QUANTIZED COORDINATE RINGS AND RELATED NOETHERIAN ALGEBRAS

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Abstract. This paper contains a survey of some ring-theoretic aspects of quantized coordinate rings, with primary focus on the prime and primitive spectra. For these algebras, the overall structure of the prime spectrum is governed by a partition into strata determined by the action of a suitable group of automorphisms of the algebra. We discuss this stratification in detail, as well as its use in determining the primitive spectrum – under suitable conditions, the primitive ideals are precisely those prime ideals which are maximal within their strata. The discussion then turns to the global structure of the primitive spectra of quantized coordinate rings, and to the conjecture that these spectra are topological quotients of the corresponding classical affine varieties. We describe the solution to the conjecture for quantized coordinate rings of full affine spaces and (somewhat more generally) affine toric varieties. The final part of the paper is devoted to the quantized coordinate ring of $n \times n$ matrices. We mention parallels between this algebra and the classical coordinate ring, such as the primeness of quantum analogs of determinantal ideals. Finally, we describe recent work which determined, for the 3×3 case, all prime ideals invariant under the group of winding automorphisms governing the stratification mentioned above.

Introduction

First, a *caveat* concerning the title: This survey is not designed to be either an introduction to or a discussion of quantum groups. Rather, we present some of the ring theory that has arisen in studying the structure of certain algebras found among quantum groups. Here we only give a few words of background, and later we present some representative examples. An introduction to the general theory of quantum groups can be found in many books; as a small sample, we mention [**3**, **6**, **28**, **29**].

The term 'quantized coordinate rings' refers to certain algebras that, loosely speaking, are deformations of the classical coordinate rings of affine algebraic varieties or algebraic groups. These algebras are typically not commutative, but they turn out to have many other properties analogous to the classical case – for example, they are noetherian, and most of the ones that have been introduced to date are integral domains, with finite global dimension. To take the most basic case, recall that the classical coordinate ring of affine n-space over a field k is just a polynomial ring in n indeterminates over k. Thus, a 'quantized' coordinate ring of affine n-space should be some type of noncommutative polynomial ring in n indeterminates, such as an n-fold iterated skew polynomial extension of k. For the canonical examples, see Section 1.1.

This is an expository paper, based on work published elsewhere.

These notes are arranged in three parts, which focus on prime ideals, primitive ideals, and the quantized coordinate rings of matrices, respectively. Most of the material in Parts I and II is excerpted from [**3**], where the reader can find a much more detailed development. The aim of Part III is to illustrate how the general picture developed in the first two parts applies to a particularly interesting quantized coordinate ring; the discussion is taken partly from [**3**] and partly from the recent paper [**15**].

Throughout, we work over a base field k, and our parameters will be elements of $k^\times$, that is, nonzero scalars from k. The characteristic of k may be arbitrary, and for many results, it does not matter whether or not k is algebraically closed. We will often concentrate on the so-called *generic* case, meaning that our parameters are not roots of unity, but when not specified, the parameters may be arbitrary. The key difference is that when sufficiently many parameters are roots of unity, quantized coordinate rings are finitely generated modules over their centers, and their study proceeds via the theory of rings with polynomial identity. Our aim here is to concentrate on the non-PI case, which requires very different tools (some not yet invented).

I. Prime Ideals

In Part I, we concentrate on prime ideals in quantized coordinate rings and related algebras, more precisely, on ways to organize the *prime spectrum* – the set $\operatorname{spec} A$ of prime ideals in an algebra A. We view $\operatorname{spec} A$ not just as a set, but as a topological space, equipped with the standard Zariski topology.

In order to have available a few examples with which to illustrate the results and techniques, we begin by presenting some of the standard quantized coordinate rings. For a survey of most of the known types, see [**10**].

1.1. Some quantized coordinate rings. Let $q \in k^\times$. The *quantized coordinate ring of the xy-plane* with parameter q is the k-algebra

$$\mathcal{O}_q(k^2) \stackrel{\text{def}}{=} k\langle x, y \mid xy = qyx\rangle.$$

This algebra is often called a *quantum plane* for short.

Quantized coordinate rings for higher-dimensional spaces are defined similarly, except that more choices of parameters are allowed. Let $\mathbf{q} = (q_{ij})$ be a *multiplicatively antisymmetric* $n \times n$ matrix over k, meaning that $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i, j. The *quantized coordinate ring of affine n-space* with parameter matrix $\mathbf{q}$ is the k-algebra

$$\mathcal{O}_{\mathbf{q}}(k^n) \stackrel{\text{def}}{=} k\langle x_1, \dots, x_n \mid x_i x_j = q_{ij} x_j x_i \text{ for all } i, j\rangle.$$

There is a single-parameter version of this algebra, defined for $q \in k^\times$ as follows:

$$\mathcal{O}_q(k^n) \stackrel{\text{def}}{=} k\langle x_1, \dots, x_n \mid x_i x_j = q x_j x_i \text{ for all } i < j\rangle.$$

This is the special case of $\mathcal{O}_{\mathbf{q}}(k^n)$ for which the matrix $\mathbf{q}$ has the form

$$\begin{pmatrix} 1 & q & q & \cdots & q & q \\ q^{-1} & 1 & q & \cdots & q & q \\ q^{-1} & q^{-1} & 1 & \cdots & q & q \\ & \vdots & & & \vdots & \\ q^{-1} & q^{-1} & q^{-1} & \cdots & 1 & q \\ q^{-1} & q^{-1} & q^{-1} & \cdots & q^{-1} & 1 \end{pmatrix}.$$

The *quantized coordinate ring of* 2×2 *matrices* with parameter q is the k-algebra $\mathcal{O}_q(M_2(k))$ given by four generators X_{11}, X_{12}, X_{21}, X_{22} and six relations

$$\begin{aligned} X_{11}X_{12} &= qX_{12}X_{11} & X_{12}X_{22} &= qX_{22}X_{12} \\ X_{11}X_{21} &= qX_{21}X_{11} & X_{21}X_{22} &= qX_{22}X_{21} \\ X_{12}X_{21} &= X_{21}X_{12} & X_{11}X_{22} - X_{22}X_{11} &= (q-q^{-1})X_{12}X_{21}. \end{aligned}$$

The first five relations, which are all of the form $xy = ryx$ for generators x and y and scalars r, can be summarized in the following mnemonic diagram:

$$\begin{tikzcd} X_{11} \arrow[r, "q"] \arrow[d, "q"'] & X_{12} \arrow[d, "q"] \arrow[dl, leftrightarrow, "1"'] \\ X_{21} \arrow[r, "q"] & X_{22} \end{tikzcd}$$

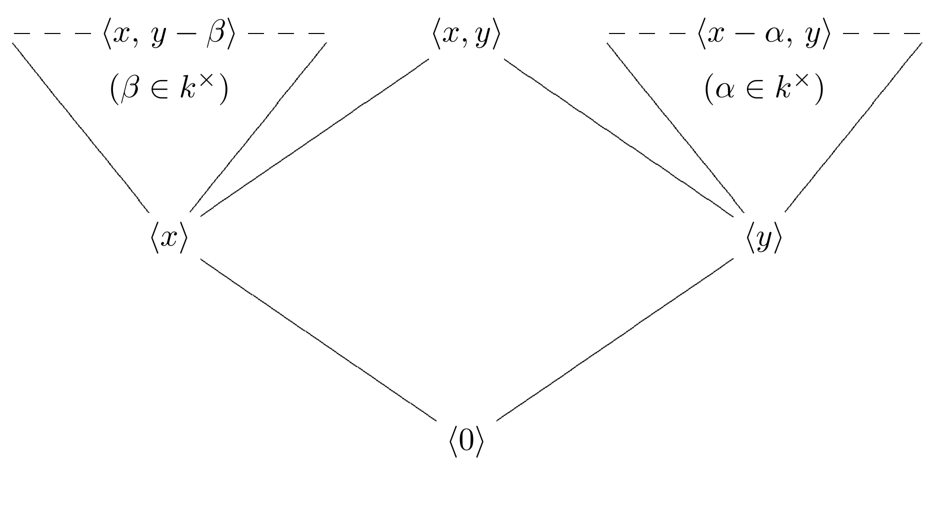
The element $D_q \overset{\text{def}}{=} X_{11}X_{22} - qX_{12}X_{21}$ in $\mathcal{O}_q(M_2(k))$ is called the (2×2) *quantum determinant*; as is easily checked, D_q lies in the center of $\mathcal{O}_q(M_2(k))$. The *quantized coordinate rings of* $GL_2(k)$ *and* $SL_2(k)$ are the algebras

$$\mathcal{O}_q(GL_2(k)) \overset{\text{def}}{=} \mathcal{O}_q(M_2(k))[D_q^{-1}] \qquad \text{and} \qquad \mathcal{O}_q(SL_2(k)) \overset{\text{def}}{=} \mathcal{O}_q(M_2(k))/\langle D_q - 1\rangle.$$

Analogous algebras $\mathcal{O}_q(M_n(k))$, $\mathcal{O}_q(SL_n(k))$, and $\mathcal{O}_q(GL_n(k))$ have been defined for arbitrary n, but we shall not give their definitions until later (Section 3.1).

A general principle from the study of quantum phenomena in physics, which holds equally well in mathematical studies of quantum algebras, is that *quantization destroys symmetry*, meaning that a quantized version of a classical system (physical or mathematical) tends to be more rigid, with less symmetry. We illustrate this principle with coordinate rings of the plane k^2. For instance, the classical coordinate ring $k[x, y]$ has a huge supply of prime ideals, but the quantized coordinate ring has far fewer, as the following example shows. The same can also be said for automorphisms, as we shall see shortly.

1.2. Example. When k is algebraically closed and q is not a root of unity, the prime spectrum of $\mathcal{O}_q(k^2)$ can be displayed as follows:



Another difference in symmetry between the classical and quantized coordinate rings of the plane is found in the automorphisms of these algebras. As an algebraic variety, the plane is completely homogeneous, in that any point can be moved to any other point by a translation. These translations induce automorphisms of $k[x, y]$ of the following form: For any scalars $a, b \in k$, there is a k-algebra automorphism of $k[x, y]$ such that $x \mapsto x+a$ and $y \mapsto y + b$. In the quantum case, however, $\mathcal{O}_q(k^2)$ has no such automorphisms except the identity (corresponding to $a = b = 0$). Fortunately, $\mathcal{O}_q(k^2)$ is not bereft of automorphisms – there are multiplicative analogs of the translation automorphisms, mapping x and y to scalar multiples of themselves. In fact, all of our standard examples have a supply of automorphisms of this type, as follows.

1.3. Some automorphisms. We define some families of k-algebra automorphisms on the quantized coordinate rings discussed above. Each family of automorphisms is parametrized by tuples of nonzero scalars, i.e., by elements from one of the multiplicative groups $(k^\times)^r$.

For $(\alpha, \beta) \in (k^\times)^2$, there is an automorphism of $\mathcal{O}_q(k^2)$ such that $x \mapsto \alpha x$ and $y \mapsto \beta y$.

For $(\alpha_1, \dots, \alpha_n) \in (k^\times)^n$, there is an automorphism of $\mathcal{O}_{\mathbf{q}}(k^n)$ such that $x_i \mapsto \alpha_i x_i$ for all i.

For $(\alpha_1, \alpha_2, \beta_1, \beta_2) \in (k^\times)^4$, there are automorphisms of $\mathcal{O}_q(M_2(k))$ and $\mathcal{O}_q(GL_2(k))$ such that $X_{ij} \mapsto \alpha_i\beta_j X_{ij}$ for all i, j. In other words,

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \longmapsto \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}.$$

The automorphisms above do not all carry over to $\mathcal{O}_q(SL_2(k))$ – we must restrict attention to those which fix the quantum determinant. For $(\alpha, \beta) \in (k^\times)^2$, there is an automorphism of $\mathcal{O}_q(SL_2(k))$ such that

$$\overline{X}_{ij} \longmapsto \alpha^{3-2i}\beta^{3-2j}\,\overline{X}_{ij}$$

for all i, j, that is,

$$\begin{pmatrix} \overline{X}_{11} & \overline{X}_{12} \\ \overline{X}_{21} & \overline{X}_{22} \end{pmatrix} \longmapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} \overline{X}_{11} & \overline{X}_{12} \\ \overline{X}_{21} & \overline{X}_{22} \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}.$$

1.4. Example. The homogeneity of the plane in the classical case carries over to its coordinate ring in the following way – if M_1 and M_2 are any maximal ideals of $k[x, y]$ of codimension 1 (these are the maximal ideals corresponding to points in the plane with coordinates in k), there is an automorphism ϕ of $k[x, y]$ such that $\phi(M_1) = M_2$. Thus, if k is algebraically closed, the maximal ideals of $k[x, y]$ form a single orbit with respect to the automorphisms of this algebra.

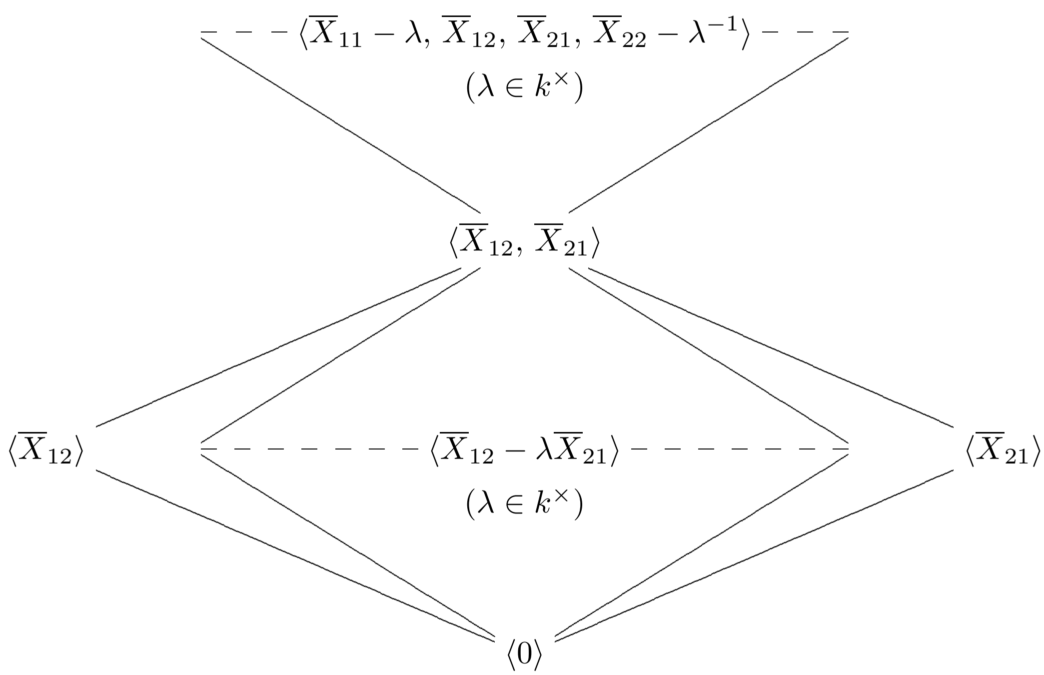
While $\mathcal{O}_q(k^2)$ does not have enough automorphisms to map any maximal ideal onto any other, there are still relatively large orbits. Assuming that k is algebraically closed and q is not a root of unity, the maximal ideals of $\mathcal{O}_q(k^2)$ can be seen in Example 1.2. Using just the automorphisms defined in (1.3), there are three orbits of maximal ideals:

$$\big\{\langle x,\, y-\beta\rangle \bigm| \beta \in k^\times\big\} \qquad \big\{\langle x,\, y\rangle\big\} \qquad \big\{\langle x-\alpha,\, y\rangle \bigm| \alpha \in k^\times\big\}.$$

Note that each of the orbits above intersects to a prime ideal which is stable under these automorphisms. The maximal ideals together with these orbit-intersections account for all but one prime ideal of $\mathcal{O}_q(k^2)$; for completeness, note that the remaining prime, namely $\langle 0 \rangle$, is also stable under the automorphisms.

A similar pattern can be observed in $\mathcal{O}_q(SL_2(k))$, as follows.

1.5. Example. When k is algebraically closed and q is not a root of unity, the prime spectrum of $\mathcal{O}_q(SL_2(k))$ can be displayed as shown below:



In this case, the maximal ideals form a single orbit under the automorphisms described in (1.3), while the prime ideals of height 1 form three orbits. The two remaining primes can be described as intersections of infinite orbits.

Patterns analogous to those discussed in Examples 1.4 and 1.5 have been found in all the other quantized coordinate rings introduced so far, assuming that k is algebraically closed and the parameters are generic. With some modifications, the picture can be expanded to include arbitrary infinite base fields. There is some interesting ring theory which explains and predicts this behavior, and our main goal in Part I is to present this theory. Some of the concepts used to describe the picture only involve an arbitrary group of automorphisms of a ring, but the key results hold when the group is an *algebraic torus*, that is, a product of copies of the multiplicative group $k^\times$. The two examples above exhibit orbits of prime ideals which intersect to stable prime ideals, which hints at the importance of such orbit intersections. This hint leads to the key idea – to group prime ideals according to the intersections of their orbits with respect to a specific group of automorphisms.

We begin with arbitrary actions of groups on rings. Whenever we refer to a group acting on a ring, we shall assume that it is acting by means of ring automorphisms (rather than just by permutations or by invertible linear transformations, for instance); similarly, actions on algebras are assumed to be actions by algebra automorphisms.

1.6. *H*-prime ideals. Let A be a ring, and let H be a group acting on A (by automorphisms). Thus, we are given a homomorphism $\phi : H \to \operatorname{Aut} A$, and we abbreviate $\phi(h)(a)$ to $h(a)$ for $h \in H$ and $a \in A$. (Many authors write $h.a$ for $\phi(h)(a)$.) For any ideal $P \lhd A$, set

$$(P : H) \overset{\text{def}}{=} \bigcap_{h \in H} h(P),$$

the largest H-stable ideal of A contained in P.

By restricting the usual definition of a prime ideal to H-stable ideals, we obtain the concept of an *H-prime* ideal of A, namely any proper H-stable ideal J of A such that $I_1 I_2 \not\subseteq J$ for all H-stable ideals $I_1, I_2 \not\subseteq J$. In parallel with the notation $\operatorname{spec} A$, we write $H\text{-spec}\, A$ to denote the set of all H-prime ideals of A. For example, if q is not a root of unity, $k = \overline{k}$, and $H = (k^\times)^2$ acts as in (1.3), then

$$H\text{-spec}\, \mathcal{O}_q(k^2) = \{ \langle x, y \rangle, \langle x \rangle, \langle y \rangle, \langle 0 \rangle \}.$$

1.7. Lemma. *Let H be a group acting on a ring A.*

(a) *If P is any prime ideal of A, then $(P : H)$ is an H-prime ideal of A.*

(b) *Now assume that A is noetherian. Then a proper ideal J of A is H-prime if and only if J equals the intersection of some finite H-orbit of prime ideals.*

In particular, it follows that all H-prime ideals of A are semiprime in this case.

Proof. (a) Easy.

(b) E.g., see [**3**, Lemma II.1.10]. $\square$

1.8. *H*-stratifications. Let H be a group acting on a ring A. For each H-prime ideal J of A, let

$$\operatorname{spec}_J A \overset{\text{def}}{=} \{P \in \operatorname{spec} A \mid (P : H) = J\}.$$

This set is called *the H-stratum of* $\operatorname{spec} A$ *corresponding to J*. In view of Lemma 1.7(a),

$$\operatorname{spec} A = \bigsqcup_{J \in H\text{-spec}\, A} \operatorname{spec}_J A,$$

a partition that we call the *H-stratification* of $\operatorname{spec} A$.

The H-stratifications just defined have similar properties to the stratifications used in algebraic geometry, as follows.

1.9. Lemma. *Let H be a group acting on a ring A.*

(a) *The closure of each H-stratum in* $\operatorname{spec} A$ *is a union of H-strata.*

(b) *If $H\text{-spec}\, A$ is finite, then each H-stratum is locally closed in* $\operatorname{spec} A$.

Proof. [**10**, Lemma 3.4]. $\square$

The stratification setup so far is extremely general, and we cannot expect to prove much about it without specializing to cases with additional hypotheses. One key specialization is to assume that H is an *affine algebraic group* over k, by which we just mean that H is isomorphic to a Zariski-closed subgroup of $GL_n(k)$ for some n. Thus, H is an affine algebraic variety as well as a group, and the group operations are morphisms of varieties. We will not need much at all of the general theory of algebraic groups, since we will concentrate on one of the simplest kind, namely algebraic tori. To see that a torus $(k^\times)^r$ is an algebraic group, note that it is isomorphic to the subgroup of $GL_{r+1}(k)$ consisting of matrices (a_{ij}) satisfying the equations $a_{ij} = 0$ for $i \neq j$ and $a_{11}a_{22}\cdots a_{r+1,r+1} = 1$.

1.10. Rational actions. Let A be a k-algebra, and H a group acting on A. (As noted above, in this situation we assume that H acts on A via k-algebra automorphisms.) Moreover, let us assume that H is an algebraic group over k.

The action of H on A is said to be *rational* provided A is a directed union of finite dimensional H-stable k-subspaces V_i such that the restriction maps $H \longrightarrow GL(V_i)$ are morphisms (of algebraic groups), i.e., group homomorphisms which are also morphisms of varieties. Fortunately for our purposes, the theory of algebraic groups provides a nice criterion that allows us to see quite easily when an action of a torus is rational, as follows.

1.11. Rational characters. Suppose that H is an algebraic torus. Recall that a *character* of H (with respect to the base field k) is any group homomorphism $H \to k^\times$. Characters appear whenever H acts on a k-algebra A: If $x \in A$ is an H-eigenvector (i.e., a simultaneous eigenvector for the actions of all the automorphisms from H), then there is a character ϕ of H such that $h(x) = \phi(h)x$ for all $h \in H$. Of course, ϕ is then called the *H-eigenvalue* of x.

A character of H is called *rational* if it is also a morphism of varieties. Let $X(H)$ denote the set of all rational characters of H; this is an abelian group under pointwise multiplication, and it is easily described. Namely, if $H = (k^\times)^r$, then $X(H)$ is a free abelian group in which the r coordinate projections $(k^\times)^r \to k^\times$ form a basis.

1.12. Theorem. *Let H be a torus acting on a k-algebra A, and assume that k is infinite. The action of H on A is rational if and only if*

(a) *The action is semisimple (i.e., A is spanned by H-eigenvectors); and*
(b) *The H-eigenvalues for the H-eigenvectors in A are <u>rational</u> characters.*

Proof. [**34**, Chapter 5, Corollary to Theorem 36]. □

From a ring-theoretic point of view, conditions (a) and (b) of Theorem 1.12 are the natural and useful conditions. Thus, we could take them as our definition of a rational action of a torus, if desired.

The next lemma illustrates one useful aspect of having a rational action. (Recall that in general, an H-prime ideal in a noetherian ring need only be semiprime.)

1.13. Lemma. *Suppose that H is a torus, acting rationally on a noetherian k-algebra A. Then every H-prime ideal of A is prime.*

Proof. If J is an H-prime ideal of A, then Lemma 1.7(b) implies that $A = (P : H)$ for some prime ideal P whose H-orbit is finite. Hence, the stabilizer subgroup $\mathrm{Stab}_H(P)$ has finite index in H. Since H acts rationally, the map $H \to \operatorname{spec} A$ given by $h \mapsto h(P)$ is continuous with respect to the Zariski topologies on H and $\operatorname{spec} A$ [**3**, Lemma II.2.8].

Consequently, the set $V = \{h \in H \mid h(P) \supseteq P\}$ is closed in H. (We cannot say immediately that $\mathrm{Stab}_H(P)$ is closed in H because $\{P\}$ need not be closed in $\operatorname{spec} A$.) However, because A is noetherian, any automorphism ϕ of A for which $\phi(P) \supseteq P$ must map P onto itself. Hence, $V = \mathrm{Stab}_H(P)$, and thus $\mathrm{Stab}_H(P)$ is indeed closed in H.

Now H is the disjoint union of the cosets of $\mathrm{Stab}_H(P)$. There are only finitely many cosets, and they are all closed. However, as a variety H is irreducible (because its coordinate ring is a Laurent polynomial ring, hence a domain), so it cannot be a finite union of proper closed subsets. Thus $\mathrm{Stab}_H(P) = P$, that is, the H-orbit of P consists of P alone. Therefore $J = P$, proving that J is prime. $\square$

We can now present a general theorem which provides a picture of the structure of the H-stratification in our current setting. Recall that a *regular* element in a noetherian ring is any non-zero-divisor. We write $\operatorname{Fract} R$ to denote the Goldie quotient ring of a semiprime noetherian ring R, and $Z(R)$ for the center of a ring R.

1.14. Stratification Theorem. [**17**, **10**] *Let A be a noetherian k-algebra, with k infinite, and let $H = (k^\times)^r$ be a torus acting rationally on A. For $J \in H\text{-}\operatorname{spec} A$, let $\mathcal{E}_J$ denote the set of all regular H-eigenvectors in A/J.*

(a) *$\mathcal{E}_J$ is a denominator set, and the localization $A_J \overset{\text{def}}{=} (A/J)[\mathcal{E}_J^{-1}]$ is an H-simple ring (with respect to the induced H-action).*

(b) $\operatorname{spec}_J A$ *is homeomorphic to* $\operatorname{spec} A_J$ *via localization and contraction, and* $\operatorname{spec} A_J$ *is homeomorphic to* $\operatorname{spec} Z(A_J)$ *via contraction and extension.*

(c) *$Z(A_J)$ is a Laurent polynomial ring of the form $K_J[z_1^{\pm 1}, \dots, z_{n(J)}^{\pm 1}]$, with $n(J) \le r$, over the fixed field $K_J \overset{\text{def}}{=} Z(A_J)^H = Z(\operatorname{Fract} A/J)^H$.*

Proof. [**3**, Chapter II.3]. $\square$

Of course, the theorem above does not say much if the H-strata are very small and there are many of them. For instance, in the extreme case the H-strata might be singletons, in which case the theorem is trivial. To get the most information out of this picture, we would like there to be only finitely many H-strata, so that the H-stratification breaks up the prime spectrum into relatively large sets. Many quantized coordinate rings are iterated skew polynomial extensions of k, and the following theorem can be applied to those algebras.

1.15. Theorem. [**17**, **3**] *Let A be an iterated skew polynomial algebra*

$$k[x_1][x_2; \tau_2, \delta_2] \cdots [x_n; \tau_n, \delta_n],$$

and let H be a group acting on A, such that $x_1, \dots, x_n$ are H-eigenvectors. Assume that there exist $h_1, \dots, h_n \in H$ such that:

(a) *$h_i(x_j) = \tau_i(x_j)$ for $i > j$; and*

(b) *The h_i-eigenvalue of x_i is not a root of unity for any i.*

Then A has at most 2^n H-prime ideals. Moreover, if H is a torus acting rationally on A, then for each $J \in H\text{-}\operatorname{spec} A$, the field K_J (from part (c) *of the Stratification Theorem) equals k.*

Proof. [**3**, Theorems II.5.12 and II.6.4]. $\square$

One of the tools involved in proving the Stratification Theorem is an equivalence between rational $(k^\times)^r$-actions and $\mathbb{Z}^r$-gradings, part of which we now sketch. Since this is intended to be applied to the H-prime factor algebras A/J, we work with an algebra called B rather than A.

1.16. Actions versus gradings. Suppose that B is a noetherian k-algebra, with k infinite, and that a torus $H = (k^\times)^r$ acts rationally on B. Because of Theorem 1.12,

$$B = \bigoplus_{g \in X(H)} B_g,$$

where B_g denotes the H-eigenspace of B with eigenvalue g. Since H acts by automorphisms, $B_g B_{g'} \subseteq B_{gg'}$ for all $g, g' \in G$, that is, B is graded by the group $X(H) \cong \mathbb{Z}^r$. (Conversely, any grading of a k-algebra by $\mathbb{Z}^r$ corresponds to a rational action of $(k^\times)^r$ on the algebra.) Problems concerning the H-action translate into problems concerning the grading in the following way:

$$\begin{array}{lcl} H\text{-eigenvectors} & \longleftrightarrow & \text{homogeneous elements} \\ H\text{-stable ideals} & \longleftrightarrow & \text{homogeneous ideals} \\ H\text{-prime ideals} & \longleftrightarrow & \text{graded-prime ideals.} \end{array}$$

To prove part (a) of the Stratification Theorem, we need to be able to localize an H-prime ring B with respect to its regular H-eigenvectors and obtain an H-simple ring. Translating to the graded case, we need to localize a graded-prime ring with respect to its homogeneous regular elements and obtain a graded-simple ring. In other words, what is required is a version of Goldie's Theorem for the setting of graded rings. This cannot be obtained in general – there are easy examples of commutative, noetherian, semiprime $\mathbb{Z}$-graded rings where the localization with respect to all homogeneous regular elements is not graded-simple. For our present purposes, it suffices to consider prime graded rings, for which the following theorem is available.

1.17. Graded Goldie Theorem. *Let G be an abelian group, and let R be a G-graded, graded-prime, right graded-Goldie ring. Let $\mathcal{E}$ be the set of all homogeneous regular elements in R. Then $\mathcal{E}$ is a right denominator set, and $R[\mathcal{E}^{-1}]$ is a graded-simple, graded-artinian ring.*

Proof. [**19**, Theorem 1]. □

Theorem 1.17 moves us to the setting of graded-simple rings, and the prime ideals in such rings can be analyzed as follows.

1.18. Proposition. *Let G be an abelian group, and let R be a G-graded, graded-simple ring.*

(a) $\operatorname{spec} R$ *is homeomorphic to* $\operatorname{spec} Z(R)$ *via contraction and extension.*

(b) *If $G \cong \mathbb{Z}^r$, then $Z(R)$ is a Laurent polynomial ring, in at most r indeterminates, over the field $Z(R)_1$ (the identity component of $Z(R)$).*

Proof. [**3**, Lemma II.3.7 and Proposition II.3.8]. □

Let us conclude Part I by presenting an open problem.

1.19. Problem. Suppose that A is a noetherian k-algebra, and that a torus $H = (k^\times)^r$ acts rationally on A. Find conditions which imply that A has only finitely many H-primes. These conditions should be

- Reasonably easy to verify; and
- Satisfied by all the standard examples.

In other words, we would like to have a theorem which we can apply to quantized coordinate rings without masses of long calculations. In seeking such a theorem, a warning is in order: When the parameters are roots of unity, quantized coordinate rings usually have infinitely many H-primes. Thus, whatever hypotheses might be used in a solution to this problem will have to correspond to the generic situation when applied to quantized coordinate rings.

II. Primitive Ideals

We now concentrate on primitive ideals as opposed to general prime ideals, and on ways to organize the *primitive spectrum* of a ring A. This set, denoted $\operatorname{prim} A$, is the set of all left primitive ideals of A. We view $\operatorname{prim} A$ as a topological space equipped with the Zariski topology, so that $\operatorname{prim} A$ is a subspace of $\operatorname{spec} A$.

The question whether the left primitive ideals and the right primitive ideals coincide in a noetherian ring remains open. To avoid this problem, we shall use the term *primitive ideal* to refer only to *left* primitive ideals.

In a classical coordinate ring over an algebraically closed field, the maximal ideals correspond to points of the underlying variety. A naive geometric analogy in the non-commutative world would be to view the maximal ideals in a ring as points of a 'non-commutative variety'. However, experience in ring theory teaches us that there are too few maximal ideals in general to hold sufficient information. Further, the influence of representation theory leads us to study the primitive ideals, as one key to irreducible representations (i.e., simple modules). Let us consider our simplest example, $\mathcal{O}_q(k^2)$.

2.1. Example. Assume that k is algebraically closed and q is not a root of unity. The prime ideals of $\mathcal{O}_q(k^2)$ are displayed in Example 1.2. Observe that the only maximal ideals are the ideals

$$\langle x-\alpha,\, y\rangle \qquad \text{and} \qquad \langle x,\, y-\beta\rangle,$$

for $\alpha, \beta \in k$. Comparing these with the maximal ideals in the classical coordinate ring $\mathcal{O}(k^2)$, we see that the maximal ideals of $\mathcal{O}_q(k^2)$ correspond only to points on the x- and y-axes of k^2. From this point of view, the remainder of the xy-plane has been 'lost'.

As suggested above, let us widen our view to include all the primitive ideals. In $\mathcal{O}_q(k^2)$, there is one non-maximal primitive ideal, namely $\langle 0\rangle$. Thus, comparing k^2 with $\operatorname{prim} \mathcal{O}_q(k^2)$, we can now say that the points on the x- and y-axes correspond precisely to the maximal ideals of $\mathcal{O}_q(k^2)$, while all other points of k^2 correspond (not bijectively, of course) to the zero ideal. We could say that the off-axis part of k^2 has 'collapsed' to a single point. Later, we shall elaborate this point of view further.

Since the primitive ideals of an algebra A are (by definition) the annihilators of the simple A-modules, it would seem that to determine these primitive ideals, we should find all the simple A-modules and then calculate their annihilators. However, to find all the simple modules over an infinite dimensional algebra is usually an impossible problem. As a substitute, Dixmier promulgated the following program for enveloping algebras of Lie algebras: Find the primitive ideals first, and then for each primitive ideal, find at least one simple module having that annihilator. In order to carry out this program, we must be able to detect the primitive ideals without knowing the simple modules in advance, and so some criterion other then the definition is required. Since all primitive ideals are prime, the question becomes, how can we tell which prime ideals are primitive? Dixmier developed two criteria, one of which is purely algebraic and one of which is phrased topologically, as follows.

2.2. Rational and locally closed primes. Let P be a prime ideal in a noetherian k-algebra A. First, we say that P is *rational* if and only if $Z(\operatorname{Fract} A/P)$ is algebraic over k. Secondly, we say that P is *locally closed* provided P is a locally closed point in $\operatorname{spec} A$, i.e., the singleton set $\{P\}$ is closed in some neighborhood of P. This condition may be rephrased as follows: P is locally closed if and only if

$$\bigcap\{Q \in \operatorname{spec} A \mid Q \supsetneqq P\} \supsetneqq P.$$

Thus, P is locally closed if and only if in the prime ring R/P, the intersection of all nonzero prime ideals is nonzero. Prime rings with the latter property are sometimes called *G-rings*, in which case locally closed primes are called *G-ideals*.

2.3. Theorem. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field of characteristic zero. Then*

$$\begin{aligned} \operatorname{prim} U(\mathfrak{g}) &= \{\text{locally closed prime ideals of } U(\mathfrak{g})\} \\ &= \{\text{rational prime ideals of } U(\mathfrak{g})\} \end{aligned}$$

Proof. This theorem was originally proved by Dixmier [**9**] and Moeglin [**32**] assuming an algebraically closed base field. Their result was extended to the non-algebraically closed case by Irving and Small [**26**]. □

2.4. The Dixmier-Moeglin equivalence. We say that an algebra A satisfies the *Dixmier-Moeglin equivalence* if the conclusion of Theorem 2.3 holds in A, that is, the primitive, locally closed, and rational prime ideals of A all coincide.

There are some relations among these three types of prime ideals which hold under fairly general hypotheses. One such hypothesis is the following adaptation of Hilbert's Nullstellensatz to noncommutative noetherian algebras.

2.5. The noncommutative Nullstellensatz. A k-algebra A is said to satisfy the *Nullstellensatz over k* if and only if

(a) The Jacobson radical of every factor ring of A is nil; and
(b) $\operatorname{End}_A(M)$ is algebraic over k for all simple A-modules M.

If A is noetherian, condition (a) is equivalent to A being a Jacobson ring, i.e., $J(A/P) = 0$ for all $P \in \operatorname{spec} A$.

The Nullstellensatz is essentially automatic if the field is large enough. In particular:

2.6. Proposition. [**1**] *If k is uncountable, then every countably generated k-algebra satisfies the Nullstellensatz over k.*

Proof. [**31**, Corollary 9.1.8]. $\square$

For algebras over countable fields, the following theorem is often useful.

2.7. Theorem. *Suppose that a k-algebra A has subalgebras $A_0 = k \subseteq A_1 \subseteq \cdots \subseteq A_t = A$ such that for all $i > 0$, either A_i is a finitely generated A_{i-1}-module on each side, or A_i is a homomorphic image of a skew polynomial ring $A_{i-1}[x_i; \tau_i, \delta_i]$. Then A satisfies the Nullstellensatz over k.*

Proof. This is a special case of [**31**, Theorem 9.4.21]. $\square$

For prime ideals in a noetherian algebra satisfying the Nullstellensatz, the following general implications are known:

$$\text{locally closed} \implies \text{primitive} \implies \text{rational}$$

[**3**, Lemma II.7.15]. Closing the loop (i.e., proving that 'rational $\implies$ locally closed') is usually the most difficult part of establishing that an algebra satisfies the Dixmier-Moeglin equivalence. In the situation of the Stratification Theorem, it is advantageous to bring the torus action into the loop – this helps in the proofs, and supplies an additional criterion for primitivity, namely the condition that a prime ideal be a maximal element of its H-stratum. Then, two implications need to be proved to close the loop, namely

$$\text{rational} \implies \text{maximal in stratum} \implies \text{locally closed},$$

but the second is quite easy. The precise theorem is as follows.

2.8. Theorem. [**17**] *Let A be a noetherian k-algebra with k infinite, and let $H = (k^\times)^r$ be a torus acting rationally on A. Assume that H-spec A is finite, and that A satisfies the Nullstellensatz over k. Then*

$$\begin{aligned} \operatorname{prim} A &= \{\text{locally closed prime ideals of } A\} \\ &= \{\text{rational prime ideals of } A\} \\ &= \bigsqcup_{J \in H\text{-spec} A} \{\text{maximal elements of } \operatorname{spec}_J A\}. \end{aligned}$$

Moreover, if k is algebraically closed, the H-orbits in prim A *coincide with the H-strata* $\operatorname{prim}_J A \overset{\text{def}}{=} (\operatorname{prim} A) \cap (\operatorname{spec}_J A)$.

Proof. [**3**, Theorems II.8.4 and II.8.14]. $\square$

Of the three criteria for primitivity given in this theorem, the third is typically easiest to apply. Here is an illustration.

2.9. Example. Let us return to our second basic example, $\mathcal{O}_q(SL_2(k))$, assuming that k is algebraically closed and q is not a root of unity. Recall from (1.3) the rational action of $H = (k^\times)^2$ on this algebra. The prime spectrum of $\mathcal{O}_q(SL_2(k))$ was displayed in Example 1.5, and the action of H on these prime ideals is easy to determine. In particular, there are only four H-prime ideals in $\mathcal{O}_q(SL_2(k))$, namely $\langle \overline{X}_{12}, \overline{X}_{21}\rangle$, $\langle \overline{X}_{12}\rangle$, $\langle \overline{X}_{21}\rangle$, and $\langle 0\rangle$. Thus, there are four H-strata in $\operatorname{spec} \mathcal{O}_q(SL_2(k))$, which we display as follows.

$$\begin{tikzcd}
 & \langle \overline{X}_{11}-\lambda,\, \overline{X}_{12},\, \overline{X}_{21},\, \overline{X}_{22}-\lambda^{-1}\rangle \\ & (\lambda\in k^\times) \arrow[d, dash] & \\
 & \langle \overline{X}_{12},\, \overline{X}_{21}\rangle & \\
\langle \overline{X}_{12}\rangle & \langle \overline{X}_{12}-\lambda\overline{X}_{21}\rangle \\ & (\lambda\in k^\times) \arrow[d, dash] & \langle \overline{X}_{21}\rangle \\
 & \langle 0\rangle &
\end{tikzcd}$$

The Nullstellensatz holds for $\mathcal{O}_q(SL_2(k))$ by Theorem 2.7. Hence, Theorem 2.8 implies that $\operatorname{prim} \mathcal{O}_q(SL_2(k))$ consists of all prime ideals except for $\langle \overline{X}_{12}, \overline{X}_{21}\rangle$ and $\langle 0\rangle$. Moreover, there are precisely four H-orbits in $\operatorname{prim} \mathcal{O}_q(SL_2(k))$:

$$\bigl\{\langle \overline{X}_{11}-\lambda,\, \overline{X}_{12},\, \overline{X}_{21},\, \overline{X}_{22}-\lambda^{-1}\rangle \bigm| \lambda\in k^\times\bigr\}$$

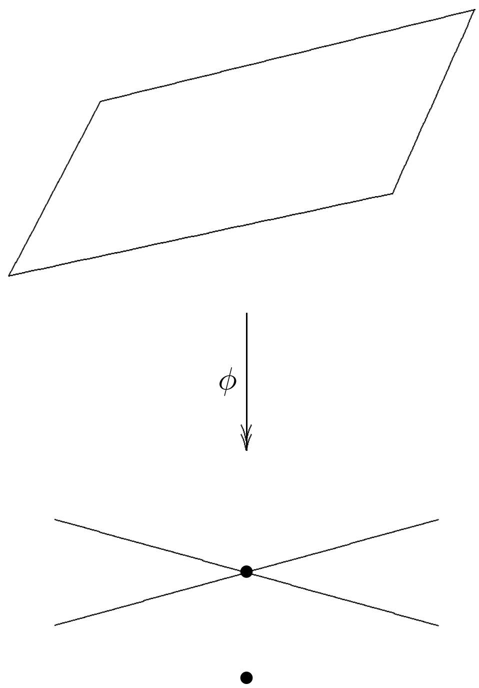
$$\bigl\{\langle \overline{X}_{12}\rangle\bigr\} \qquad \bigl\{\langle \overline{X}_{12}-\lambda\overline{X}_{21}\rangle \bigm| \lambda\in k^\times\bigr\} \qquad \bigl\{\langle \overline{X}_{21}\rangle\bigr\}$$

Now that we have access to finding the primitive ideals in quantized coordinate rings, let us turn to the global problem – trying to understand the primitive spectrum of such an algebra A as a whole. We would like $\operatorname{prim} A$ to reflect some kind of 'noncommutative geometry'. Since there is as yet no indication of what might play the role of regular functions on $\operatorname{prim} A$, we focus for now on the topological structure of this space.

For the remainder of Part II, assume that k is algebraically closed.

2.10. Problem. Let V be an affine variety over k, with classical coordinate ring $\mathcal{O}(V)$, and suppose that A is some quantized coordinate ring of V. Since $\max \mathcal{O}(V) \approx V$, we may view $\operatorname{prim} A$ as a 'quantization of V'. Then the problem arises: How are $\operatorname{prim} A$ and V related?

2.11. Example. Assume that q is not a root of unity. In Example 2.1, we suggested that $\operatorname{prim} \mathcal{O}_q(k^2)$ could be viewed as the union of the x-axis, the y-axis, and one other point obtained from collapsing the rest of the xy-plane. This leads to a map ϕ from k^2 onto $\operatorname{prim} \mathcal{O}_q(k^2)$, as in the following sketch:



To describe this map more precisely, recall that $\operatorname{prim} \mathcal{O}_q(k^2)$ consists of the maximal ideals $\langle x - \alpha,\, y\rangle$ and $\langle x,\, y - \beta\rangle$, for $\alpha, \beta \in k$, together with $\langle 0\rangle$. Thus, ϕ is given as follows:

$$\begin{aligned} (\alpha, 0) &\longmapsto \langle x - \alpha,\, y\rangle \\ (0, \beta) &\longmapsto \langle x,\, y - \beta\rangle \\ \text{other } (\alpha, \beta) &\longmapsto \langle 0\rangle\ . \end{aligned}$$

It is easy to check that ϕ is continuous. In fact, the topology on $\operatorname{prim} \mathcal{O}_q(k^2)$ equals the quotient topology induced by ϕ. Thus, $\operatorname{prim} \mathcal{O}_q(k^2)$ is a topological quotient of k^2.

2.12. Conjecture. *If an algebra A is one of the 'standard' quantized coordinate rings of an affine variety V, then* $\operatorname{prim} A$ *is a topological quotient of V.*

This conjecture is known to hold in several cases:

(1) $A = \mathcal{O}_q(SL_2(k))$, when q is not a root of unity. We invite the reader to try this as an exercise.

(2) $A = \mathcal{O}_{\mathbf{q}}((k^\times)^n) \overset{\text{def}}{=} \mathcal{O}_{\mathbf{q}}(k^n)[x_1^{-1}, \dots, x_n^{-1}]$, for arbitrary $\mathbf{q}$. This follows from work of De Concini-Kac-Procesi [**8**], Hodges [**20**], Vancliff [**36**], Brown-Goodearl [**2**], Goodearl-Letzter [**16**], and others.

(3) $A = \mathcal{O}_{\mathbf{q}}(k^n)$ and a more general type of algebra known as a 'quantum toric variety' (which we will describe below), when the subgroup of $k^\times$ generated by the entries of $\mathbf{q}$ does not contain -1. This is work of Goodearl and Letzter [**18**].

(4) $A = \mathcal{O}_q(\mathfrak{sp}\, k^4)$, the single-parameter quantized coordinate ring of symplectic 4-space, when q is not a root of unity. This algebra was first defined in [**35**]; for somewhat simpler presentations see [**33**] and [**24**]. The topological quotient here was established by Horton [**23**, Theorem 7.9].

2.13. Quantum tori. Given a multiplicatively antisymmetric $n \times n$ matrix $\mathbf{q} = (q_{ij})$ over k, the corresponding *quantized coordinate ring of* $(k^\times)^n$ is the k-algebra

$$\mathcal{O}_{\mathbf{q}}((k^\times)^n) \overset{\text{def}}{=} k\langle x_1^{\pm 1}, \dots, x_n^{\pm 1} \mid x_i x_j = q_{ij} x_j x_i \text{ for all } i, j\rangle.$$

The torus $H = (k^\times)^n$ acts rationally on the algebra $A = \mathcal{O}_{\mathbf{q}}((k^\times)^n)$ in the same way as it acts on $\mathcal{O}_{\mathbf{q}}(k^n)$. As is easily checked, $\langle 0\rangle$ is the only H-prime in A, and $\operatorname{prim} A$ is a single H-stratum as well as a single H-orbit. Hence, for any primitive ideal P, there is a bijection

$$H/\operatorname{Stab}_H(P) \longleftrightarrow \operatorname{prim} A.$$

This bijection is a homeomorphism, assuming that $H/\operatorname{Stab}_H(P)$ is given the quotient topology. Thus, the fact that $\operatorname{prim} A$ is a topological quotient of H is easily established in this case.

2.14. Quantum affine spaces. Now let $A = \mathcal{O}_{\mathbf{q}}(k^n)$, and recall that $H = (k^\times)^n$ acts rationally on A by k-algebra automorphisms such that $(\alpha_1, \dots, \alpha_n).x_i = \alpha_i x_i$ for $(\alpha_1, \dots, \alpha_n) \in H$ and $i = 1, \dots, n$. Let W be the collection of subsets of $\{1, \dots, n\}$. There is a bijection

$$\begin{aligned} W &\longrightarrow H\text{-spec}\, A \\ w &\longmapsto J_w \overset{\text{def}}{=} \langle x_i \mid i \in w\rangle. \end{aligned}$$

Thus, the H-stratifications of $\operatorname{spec} A$ and $\operatorname{prim} A$, and the localizations of A appearing in the Stratification Theorem, are indexed by the H-primes J_w. To simplify notation, we re-index using W. In particular, we write

$$\begin{aligned} \operatorname{prim}_w A &\overset{\text{def}}{=} \operatorname{prim}_{J_w} A = \{P \in \operatorname{prim} A \mid x_i \in P \iff i \in w\} \\ A_w &\overset{\text{def}}{=} A_{J_w} = (A/J_w)[x_j^{-1} \mid j \notin w] \end{aligned}$$

for $w \in W$. Note that each A_w is a quantum torus.

The torus H acts on $\mathcal{O}(k^n)$ exactly as it does on A; this action is induced from the action of H on k^n by the rule

$$(\alpha_1, \dots, \alpha_n).(a_1, \dots, a_n) \overset{\text{def}}{=} (\alpha_1^{-1} a_1, \dots, \alpha_n^{-1} a_n).$$

There are 2^n H-orbits in k^n, which we index by W as follows:

$$(k^n)_w \overset{\text{def}}{=} \{(a_1, \dots, a_n) \in k^n \mid a_i = 0 \iff i \in w\}$$

for $w \in W$. We may note that $(k^n)_w$ is isomorphic to a torus of rank $n-|w|$. The results discussed in (2.13) imply that $\operatorname{prim}_w A$ is a topological quotient of $(k^n)_w$ for each w. Thus, the problem is to patch individual topological quotient maps $(k^n)_w \twoheadrightarrow \operatorname{prim}_w A$ together, to obtain a topological quotient map $k^n \twoheadrightarrow \operatorname{prim} A$. Our solution to this problem requires a small technical condition, phrased in terms of $\langle q_{ij} \rangle$, the subgroup of $k^\times$ generated by the entries q_{ij} of $\mathbf{q}$.

2.15. Theorem. *Assume that either $-1 \notin \langle q_{ij} \rangle$ or $\operatorname{char} k = 2$. Then there exist compatible, H-equivariant topological quotient maps*

$$k^n \twoheadrightarrow \operatorname{prim} \mathcal{O}_{\mathbf{q}}(k^n) \qquad \text{and} \qquad \operatorname{spec} \mathcal{O}(k^n) \twoheadrightarrow \operatorname{spec} \mathcal{O}_{\mathbf{q}}(k^n)$$

such that for $w \in W$, the inverse image of $\operatorname{prim}_w \mathcal{O}_{\mathbf{q}}(k^n)$ is $(k^n)_w$. Moreover, the fibres over points in $\operatorname{prim}_w \mathcal{O}_{\mathbf{q}}(k^n)$ are G_w-orbits in $(k^n)_w$ for certain subgroups $G_w \subseteq H$.

Proof. [**18**, Theorem 4.11; **11**, Theorem 3.5]. These papers also describe how to calculate the subgroups G_w. $\square$

To illustrate this theorem, we use a single parameter quantum affine 3-space, this being the simplest case in which the topological quotient map differs from what one might naively write down.

2.16. Example. Choose a non-root of unity $q \in k^\times$, and let $A = \mathcal{O}_q(k^3)$. Then the entries q_{ij} in $\mathbf{q}$ consist of q, q^{-1}, and 1, and so the group $\langle q_{ij} \rangle$ is infinite cyclic. In particular, $-1 \notin \langle q_{ij} \rangle$ unless $\operatorname{char} k = 2$. Now let p be one of the square roots of q in $k^\times$. The topological quotient map $k^3 \twoheadrightarrow \operatorname{prim} A$ given by Theorem 2.15 can be described as shown below, where all $\lambda_i \in k^\times$.

$$\begin{gathered}
(0,0,0) \longmapsto \langle x_1,\, x_2,\, x_3 \rangle \\
(\lambda_1,0,0) \longmapsto \langle x_1-\lambda_1,\, x_2,\, x_3 \rangle \qquad (\lambda_1,\lambda_2,0) \longmapsto \langle x_3 \rangle \\
(0,\lambda_2,0) \longmapsto \langle x_1,\, x_2-\lambda_2,\, x_3 \rangle \qquad (\lambda_1,0,\lambda_3) \longmapsto \langle x_2 \rangle \\
(0,0,\lambda_3) \longmapsto \langle x_1,\, x_2,\, x_3-\lambda_3 \rangle \qquad (0,\lambda_2,\lambda_3) \longmapsto \langle x_1 \rangle \\
(\lambda_1,\lambda_2,\lambda_3) \longmapsto \langle \lambda_2 x_1 x_3 - p\lambda_1\lambda_3 x_2 \rangle \; .
\end{gathered}$$

Note the appearance of p in the final line – without that factor, the resulting map from k^3 to $\operatorname{prim} A$ will still be surjective, but not Zariski-continuous.

2.17. We indicate one basic mechanism from the proof of Theorem 2.15. Given a multiplicatively antisymmetric $n \times n$ matrix $\mathbf{q}$, we write parallel, coordinate-free descriptions of the algebras $R = \mathcal{O}(k^n)$ and $A = \mathcal{O}_{\mathbf{q}}(k^n)$ as follows. Namely, $R = k(\mathbb{Z}^+)^n$, a semigroup algebra, and $A = k^c(\mathbb{Z}^+)^n$, a twisted semigroup algebra for a suitable cocycle $c : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow k^\times$. There are many choices of c; we just need to have $c(\epsilon_i,\epsilon_j)c(\epsilon_j,\epsilon_i)^{-1} = q_{ij}$ for all i,j, where $\epsilon_1,\dots,\epsilon_n$ is the standard basis for $\mathbb{Z}^n$. Now both R and A have bases identified with $(\mathbb{Z}^+)^n$, and so there is a vector space isomorphism $\Phi_c : A \longrightarrow R$ which is the identity on $(\mathbb{Z}^+)^n$. Similarly, Φ_c extends to a vector space isomorphism from the group algebra $k\mathbb{Z}^n$ onto the twisted group algebra $k^c\mathbb{Z}^n$, and so for each $w \in W$ we obtain a vector space isomorphism Φ_c from A_w onto a subalgebra R_w of $k\mathbb{Z}^n$. The key to Theorem 2.15 is to choose c so that the above maps behave well:

2.18. Key Lemma. *There is a choice of cocycle c such that Φ_c yields* <u>*k-algebra*</u> *maps $Z(A_w) \longrightarrow R_w$ for all w.*

For this choice of c, the topological quotient maps $\max R \twoheadrightarrow \operatorname{prim} A$ *and* $\operatorname{spec} R \twoheadrightarrow \operatorname{spec} A$ *can be described by the rule*

$$Q \longmapsto \bigl(\text{the largest ideal of } A \text{ contained in } \Phi_c^{-1}(Q)\bigr).$$

Proof. The first statement follows from [**18**, (4.2), (4.6–4.8), (3.5)], while the second is [**11**, Lemma 3.6]. □

For a more precise description of this map in terms of operations within the commutative algebra R, see [**18**, **11**].

Since the method just sketched is based on twisting the polynomial ring $k(\mathbb{Z}^+)^n$ by a cocycle, it readily extends to a somewhat more general class of algebras twisted by cocycles.

2.19. Cocycle twists. Suppose that G is a group, and that R is a G-graded k-algebra. Let $c : G \times G \to k^\times$ be a 2-cocycle, normalized so that $c(1,1) = 1$ (or $c(0,0) = 1$, in case G is written additively). The *twist of R by c* is a k-algebra based on the same G-graded vector space as R, but with a new multiplication $*$ defined on homogeneous elements as follows: $r * s \overset{\text{def}}{=} c(\alpha,\beta)rs$ for $r \in R_\alpha$ and $s \in R_\beta$.

Now specialize to the case that R is a commutative affine G-graded k-algebra, and A is the twist of R by a 2-cocycle c. Then R is generated by finitely many homogeneous elements, say $y_1, \dots, y_n$, of degrees $\alpha_1, \dots, \alpha_n$. The algebra A is generated by the same elements $y_1, \dots, y_n$, and $y_i * y_j = q_{ij} y_j * y_i$ for all i, j, where $q_{ij} = c(\alpha_i, \alpha_j)c(\alpha_j, \alpha_i)^{-1}$. Consequently, $A \cong \mathcal{O}_{\mathbf{q}}(k^n)/I$ for $\mathbf{q} = (q_{ij})$ and some ideal I.

In particular, if $G = \mathbb{Z}^n$ and $\dim R_\alpha = 1$ for all $\alpha \in G$, then R is the coordinate ring of an affine toric variety V, and we regard A as a quantized coordinate ring of V. This case was studied by Ingalls [**25**], who introduced the term *quantum toric variety* to describe the resulting algebras A.

The constructions behind Theorem 2.15 adapt well to factor algebras $\mathcal{O}_{\mathbf{q}}(k^n)/I$, and that theorem extends to the cocyle twisted setting as follows.

2.20. Theorem. *Let G be a torsionfree abelian group, and let R be a commutative, affine, G-graded k-algebra. Let A be the twist of R by a 2-cocycle $c : G \times G \to k^\times$. Assume that $-1 \notin \langle \mathrm{image}(c) \rangle \subseteq k^\times$, or that* $\operatorname{char} k = 2$.

Then there exist compatible topological quotient maps

$$\max R \twoheadrightarrow \operatorname{prim} A \qquad \textit{and} \qquad \operatorname{spec} R \twoheadrightarrow \operatorname{spec} A,$$

which are equivariant with respect to the action of a suitable torus.

Proof. [**18**, Theorem 6.3; **11**, Theorem 4.5]. □

It is not clear whether the hypothesis concerning -1 can be removed from Theorems 2.15 and 2.20. We end Part II by putting an extreme case forward as an open problem.

2.21. Problem. Assume that $\operatorname{char} k \neq 2$. Consider the single parameter algebras

$$\mathcal{O}_{-1}(k^n) = k\langle x_1, \dots, x_n \mid x_i x_j = -x_j x_i \text{ for all } i \neq j \rangle.$$

The methods used to prove Theorem 2.15 still work for $\mathcal{O}_{-1}(k^2)$ and $\mathcal{O}_{-1}(k^3)$. These methods break down for $\mathcal{O}_{-1}(k^4)$, but extensive ad hoc calculations lead to a Zariski-continuous surjection $k^4 \twoheadrightarrow \operatorname{prim} \mathcal{O}_{-1}(k^4)$; in higher dimensions, the problem is completely open. Thus, we ask:

For $n \geq 4$, is the space $\operatorname{prim} \mathcal{O}_{-1}(k^n)$ a topological quotient of k^n?

III. Quantum Matrices

The focus on topological quotients in Part II was chosen to emphasize one way in which the quantized coordinate ring of a variety can be geometrically similar to the classical coordinate ring. We can also ask about algebraic similarities, of which there are many – chain conditions, homological conditions, etc. In fact, there exist much tighter similarities – many classical theorems have surprisingly close quantum analogs, once they are properly rephrased. We illustrate this principle by discussing quantum matrices, that is, the quantized coordinate rings of varieties of matrices. The 2×2 case was presented in (1.1); we now give the general definition.

3.1. Generators and relations. Let n be a positive integer and $q \in k^\times$. The *quantized coordinate ring of $n \times n$ matrices* with parameter q is the k-algebra with generators X_{ij} for $i, j = 1, \dots, n$ such that for all $i < l$ and $j < m$, the generators $X_{ij}, X_{im}, X_{lj}, X_{lm}$ satisfy the defining relations for $\mathcal{O}_q(M_2(k))$. As in (1.1), five of these relations can be summarized in the following mnemonic diagram:

$$\begin{tikzcd} X_{ij} \arrow[r, "q"] \arrow[d, "q"'] & X_{im} \arrow[d, "q"] \arrow[dl, leftrightarrow, "1"'] \\ X_{lj} \arrow[r, "q"] & X_{lm} \end{tikzcd}$$

The remaining relation is $X_{ij}X_{lm} - X_{lm}X_{ij} = (q - q^{-1})X_{im}X_{lj}$.

The $n \times n$ quantum determinant is modelled on the usual determinant, but with powers of -1 replaced by powers of $-q$. More precisely, the $n \times n$ *quantum determinant* is the element

$$D_q \overset{\text{def}}{=} \sum_{\pi \in S_n} (-q)^{\ell(\pi)} X_{1,\pi(1)} X_{2,\pi(2)} \cdots X_{n,\pi(n)} \quad \in \quad \mathcal{O}_q(M_n(k)),$$

where S_n denotes the symmetric group and $\ell(\pi)$, the *length* of a permutation π, is the minimum length for an expression of π as a product of simple transpositions $(i, i{+}1)$. It is known that D_q lies in the center of $\mathcal{O}_q(M_n(k))$. Hence, one defines quantized coordinate rings $\mathcal{O}_q(GL_n(k)) \overset{\text{def}}{=} \mathcal{O}_q(M_n(k))[D_q^{-1}]$ and $\mathcal{O}_q(SL_n(k)) \overset{\text{def}}{=} \mathcal{O}_q(M_n(k))/\langle D_q - 1 \rangle$ as before.

The algebra $\mathcal{O}_q(M_n(k))$ is a bialgebra with comultiplication and counit maps

$$\Delta : \mathcal{O}_q(M_n(k)) \longrightarrow \mathcal{O}_q(M_n(k)) \otimes \mathcal{O}_q(M_n(k)) \qquad \text{and} \qquad \varepsilon : \mathcal{O}_q(M_n(k)) \longrightarrow k$$

such that $\Delta(X_{ij}) = \sum_{l=1}^{n} X_{il} \otimes X_{lj}$ and $\varepsilon(X_{ij}) = \delta_{ij}$ for all i, j. In particular, $\Delta(D_q) = D_q \otimes D_q$ and $\varepsilon(D_q) = 1$.

All this structure is exactly parallel to the classical case, which we get if $q = 1$. Much interesting geometry has resulted from viewing sets of matrices of a given size as algebraic varieties and focusing on constructs from linear algebra as geometric processes. One such line leads to determinantal ideals, as follows.

3.2. Classical determinantal ideals. Let $t \leq n$ be positive integers, and consider the variety

$$V_t \overset{\text{def}}{=} \{n \times n \text{ matrices of rank } < t\},$$

the closed subvariety of the affine space $M_n(k)$ defined by the vanishing of all $t \times t$ minors. From linear algebra, V_t is the image of the matrix multiplication map

$$M_{n,t-1}(k) \times M_{t-1,n}(k) \longrightarrow M_n(k).$$

Since $M_{n,t-1}(k)$ and $M_{t-1,n}(k)$ are irreducible varieties, it follows that V_t is irreducible.

Let $I_t \lhd \mathcal{O}(M_n(k))$ be the ideal of polynomial functions vanishing on V_t, so that $\mathcal{O}(M_n(k))/I_t = \mathcal{O}(V_t)$. On the geometric side, V_t is defined by the vanishing of all $t \times t$ minors. However, this only tells us that I_t equals the radical of the ideal generated by the $t \times t$ minors. It is a classical theorem that these minors actually generate this ideal:

3.3. Theorem. *I_t equals the ideal of $\mathcal{O}(M_n(t))$ generated by all $t \times t$ minors.*

Proof. See, e.g., [**4**, **7**]. □

3.4. Corollary. *The set of all $t \times t$ minors in $\mathcal{O}(M_n(k))$ generates a prime ideal.* □

In the quantum world, there is no variety V_t, and so we cannot ask for a direct analog of Theorem 3.3. However, there are analogs of minors, which means that we can look for an analog of Corollary 3.4.

3.5. Quantum minors. Let $I, J \subseteq \{1, \dots, n\}$ be index sets with $|I| = |J| = t$. We may write the elements of these sets in ascending order, say $I = \{i_1 < \cdots < i_t\}$ and $J = \{j_1 < \cdots < j_t\}$ for short. There is a natural k-algebra embedding $\phi_{I,J} : \mathcal{O}_q(M_t(k)) \to \mathcal{O}_q(M_n(k))$ such that $\phi_{I,J}(X_{lm}) = X_{i_l j_m}$ for all l, m. The *quantum minor with index sets I and J* is the element

$$[I|J] \overset{\text{def}}{=} \phi_{I,J}(D_q^{t \times t}) \quad \in \quad \mathcal{O}_q(M_n(k)),$$

where $D_q^{t \times t}$ denotes the quantum determinant in $\mathcal{O}_q(M_t(k))$.

3.6. Theorem. *The ideal I_t of $\mathcal{O}_q(M_n(k))$ generated by all $t \times t$ quantum minors is completely prime, i.e., $\mathcal{O}_q(M_n(k))/I_t$ is an integral domain.*

Proof. [**13**, Theorem 2.5]. □

Although many steps in the proof of the classical result have no analogs in the quantum case, one part of the classical pattern does carry over, as we now summarize.

3.7. As noted above, V_t is the image of the multiplication map

$$\mu : M_{n,t-1}(k) \times M_{t-1,n}(k) \longrightarrow M_n(k).$$

Hence, the ideal I_t is the kernel of the comorphism

$$\mu^* : \mathcal{O}(M_n) \longrightarrow \mathcal{O}(M_{n,t-1} \times M_{t-1,n}).$$

We may identify $\mathcal{O}(M_{n,t-1} \times M_{t-1,n})$ with $\mathcal{O}(M_{n,t-1}) \otimes \mathcal{O}(M_{t-1,n})$, which allows us to describe μ^* as the composition of the maps

$$\mathcal{O}(M_n) \xrightarrow{\Delta} \mathcal{O}(M_n) \otimes \mathcal{O}(M_n) \xrightarrow{\text{quo}\otimes\text{quo}} \mathcal{O}(M_{n,t-1}) \otimes \mathcal{O}(M_{t-1,n}).$$

3.8. A quantum analog. Quantized coordinate rings for the rectangular matrix varieties $M_{n,t-1}(k)$ and $M_{t-1,n}(k)$ may be defined as the subalgebras of $\mathcal{O}_q(M_n(k))$ generated by those X_{ij} with $j < t$ (respectively, $i < t$). There are natural k-algebra retractions of $\mathcal{O}_q(M_n(k))$ onto these subalgebras, and so

$$\mathcal{O}_q(M_{n,t-1}(k)) \cong \mathcal{O}_q(M_n(k))/\langle X_{ij} \mid j \geq t\rangle$$
$$\mathcal{O}_q(M_{t-1,n}(k)) \cong \mathcal{O}_q(M_n(k))/\langle X_{ij} \mid i \geq t\rangle.$$

Thus, the quantum analog of the comorphism μ^* in (3.7) is the k-algebra map

$$\mu_q^* \overset{\text{def}}{=} \mathcal{O}_q(M_n) \xrightarrow{\Delta} \mathcal{O}_q(M_n) \otimes \mathcal{O}_q(M_n) \xrightarrow{\text{quo}\otimes\text{quo}} \mathcal{O}_q(M_{n,t-1}) \otimes \mathcal{O}_q(M_{t-1,n}).$$

It is easy to check that $\mathcal{O}_q(M_{n,t-1}(k)) \otimes \mathcal{O}_q(M_{t-1,n}(k))$ is an iterated skew polynomial algebra over k, and therefore a domain. Thus, to prove that the ideal I_t of $\mathcal{O}_q(M_n(k))$ is completely prime, one just has to show that $I_t = \ker(\mu_q^*)$. This is the heart of the proof of Theorem 3.6.

To understand quantum analogs of other geometric aspects of matrices, and also to understand the quantum matrix algebra better, we would like to know its prime and primitive ideals. We approach this problem via the Stratification Theorem, as discussed in Parts I and II.

For the remainder of Part III, assume that q is not a root of unity, and set $A = \mathcal{O}_q(M_n(k))$.

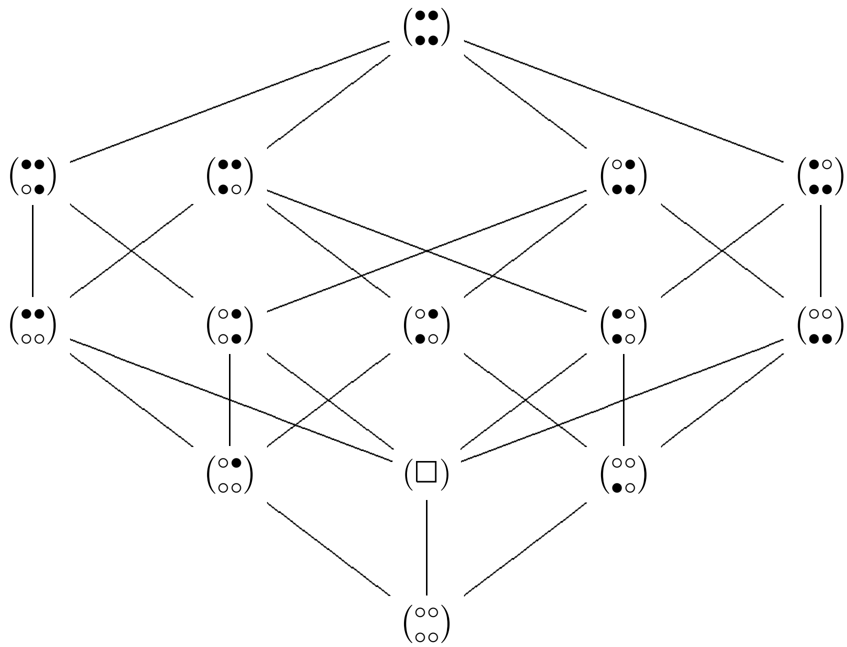
3.9. Problem. In parallel with the 2×2 case discussed in (1.3), the torus $H = (k^\times)^{2n}$ acts on A by k-algebra automorphisms so that

$$(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n).X_{ij} = \alpha_i \beta_j X_{ij}$$

for all i, j. These automorphisms are called 'winding automorphisms', because they arise from the bialgebra structure on A in a manner analogous to the classical winding automorphisms on enveloping algebras of Lie algebras. According to Theorem 1.15, there are at most 2^{n^2} H-prime ideals in A. The basic problem is:

Determine the H-primes of A.

3.10. Example. The 2×2 case of Problem 3.9 is easily solved – there are exactly 14 H-prime ideals in $\mathcal{O}_q(M_2(k))$, as displayed in the following diagram. Each 2×2 pattern here is shorthand for a set of generators of an ideal – a bullet $\bullet$ in position (i, j) corresponds to a generator X_{ij}; a circle $\circ$ in a given position is a placeholder; and the square $\square$ denotes the 2×2 quantum determinant.



3.11. Certain types of H-primes in A are already known. For convenient labelling, we carry over the term 'rank' from ordinary matrices to the quantum case, as follows. We define the *rank* of a prime ideal P in A to be the minimum r such that P contains all $(r+1) \times (r+1)$ quantum minors.

The H-primes of A of rank n are those which do not contain the quantum determinant D_q. By localization, these correspond to the H-primes of $\mathcal{O}_q(GL_n(k))$, and it is known that those, in turn, correspond to the H-primes of $\mathcal{O}_q(SL_n(k))$. The latter can be determined using results of Hodges and Levasseur [**21**, **22**]. In particular, $\mathcal{O}_q(SL_n(k))$ has $(n!)^2$ H-primes, parametrized by $S_n \times S_n$, and it follows from work of Joseph [**27**, Théorème 3] that each of these H-primes is generated by a set of quantum minors. We conclude that back in A, the H-primes of rank n are generated – up to localization at the powers of D_q – by sets of quantum minors.

At the other extreme, the H-primes of rank at most 1 are the H-primes of A which contain all 2×2 quantum minors. These were determined by Goodearl and Lenagan [**12**, Proposition 3.4]. There are $(2^n - 1)^2 + 1$ such H-primes, all having the form

$$\big\langle [I|J] \bigm| |I| = |J| = 2 \big\rangle + \big\langle X_{ij} \mid i \in R \big\rangle + \big\langle X_{ij} \mid j \in C \big\rangle$$

for $R, C \subseteq \{1, \ldots, n\}$. For these H-primes, we have generating sets consisting of quantum minors, since each X_{ij} is a 1×1 quantum minor.

3.12. Conjecture. *Every H-prime of A is generated by a set of quantum minors.*

It is easily seen that the conjecture holds when $n = 2$, in view of (3.10). Cauchon proved that there are enough quantum minors to separate the H-primes in A: For any H-primes $P \subsetneq Q$, there is a quantum minor in $Q \backslash P$ [**5**, Proposition 6.2.2 and Théorème 6.3.1]. The 3×3 case of the conjecture has been established by Goodearl and Lenagan [**15**, Theorem 7.4], and the $n \times n$ case, assuming that $k = \mathbb{C}$ and q is transcendental over $\mathbb{Q}$, has been proved by Launois [**30**, Théorème 3.7.2]. We shall display the solution to the 3×3 case below.

Cauchon's and Launois's results are existence theorems – they do not provide descriptions of which sets of quantum minors generate H-primes. Such descriptions are needed not only for completeness, but also to get full benefit from the stratification, e.g., to determine the prime and primitive ideals in each H-stratum via Theorems 1.14 and 2.8. Thus, we accompany Conjecture 3.12 with the following problems.

3.13. Problems. If $J \in H\text{-spec}\, A$, then Theorems 1.14 and 1.15 tell us that the center of the localization $A_J = (A/J)[\mathcal{E}_J^{-1}]$ is a Laurent polynomial ring $k[z_1^{\pm 1}, \dots, z_{n(J)}^{\pm 1}]$.

(a) Assuming J is generated by a set $\mathcal{M}$ of quantum minors, find a formula for $n(J)$ in terms of $\mathcal{M}$.

(b) Find explicit descriptions of the indeterminates $z_i \in A_J$.

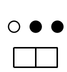
We now summarize the solution to the 3×3 case of Conjecture 3.12 given in [**15**].

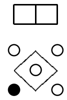
3.14. As in (3.11), we may divide up the H-primes in $\mathcal{O}_q(M_3(k))$ according to their ranks. Those of ranks 0, 1, and 3 were known earlier, while the ones of rank 2 were first determined in [**15**]. The numerical count is as follows:

rank 0	:	1
rank 1	:	49
rank 2	:	144
rank 3	:	36
total	:	230.

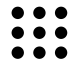
The determination of these H-primes was done partly by *ad hoc* methods, which are unlikely to work in the general case. In particular, Cauchon has given a formula for the total number of H-primes in $\mathcal{O}_q(M_n(k))$ [**5**, Théorème 3.2.2 and Proposition 3.3.2], which shows that $\mathcal{O}_q(M_4(k))$ has 6902 H-primes!

3.15. The H-primes in $\mathcal{O}_q(M_3(k))$ can be displayed as in the following diagrams, where each 3×3 pattern represents a set of generators for an H-prime. As in (3.10), circles are placeholders and bullets represent generators X_{ij}. This time, squares and rectangles represent 2×2 quantum minors whose row and column index sets correspond to the edges. Finally, the diamond that appears in four patterns represents the 3×3 quantum determinant. Below are samples showing the ideals corresponding to two patterns.

 $\longleftrightarrow \quad \langle\, X_{12},\ X_{13},\ [23|12],\ [23|13],\ [23|23]\,\rangle$

 $\longleftrightarrow \quad \langle\, [123|123],\ X_{31}\,\rangle.$

The case of rank 0 is trivial – there is only one H-prime of this rank, corresponding to the following pattern:



The 49 H-primes of rank 1 correspond to the patterns in Figure A below.

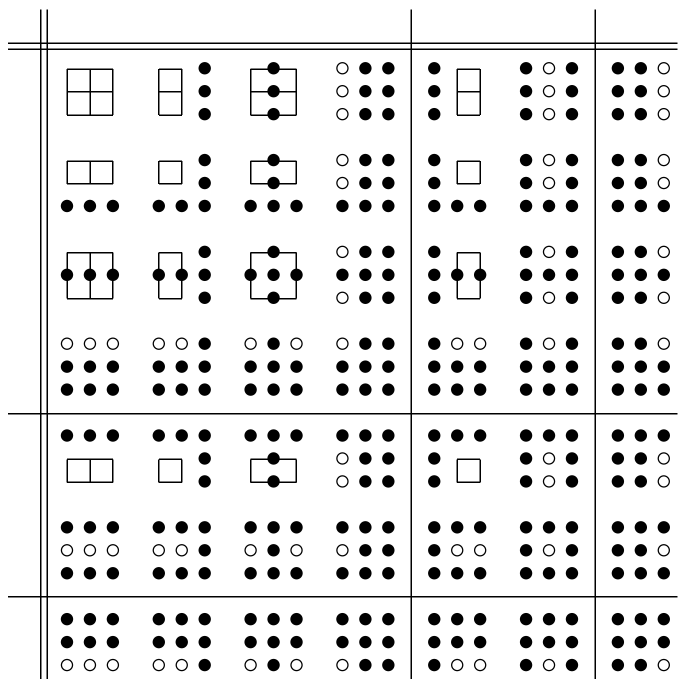


Figure A

As indicated in (3.11), the H-primes of maximal rank were known, up to localization, from the results of [**21**, **22**, **27**]. The sets of quantum minors which generate the corresponding H-primes in $\mathcal{O}_q(SL_3(k))$ also, as it turns out, generate the H-primes of rank 3 in $\mathcal{O}_q(M_3(k))$. These 36 ideals correspond to the patterns in Figure B.

The 144 patterns for generating sets of the H-primes of rank 2 in $\mathcal{O}_q(M_3(k))$ are given in our final display, Figure C. The procedure used in [**15**] to determine these H-primes involved three steps. First, some general theory developed in [**14**] provided a reduction mechanism relating the H-primes in $\mathcal{O}_q(M_n(k))$ to pairs of H-primes from smaller quantum matrix algebras. Consequently, we could find the H-primes in $\mathcal{O}_q(M_3(k))$ from the (known) H-primes in $\mathcal{O}_q(M_2(k))$, but only as kernels of certain algebra homomorphisms. This process also gave precise counts for the number of H-primes of each rank. In the second step, the information from Step 1 was used to determine the quantum minors contained in each H-prime, thus yielding at least potential sets of generators. Finally, the third step consisted of proving that each set of quantum minors appearing in Step 2 does generate an H-prime, and that the resulting H-primes are distinct. Since that yielded a list of the correct number of H-primes, we were done.

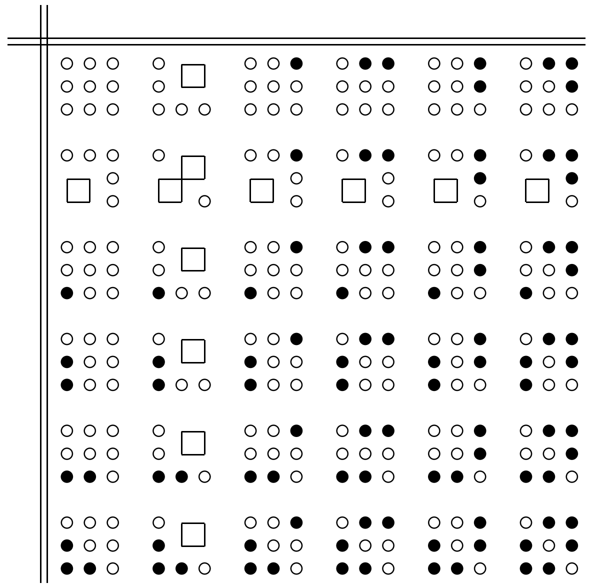


Figure B

It may be useful to break the general problem down into steps of similar type, as follows.

3.16. Problems. We return to the general $n \times n$ situation, keeping the field k arbitrary but still requiring q to be a non-root of unity.

(a) Which sets $\mathcal{M}$ of quantum minors in $\mathcal{O}_q(M_n(k))$ generate prime ideals?

(b) Develop general theorems to prove that suitable ideals of the form $\langle \mathcal{M} \rangle$ are prime.

(c) Find combinatorial data to parametrize the sets $\mathcal{M}$ in (a).

We conclude by stating a result which illustrates one pattern which solutions to the above problems might take. Part (a) is an easy exercise involving the relations among the X_{ij}, part (b) can be proved by showing that $\mathcal{O}_q(M_n(k))/\langle \mathcal{X} \rangle$ is an iterated skew polynomial extension of k, and part (c) is another easy exercise.

3.17. Sample result. (a) *If P is an H-prime ideal of $\mathcal{O}_q(M_n(k))$, then the set $\mathcal{X} = P \cap \{X_{ij} \mid i, j = 1, \dots, n\}$ satisfies the following condition:*

(*) *If $X_{ij} \in \mathcal{X}$, then either $X_{lm} \in \mathcal{X}$ for all $l \geq i$ and $m \leq j$, or else $X_{lm} \in \mathcal{X}$ for all $l \leq i$ and $m \geq j$.*

(b) *If $\mathcal{X}$ is any subset of $\{X_{ij} \mid i, j = 1, \dots, n\}$ which satisfies (*), then $\mathcal{X}$ generates an H-prime ideal of $\mathcal{O}_q(M_n(k))$, and $\langle \mathcal{X} \rangle \cap \{X_{ij} \mid i, j = 1, \dots, n\} = \mathcal{X}$.*

(c) *Given subsets $I, J \subseteq \{1, \dots, n\}$ and nondecreasing functions $f : \{1, \dots, n\} \setminus J \to \{2, \dots, n+1\} \setminus I$ and $g : \{1, \dots, n\} \setminus I \to \{2, \dots, n+1\} \setminus J$, the set*

$$\begin{aligned} \mathcal{X}(I, J, f, g) \stackrel{\text{def}}{=} &\big\{X_{ij} \bigm| i \in I\big\} \cup \big\{X_{ij} \bigm| i \notin I;\ j \notin J;\ i \geq f(j)\big\} \\ &\cup \big\{X_{ij} \bigm| j \in J\big\} \cup \big\{X_{ij} \bigm| i \notin I;\ j \notin J;\ j \geq g(i)\big\} \end{aligned}$$

satisfies (*). Conversely, any subset $\mathcal{X} \subseteq \{X_{ij} \mid i, j = 1, \dots, n\}$ which satisfies (*) equals $\mathcal{X}(I, J, f, g)$ for some I, J, f, g. $\square$

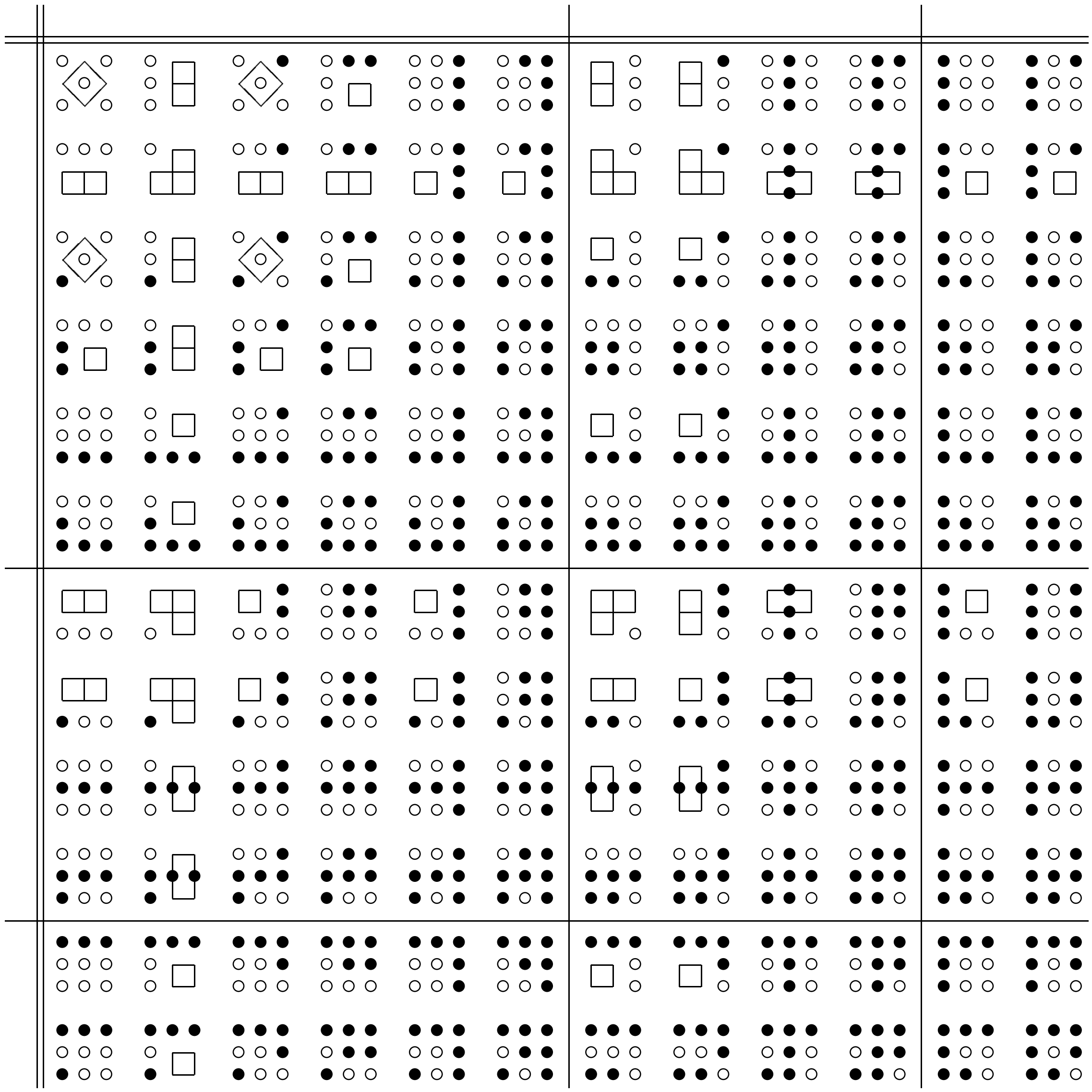


Figure C

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