that $x_\alpha x_\beta = \widetilde{c}(\alpha,\beta) x_{\alpha+\beta}$ for $\alpha,\beta \in \Gamma^+$. Let $\widetilde{\Phi} : \widetilde{A} \to \widetilde{R}$ be the k-linear isomorphism such that $\widetilde{\Phi}(x_\alpha) = y_\alpha$ for $\alpha \in \Gamma^+$. There are k-algebra epimorphisms $f : \widetilde{A} \to A$ and $g : \widetilde{R} \to R$ such that $f(x_\alpha) = a_\alpha$ and $g(y_\alpha) = r^\alpha$ for $\alpha \in \Gamma^+$. We now obtain the following commutative diagram:

$$\begin{array}{ccc} \widetilde{A} & \xrightarrow{\tilde{\Phi}} & \widetilde{R} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ A & \xrightarrow{\Phi} & R \end{array}$$

Thus if $I = \ker f$, then $\widetilde{\Phi}(I) = \ker g$.

Define $\mathcal{G}$ as in (4.9), with $\widetilde{c}$, $\widetilde{R}$, $\widetilde{A}$ playing the roles of c, R, A. By (4.11), the rule $P \mapsto \widetilde{\Phi}^{-1}(P : \mathcal{G})$ gives topological quotient maps

$$\phi_s : \operatorname{spec} \widetilde{R} \to \operatorname{spec} \widetilde{A} \qquad \text{and} \qquad \phi_m : \max \widetilde{R} \to \operatorname{prim} \widetilde{A}.$$

These maps restrict to topological quotient maps

$$\phi_s^{-1} V(I) \to V(I) \approx \operatorname{spec} A$$
$$\phi_m^{-1}\big(V(I) \cap \operatorname{prim} \widetilde{A}\big) \to V(I) \cap \operatorname{prim} \widetilde{A} \approx \operatorname{prim} A.$$

To finish the proof of the theorem, it suffices to show that $\phi_s^{-1} V(I) = V(\widetilde{\Phi}(I))$. The inclusion $\subseteq$ is easy: If $P \in \phi_s^{-1} V(I)$, then $\widetilde{\Phi}^{-1}(P : \mathcal{G}) \supseteq I$, whence $P \supseteq (P : \mathcal{G}) \supseteq \widetilde{\Phi}(I)$.

Let $P_1, \dots, P_m$ be the prime ideals of $\widetilde{A}$ minimal over I. In view of (6.2), the sets $\widetilde{\Phi}(P_1), \dots, \widetilde{\Phi}(P_m)$ are the prime ideals of $\widetilde{R}$ minimal over $\widetilde{\Phi}(I)$. By (4.10), each $\widetilde{\Phi}(P_j) \in \mathcal{G}\text{-spec}\, \widetilde{R}$. Given any $P \in V(\widetilde{\Phi}(I))$, we have $P \supseteq \widetilde{\Phi}(P_j)$ for some j. Since $\widetilde{\Phi}(P_j)$ is a $\mathcal{G}$-prime ideal, $(P : \mathcal{G}) \supseteq \widetilde{\Phi}(P_j) \supseteq \widetilde{\Phi}(I)$, and thus $\phi_s(P) \in V(I)$. Therefore $\phi_s^{-1} V(I) = V(\widetilde{\Phi}(I))$, as desired. $\square$

6.4 Remark. Retain the notation in the statement of the preceding theorem, and let $r_1, \dots, r_n$ be homogeneous generators for R. As seen in the proof, we may assume without loss of generality that $G = \mathbb{Z}^n$ and that c is an alternating bicharacter on $\mathbb{Z}^n$. In particular, it can be shown that the topological quotient maps in (6.3) are equivariant with respect to suitable actions by a subgroup of $(k^\times)^n$; details are left to the interested reader.

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