QUANTUM DETERMINANTAL IDEALS

K. R. Goodearl and T. H. Lenagan

Abstract. Quantum determinantal ideals in coordinate algebras of quantum matrices are investigated. The ideal $\mathcal{I}_t$ generated by all $(t+1)\times(t+1)$ quantum minors in $\mathcal{O}_q(M_{m,n}(k))$ is shown to be a completely prime ideal, that is, $\mathcal{O}_q(M_{m,n}(k))/\mathcal{I}_t$ is an integral domain. The corresponding result is then obtained for the multiparameter quantum matrix algebra $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$. The main idea involved in the proof is the construction of a *preferred basis* for $\mathcal{O}_q(M_n(k))$ in terms of certain products of quantum minors, together with rewriting rules for expressing elements of this algebra in terms of the preferred basis.

Introduction

Fix a base field k. The quantized coordinate ring of $n\times n$ matrices over k, denoted $\mathcal{O}_q(M_n(k))$, is a deformation of the classical coordinate ring of $n\times n$ matrices, $\mathcal{O}(M_n(k))$. As such it is a k-algebra generated by n^2 indeterminates X_{ij}, for $1\le i,j\le n$, subject to relations which we recall in (1.1). Here q is a nonzero element of the field k. When $q=1$, we recover $\mathcal{O}(M_n(k))$, which is the commutative polynomial algebra $k[X_{ij}]$. The algebra $\mathcal{O}_q(M_n(k))$ has a distinguished element D_q, the *quantum determinant*, which is a central element. Two important algebras $\mathcal{O}_q(GL_n(k))$ and $\mathcal{O}_q(SL_n(k))$ are formed by inverting D_q and setting $D_q=1$, respectively.

The structures of the primitive and prime ideal spectra of the algebras $\mathcal{O}_q(GL_n(k))$ and $\mathcal{O}_q(SL_n(k))$ have been investigated recently; see, for example, [**2**], [**6**] and [**9**]. Results obtained in these investigations can be pulled back to partial results about the primitive and prime ideal spectra of $\mathcal{O}_q(M_n(k))$. However, these techniques give no information about the closed subset of the spectrum determined by D_q. In this paper, we begin the study of this portion of the spectrum.

In the classical commutative setting, much attention has been paid to *determinantal ideals*, that is, the ideals generated by the minors of a given size. In particular, these are special prime ideals of $\mathcal{O}(M_n(k))$ containing the determinant. Moreover, there are interesting geometrical and invariant theoretical reasons for the importance of these ideals; see, for example, [**4**]. In order to put our results into context, it may be useful to review some highlights of the commutative theory.

Let $M_{l,m}(k)$ denote the algebraic variety of $l\times m$ matrices over k. For $t\le n$, the general linear group $GL_t(k)$ acts on $M_{n,t}(k)\times M_{t,n}(k)$ via

$$g\cdot(A,B):=(Ag^{-1},gB).$$

1991 *Mathematics Subject Classification.* 16A40, 16W30, 16D30, 16S15, 13C40, 17B37.

This research was partially supported by National Science Foundation research grant DMS-9622876 and NATO Collaborative Research Grant 960250.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Matrix multiplication yields a map

$$\mu : M_{n,t}(k) \times M_{t,n}(k) \to M_n(k),$$

the image of which is the set of matrices with rank at most t.

There is an induced map

$$\mu_* : \mathcal{O}(M_n(k)) \to \mathcal{O}(M_{n,t}(k) \times M_{t,n}(k)) = \mathcal{O}(M_{n,t}(k)) \otimes \mathcal{O}(M_{t,n}(k)).$$

The First Fundamental Theorem of invariant theory identifies the fixed ring of the coordinate ring $\mathcal{O}(M_{n,t}(k) \times M_{t,n}(k))$ under the induced action of $GL_t(k)$ as precisely the image of μ_*. The Second Fundamental Theorem states that the kernel of μ_* is $\mathcal{I}_t$, the ideal generated by the $(t+1) \times (t+1)$ minors of $\mathcal{O}(M_n(k))$, so that the coordinate ring of the variety of $n \times n$ matrices of rank at most t is $\mathcal{O}(M_n(k))/\mathcal{I}_t$. As a consequence, since this variety is irreducible, the ideal $\mathcal{I}_t$ is a prime ideal of $\mathcal{O}(M_n(k))$.

Our main result, Theorem 2.5, is a quantum analog of the Second Fundamental Theorem. (We conjecture that there is also a quantum analog of the First Fundamental Theorem, but do not address that problem in the present paper.) If I, J are subsets of $\{1, \dots, n\}$ with $|I| = |J|$, then $D(I,J)$ denotes the quantum minor obtained by evaluating the quantum determinant of the subalgebra of $\mathcal{O}_q(M_n(k))$ generated by those X_{ij} with $i \in I$ and $j \in J$. The ideal $\mathcal{I}_t$ is then the ideal generated by the $(t+1) \times (t+1)$ quantum minors of $\mathcal{O}_q(M_n(k))$. Theorem 2.5 states that $\mathcal{O}_q(M_n(k))/\mathcal{I}_t$ is an integral domain, for $0 \leq t \leq n-1$. The case $(t = n-1)$ of this result has been proved by Jordan [**8**] and Levasseur-Stafford [**10**, p. 182], and the case $(t = 1)$ has recently been obtained by Rigal [**14**]. The case $(t = 0)$ holds trivially.

The classical commutative appproach to the Second Fundamental Theorem is as follows. By geometrical considerations, the variety of $n \times n$ matrices of rank at most t is an irreducible variety, and it is easy to see that the coordinate ring of this variety is $\mathcal{O}(M_n(k))/\sqrt{\mathcal{I}_t}$. Thus the main problem is to show that $\mathcal{I}_t$ is a radical ideal of $\mathcal{O}(M_n(k))$. This is achieved via the notion of *algebras with straightening laws*; see [**3**] or [**4**]. In order to simplify the problem, the algebra $\mathcal{O}(M_n(k))$ is replaced temporarily by $\mathcal{O}(M_{n,2n}(k))$. The subalgebra B of $\mathcal{O}(M_{n,2n}(k))$ generated by the maximal minors of $\mathcal{O}(M_{n,2n}(k))$ is the coordinate ring of the Grassmanian of the n-dimensional subspaces of k^{2n}. The products of maximal minors span B, but do not form a basis – the famous *Plücker relations* generate the relations between the maximal minors. The Plücker relations are used to produce straightening laws leading to a *standard basis* of B. All this is now specialised by setting the rightmost $n \times n$ block of X_{ij}'s equal to the identity matrix. The images of the maximal minors become *all* of the minors of $\mathcal{O}(M_n(k))$, and the standard basis of B induces a standard basis of $\mathcal{O}(M_n(k))$ consisting of certain products of minors of $\mathcal{O}(M_n(k))$. This establishes that $\mathcal{O}(M_n(k))$ is an algebra with a straightening law. The conclusion that $\mathcal{I}_t$ is radical then follows easily.

The classical approach breaks down completely in the quantum setting. There is no group acting, and setting noncentral elements equal to 0 or 1 produces a homomorphic image that is far too small. The action of the group $GL_t(k)$ can be replaced by a coaction of the Hopf algebra $\mathcal{O}_q(GL_t(k))$. Otherwise, the only thing that survives is the idea of a basis constructed of products of (quantum) minors and straightening laws. However, an added complication appears: as well as straightening laws to deal with linear dependencies, it is also necessary to generate commutation laws to deal with re-ordering products. This latter problem leads us to choose a different ordering of variables than that chosen in the classical case, so that we can give preference to the best approximation to central elements – the *normal* elements that occur in profusion in the quantum case. As a result, we call our basis a *preferred basis* rather than a standard basis.

The second main ingredient in the proof is the exploitation of the fact that $\mathcal{O}_q(M_n(k))$ is a bialgebra. Quantum minors behave well under the comultiplication map Δ, and using this fact we produce an embedding of $\mathcal{O}_q(M_n(k))/\mathcal{I}_t$ into the algebra $\mathcal{O}_q(M_{n,t}(k)) \otimes \mathcal{O}_q(M_{t,n}(k))$. This latter algebra is an iterated Ore extension of k and is thus a domain, establishing our theorem.

In the latter part of the paper, we show, using the twisting methods of Artin-Schelter-Tate [**1**], that our results also hold for multiparameter coordinate rings of quantum matrices.

I. A basis for quantum matrices

This section is devoted to establishing the existence of a basis for $\mathcal{O}_q(M_n(k))$ which is built from products of quantum minors. This basis is crucial to our calculations with quantum determinantal ideals. A basis of this type was constructed in [**7**] for a class of quantum matrix superalgebras which includes the $\mathcal{O}_q(M_n(k))$ for q not a root of unity. Our modification of their construction allows q to be an arbitrary nonzero scalar. For convenience of notation and in applying results from the literature, we work mainly with the quantum coordinate rings of square matrices. At the end of the section, we shall see that our basis theorem readily carries over to the case of $\mathcal{O}_q(M_{m,n}(k))$.

The calculations involved in constructing and verifying our basis rely on several general identities concerning products of quantum minors. Although some of these identities are of standard types, they are not available in the literature in precisely the forms we require, and so we derive them from known forms. In order not to disrupt the line of this section, we relegate the discussions of the identities to appendices in sections V and VI.

1.1. Throughout this section, we fix an integer $n \geq 2$, a base field k, and a nonzero scalar $q \in k^\times$. No other restrictions are assumed; in particular, k need not be algebraically closed, and q is allowed to be a root of unity. We work with the one-parameter quantized coordinate ring of $n \times n$ matrices over k, namely the algebra $\mathcal{A} = \mathcal{O}_q(M_n(k))$ with generators X_{ij} for $i, j = 1, \dots, n$ and relations

$$\begin{aligned}
X_{ij}X_{lj} &= qX_{lj}X_{ij} && \text{when } i < l;\\
X_{ij}X_{im} &= qX_{im}X_{ij} && \text{when } j < m;\\
X_{im}X_{lj} &= X_{lj}X_{im} && \text{when } i < l \text{ and } j < m;\\
X_{ij}X_{lm} - X_{lm}X_{ij} &= (q - q^{-1})X_{im}X_{lj} && \text{when } i < l \text{ and } j < m.
\end{aligned}$$

As is well known, this algebra is in fact a bialgebra, with comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and counit $\epsilon : \mathcal{A} \to k$ such that

$$\Delta(X_{ij}) = \sum_{l=1}^{n} X_{il} \otimes X_{lj} \qquad \text{and} \qquad \epsilon(X_{ij}) = \delta_{ij}$$

for all i, j.

1.2. We shall need several partial order relations on index sets. Let $A, B \subseteq \{1, \dots, n\}$, not necessarily of the same cardinality. First, we define a "row ordering", denoted $\leq_r$. To describe this, write A and B in descending order:

$$A = \{a_1 > a_2 > \cdots > a_\alpha\} \qquad \text{and} \qquad B = \{b_1 > b_2 > \cdots > b_\beta\}.$$

Define $A \leq_r B$ to mean that $\alpha \geq \beta$ and $a_i \geq b_i$ for $i = 1, \dots, \beta$. For the "column ordering", $\leq_c$, write A and B in ascending order:

$$A = \{a_1 < a_2 < \cdots < a_\alpha\} \qquad \text{and} \qquad B = \{b_1 < b_2 < \cdots < b_\beta\}.$$

Define $A \leq_c B$ to mean that $\alpha \geq \beta$ and $a_i \leq b_i$ for $i = 1, \dots, \beta$.

By an *index pair* we will mean a pair (I, J) where $I, J \subseteq \{1, \dots, n\}$ and $|I| = |J|$. Order index pairs by $(\leq_r, \leq_c)$, that is, define $(I, J) \leq (I', J')$ if and only if $I \leq_r I'$ and $J \leq_c J'$. For example, when $n = 3$ the poset of index pairs can be drawn as in Diagram 1.2, where we have abbreviated the descriptions of index sets by eliminating braces and commas.

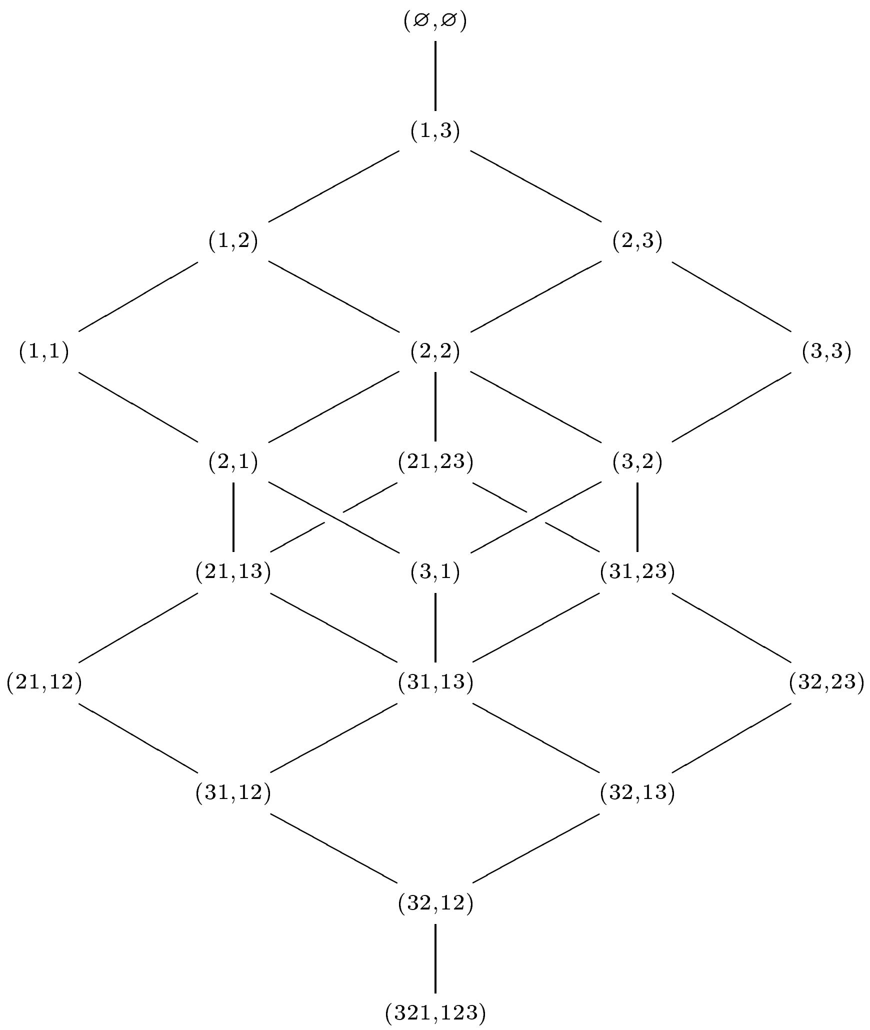

Diagram 1.2

1.3. The basis we construct will be indexed by certain bitableaux (pairs of tableaux) with specifications as below. Recall that, in general, a *tableau* consists of a Young diagram with entries in each box. We consider only tableaux with entries from $\{1,\dots,n\}$ and no repetitions in any row. Allowable bitableaux are pairs (T,T') where

(a) T and T' have the same shape;
(b) T has strictly decreasing rows;
(c) T' has strictly increasing rows.

Rows of T or T' can be identified with subsets of $\{1,\dots,n\}$ listed in descending or ascending order. Hence, allowable bitableaux can be labelled in the form

$$(\bullet\bullet) \qquad \begin{pmatrix} I_1 & J_1 \\ I_2 & J_2 \\ \vdots & \vdots \\ I_l & J_l \end{pmatrix}$$

where $(I_1,J_1),\dots,(I_l,J_l)$ are index pairs such that $|I_1| \geq |I_2| \geq \cdots \geq |I_l|$. The pair (I_1,J_1) will be called the *top row* of (T,T').

We say that a bitableau (T,T') is *preferred* if it is allowable, the columns of T are nonincreasing, and the columns of T' are nondecreasing. In the format $(\bullet\bullet)$ above, (T,T') is preferred if and only if $(I_1,J_1) \leq (I_2,J_2) \leq \cdots \leq (I_l,J_l)$.

For induction purposes, we shall also need an ordering on bitableaux. Suppose that

$$(S,S') = \begin{pmatrix} I_1 & J_1 \\ I_2 & J_2 \\ \vdots & \vdots \\ I_s & J_s \end{pmatrix} \qquad \text{and} \qquad (T,T') = \begin{pmatrix} K_1 & L_1 \\ K_2 & L_2 \\ \vdots & \vdots \\ K_t & L_t \end{pmatrix}$$

are bitableaux presented in the format $(\bullet\bullet)$. Define $(S,S') \prec (T,T')$ if and only if either the shape of S is larger than the shape of T relative to the lexicographic ordering on shapes, that is,

$$(|I_1|,\dots,|I_s|) >_{\text{lex}} (|K_1|,\dots,|K_t|);$$

or else the shapes of S and T coincide and

$$((I_1,J_1),\dots,(I_s,J_s)) <_{\text{lex}} ((K_1,L_1),\dots,(K_t,L_t)),$$

that is, there is an index l such that $(I_j,J_j) = (K_j,L_j)$ for $j<l$ and $(I_l,J_l) < (K_l,L_l)$ (relative to the ordering defined in (1.2)).

Note that if (I,J) and (K,L) are index pairs, then $(I,J) < (K,L)$ implies $(I\ J) \prec (K\ L)$ but not vice versa (unless $|I| = |K|$).

1.4. We adopt the notation of [**7**], using $[\bullet|\bullet]$ in place of $(\bullet|\bullet)$ to help distinguish quantum minors and products of these from the index pairs and bitableaux labelling them.

For any index pair (I,J), there is a quantum minor $D(I,J) \in \mathcal{A}$ defined in terms of the X_{ij} for $i \in I$ and $j \in J$. One can obtain $D(I,J)$ as the image of the quantum determinant in $\mathcal{O}_q(M_{|I|}(k))$ under the natural isomorphism of this algebra with $k\langle X_{ij} \mid i \in I,\ j \in J\rangle$ (cf.

[**13**, (4.3)]). By convention, $D(\varnothing,\varnothing)=1$. In this section and the next, we shall abbreviate $D(I,J)$ by the symbol $[I|J]$.

For any (allowable) bitableau (T,T'), write

$$(T,T')=\begin{pmatrix} I_1 & J_1 \\ I_2 & J_2 \\ \vdots & \vdots \\ I_l & J_l \end{pmatrix}$$

in the format $(\bullet\bullet)$ of (1.3). We define $[T|T']=[I_1|J_1][I_2|J_2]\cdots[I_l|J_l]$.

1.5. The *content* of a tableau T is the multiset $1^{m_1}2^{m_2}\cdots n^{m_n}$ where m_i is the number of times i appears in T. The *bicontent* of a bitableau (T,T') is the pair of multisets $(\text{content}(T),\text{content}(T'))$.

There is a natural $\mathbb{Z}^n\times\mathbb{Z}^n$ bigrading on $\mathcal{A}$, under which each X_{ij} has bidegree (ϵ_i,ϵ_j) where $\epsilon_1,\dots,\epsilon_n$ is the standard basis for $\mathbb{Z}^n$. Observe that any quantum minor $[I|J]$ is homogeneous of bidegree (χ_I,χ_J) where χ_X stands for the characteristic function of a subset X of $\{1,\dots,n\}$. More generally, if a bitableau (T,T') has bicontent

$$(1^{r_1}2^{r_2}\cdots n^{r_n},1^{c_1}2^{c_2}\cdots n^{c_n}),$$

then $[T|T']$ is homogeneous of bidegree $(r_1,\dots,r_n,c_1,\dots,c_n)$.

1.6. Lemma. *Let $(R_1,C_1),\dots,(R_l,C_l)$ be index pairs such that $|R_1|\geq|R_2|\geq\cdots\geq|R_j|$ but $|R_j|<|R_{j+1}|$ for some j. Let $g<j$ be the largest index such that $|R_g|\geq|R_{j+1}|$, or $g=0$ if $|R_1|<|R_{j+1}|$. Then the product*

$$P:=[R_1|C_1][R_2|C_2]\cdots[R_l|C_l]$$

can be expressed as a linear combination of products $[T|T']$ of the same bidegree as P, where each (T,T') is a bitableau of the form

$$\begin{pmatrix} R_1 & C_1 \\ \vdots & \vdots \\ R_g & C_g \\ K_1 & K_1' \\ K_2 & K_2' \\ \vdots & \vdots \end{pmatrix}$$

with $|K_1|=|R_{j+1}|$ and $(K_1,K_1')\leq(R_{j+1},C_{j+1})$.

Proof. By induction on j.

In view of Proposition 5.3, $[R_j|C_j][R_{j+1}|C_{j+1}]$ can be written as a linear combination of products

$$[K_1|K_1'][K_2|K_2']\cdots[K_s|K_s']$$

of the same bidegree as $[R_j|C_j][R_{j+1}|C_{j+1}]$, such that $|K_1|=|R_{j+1}|$ and $(K_1,K_1')\leq(R_{j+1},C_{j+1})$. (Here we write any X_{ij} occurring in a monomial M' as a 1×1 quantum minor $[i|j]$.) Substituting this linear combination into P, we obtain an expression for P as a linear combination of products

$$[R_1|C_1]\cdots[R_{j-1}|C_{j-1}][K_1|K_1'][K_2|K_2']\cdots[K_s|K_s'][R_{j+2}|C_{j+2}]\cdots[R_l|C_l]$$

with the same bidegree as P. After expanding each of $[R_{j+2}|C_{j+2}],\dots,[R_l|C_l]$ as a linear combination of monomials, we can express P as a linear combination of products

$$(\dagger)\qquad [R_1|C_1]\cdots[R_{j-1}|C_{j-1}][K_1|K_1'][K_2|K_2']\cdots[K_t|K_t']$$

of the same bidegree as P, such that $|K_1| = |R_{j+1}|$ and $(K_1, K_1') \leq (R_{j+1}, C_{j+1})$, while $|K_i| = 1$ for $i > 1$.

If either $j = 1$ or $|R_{j-1}| \geq |K_1|$, these products can be written in the form $[T|T']$ for bitableaux (T, T') of the desired type, and we are done. If $j > 1$ and $|R_{j-1}| < |K_1|$, the induction hypothesis applies to each of the products $(\dagger)$; collecting terms, we are again done. $\square$

1.7. Recall the (non-total) ordering $\prec$ on bitableau defined in (1.3).

Lemma. *Let (S, S') be a bitableau with bicontent γ and top row (I_1, J_1), and suppose that (S, S') is not preferred.*

(a) *(S, S') is not minimal with respect to $\prec$ among bitableau with bicontent γ.*

(b) *$[S|S']$ can be expressed as a linear combination of products $[T|T']$ where each (T, T') is a bitableau with bicontent γ such that $(T, T') \prec (S, S')$. Further, each (T, T') can be chosen with a top row (X_1, Y_1) such that either $|X_1| > |I_1|$ or $(X_1, Y_1) \leq (I_1, J_1)$.*

Although part (a) can be obtained as a consequence of part (b), we find it clearer to give an explicit proof of (a).

Proof. Let δ denote the bidegree of $[S|S']$.

Since (S, S') is not preferred, it must have at least two rows. Write

$$(S, S') = \begin{pmatrix} I_1 & J_1 \\ I_2 & J_2 \\ \vdots & \vdots \\ I_l & J_l \end{pmatrix}$$

in the format $(\bullet\bullet)$ of (1.3); then either $I_j \not\leq_r I_{j+1}$ or $J_j \not\leq_c J_{j+1}$ for some j.

Case I: Suppose that $I_j \not\leq_r I_{j+1}$ for some j. We may assume that j is minimal with respect to this property, so that $I_1 \leq_r I_2 \leq_r \cdots \leq_r I_j$. Write

$$\begin{aligned} I_j &= \{a_1 > a_2 > \cdots > a_\alpha\} \\ I_{j+1} &= \{b_1 > b_2 > \cdots > b_\beta\}; \end{aligned}$$

then $\alpha \geq \beta$ (by the shape of S) but $a_i < b_i$ for some $i \leq \beta$. We may assume that i is minimal, so that $a_1 \geq b_1, \dots, a_{i-1} \geq b_{i-1}$. Set

$$\begin{aligned} A_1 &= \{a_1 > a_2 > \cdots > a_{i-1}\} \\ A_2 &= \{b_{i+1} > \cdots > b_\beta\} \\ K &= \{b_1 > \cdots > b_i > a_i > \cdots > a_\alpha\}. \end{aligned}$$

(a) Since $\{b_1, \dots, b_i\}$ has one more element than A_1, there must be an index $p \le i$ such that $b_p \notin A_1$. In addition, $b_p \ge b_i > a_i > \cdots > a_\alpha$, and so $b_p \notin I_j$. Similarly, there is an index $q \ge i$ such that $a_q \notin I_{j+1}$, and $b_p \ge b_i > a_i \ge a_q$. Now set

$$\begin{aligned} I'_j &= I_j \cup \{b_p\} \setminus \{a_q\} \\ I'_{j+1} &= I_{j+1} \cup \{a_q\} \setminus \{b_p\} \end{aligned}$$

and observe that I'_j and I'_{j+1} have the same cardinalities as I_j and I_{j+1}, respectively. Further, $I'_j \cup I'_{j+1} = I_j \cup I_{j+1}$, and $I'_j <_r I_j$ because $b_p > a_q$. Set

$$(R, R') = \begin{pmatrix} I_1 & J_1 \\ \vdots & \vdots \\ I_{j-1} & J_{j-1} \\ I'_j & J_j \\ I'_{j+1} & J_{j+1} \\ I_{j+2} & J_{j+2} \\ \vdots & \vdots \\ I_l & J_l \end{pmatrix}$$

and note that (R, R') is a bitableau with the same shape and bicontent as (S, S'). Since $I'_j <_r I_j$, we also have $(I'_j, J_j) < (I_j, J_j)$, and therefore $(R, R') \prec (S, S')$.

(b) The exchange formula (6.2)(b) gives us a relation of the form

$$(\dagger) \qquad \sum_{K = K' \sqcup K''} \pm q^\bullet [A_1 \sqcup K' | J_j][K'' \sqcup A_2 | J_{j+1}] \\ = \sum_{J_\nu = J'_\nu \sqcup J''_\nu} \pm q^\bullet [A_1 | J'_j][K | J''_j \sqcup J''_{j+1}][A_2 | J'_{j+1}]$$

with all terms of the same bidegree. Note that $[I_j | J_j][I_{j+1} | J_{j+1}]$ occurs on the left hand side of $(\dagger)$ when $K' = \{a_i > \cdots > a_\alpha\}$ and $K'' = \{b_1 > \cdots > b_i\}$. In any other term on the left, K' contains at least one of $b_1, \dots, b_i$, from which we see that $A_1 \sqcup K' <_r I_j$.

We now have a relation of the form

$$(*) \qquad \sum_{K = K' \sqcup K''} \pm q^\bullet [I_1 | J_1] \cdots [I_{j-1} | J_{j-1}][A_1 \sqcup K' | J_j][K'' \sqcup A_2 | J_{j+1}][I_{j+2} | J_{j+2}] \cdots [I_l | J_l] \\ = \sum_{\substack{J_\nu = J'_\nu \sqcup J''_\nu \\ (\nu = j, j+1)}} \pm q^\bullet [I_1 | J_1] \cdots [I_{j-1} | J_{j-1}][A_1 | J'_j][K | J''_j \sqcup J''_{j+1}][A_2 | J'_{j+1}][I_{j+2} | J_{j+2}] \cdots [I_l | J_l]$$

with all terms of bidegree δ. On the left hand side, a term $\pm q^\bullet [S | S']$ occurs once, and all other terms are of the form $\pm q^\bullet [T | S']$ with $(T, S') \prec (S, S')$. Moreover, if $j > 1$ the top row of (T, S') equals (I_1, J_1), while if $j = 1$, the top row of (T, S') equals $(A_1 \sqcup K', J_1)$, and in this case $(A_1 \sqcup K', J_1) < (I_1, J_1)$ because $A_1 \sqcup K' <_r I_1$. Hence, to prove part (b) we just need to express the right hand side of $(*)$ in the desired form.

Note that $|A_1| = i-1 < \alpha$ while $|K| = \alpha+1$. Let $g < j$ be the largest index such that $|I_g| \geq \alpha+1$, or $g = 0$ if $|I_1| \leq \alpha$. Applying Lemma 1.6 to each term, we can express the right-hand side of (*) as a linear combination of products $[T|T']$ of bidegree δ where each (T,T') is a bitableau of the form

$$\begin{pmatrix} I_1 & J_1 \\ \vdots & \vdots \\ I_g & J_g \\ K_1 & K_1' \\ K_2 & K_2' \\ \vdots & \vdots \end{pmatrix}$$

with $|K_1| = \alpha+1$. Since $|I_{g+1}| \leq \alpha$, the shape of T is larger than the shape of S, and so $(T,T') \prec (S,S')$. If $g \geq 1$, the top row of (T,T') equals (I_1,J_1), while if $g=0$, the top row of (T,T') equals (K_1,K_1'), and in this case $|K_1| = \alpha+1 > \alpha \geq |I_1|$. This establishes part (b) in Case I.

Case II: Suppose that $J_j \not\leq_c J_{j+1}$ for some j. This case can be handled in the same manner as Case I, by using (6.2)(a) rather than (6.2)(b). $\square$

1.8. Corollary. *Let (S,S') be a bitableau with bicontent γ and top row (I_1,J_1). Then $[S|S']$ can be expressed as a linear combination of products $[T|T']$ where*

(a) *(T,T') is a preferred bitableau with bicontent γ.*

(b) *(T,T') has a top row (X_1,Y_1) such that either $|X_1| > |I_1|$ or $(X_1,Y_1) \leq (I_1,J_1)$.*

Proof. This follows from Lemma 1.7 by induction with respect to $\prec$. $\square$

Part (b) of this corollary is a weak form of the straightening law for a classical standard basis, and is useful in computing preferred bases for certain ideals of $\mathcal{O}_q(M_n(k))$.

1.9. Theorem. *Let $\delta = (r_1,\dots,r_n,c_1,\dots,c_n) \in (\mathbb{Z}^+)^n \times (\mathbb{Z}^+)^n$, let V be the homogeneous component of $\mathcal{A}$ with bidegree δ, and set $\gamma = (1^{r_1}2^{r_2}\cdots n^{r_n}, 1^{c_1}2^{c_2}\cdots n^{c_n})$. The products $[T|T']$, as (T,T') runs over all preferred bitableaux with bicontent γ, form a basis for V.*

Proof. Observe that $[S|S'] \in V$ for all bitableau (S,S') with bicontent γ, and that there are only finitely many such bitableaux. Further, such products $[S|S']$ include all monomials $X_{i_1j_1}X_{i_2j_2}\cdots X_{i_rj_r}$ with bidegree δ, and these monomials span V. Hence, Corollary 1.8 implies that V is spanned by the products $[T|T']$ as (T,T') runs over all preferred bitableaux with bicontent γ. It remains to show that these products are linearly independent. To see this, it suffices to prove that the number of preferred bitableaux with bicontent γ is equal to the dimension of V.

We may write $\mathcal{A}$ as an iterated Ore extension with the variables X_{ij} in the order

$$X_{nn}, X_{n,n-1}, \dots, X_{n1}, X_{n-1,n}, X_{n-1,n-1}, \dots, X_{n-1,1}, \dots, X_{1n}, X_{1,n-1}, \dots, X_{11}.$$

Hence, $\mathcal{A}$ has a basis consisting of monomials $X_{i_1j_1}X_{i_2j_2}\cdots X_{i_rj_r}$ satisfying the conditions

(a) $i_1 \geq i_2 \geq \cdots \geq i_r$;

(b) $j_l \geq j_{l+1}$ whenever $i_l = i_{l+1}$.

Since $X_{im}X_{ij} = q^{-1}X_{ij}X_{im}$ when $m > j$, we can reverse any product of generators with the same row index, at the cost of a nonzero scalar coefficient. Hence, $\mathcal{A}$ has a basis $\mathcal{B}$ consisting of monomials $X_{i_1j_1}X_{i_2j_2}\cdots X_{i_rj_r}$ such that

(a) $i_1 \geq i_2 \geq \cdots \geq i_r$;
(b') $j_l \leq j_{l+1}$ whenever $i_l = i_{l+1}$.

Note that under conditions (a),(b'), the list $i_1j_1, \ldots, i_rj_r$ of double indices is in lexicographic order, provided we write our row alphabet in reverse order, i.e., $n, n-1, \ldots, 1$, while keeping our column alphabet $1, 2, \ldots, n$ in the usual order. With this convention, the monomials in $\mathcal{B}$ are in bijection with those two-rowed matrices

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_r \\ j_1 & j_2 & \cdots & j_r \end{pmatrix}$$

having entries from $\{1, \ldots, n\}$ and columns in lexicographic order. Note that the monomial $X_{i_1j_1}X_{i_2j_2}\cdots X_{i_rj_r}$ has bidegree δ iff the pair of multisets $(\{i_1, \ldots, i_r\}, \{j_1, \ldots, j_r\})$ coincides with γ.

By the Robinson-Schensted-Knuth Theorem [**5**, p. 40], the two-rowed matrices corresponding to monomials from $\mathcal{B}$ with bidegree δ are in bijection with standard bitableaux (Q, P) of bicontent γ. In this result, standard tableaux are required to be nondecreasing on each row and strictly increasing on each column, relative to the total orders given on the two alphabets. Note this means that relative to the usual ordering of integers, Q has nonincreasing rows and strictly decreasing columns. Thus the ordering conditions on (Q, P) hold precisely when the pair $(Q^{\mathrm{tr}}, P^{\mathrm{tr}})$ is preferred in our sense (1.3).

Therefore there exists a bijection between the monomials of bidegree δ in $\mathcal{B}$ and the preferred bitableaux with bicontent γ. Since the former make a basis for V, we conclude that the number of preferred bitableaux with bicontent γ is precisely $\dim_k V$, as required. $\square$

1.10. Corollary. *The products $[T|T']$, as (T, T') runs over all preferred bitableaux, form a basis for $\mathcal{A} = \mathcal{O}_q(M_n(k))$.* $\square$

1.11. The existence of analogous bases for rectangular quantum matrix algebras follows easily from Corollary 1.10. For $m < n$, we may define $\mathcal{O}_q(M_{m,n}(k))$ as the k-subalgebra of $\mathcal{O}_q(M_n(k))$ generated by the X_{ij} with $i \leq m$; the case $(m > n)$ is handled by writing $\mathcal{O}_q(M_{m,n}(k))$ as a subalgebra of $\mathcal{O}_q(M_m(k))$. Note that in the first case, there is a k-algebra retraction $\pi : \mathcal{O}_q(M_n(k)) \to \mathcal{O}_q(M_{m,n}(k))$ such that $\pi(X_{ij}) = X_{ij}$ when $i \leq m$ and $\pi(X_{ij}) = 0$ when $i > m$.

Corollary. *Let m, n be any positive integers, and let $\mathcal{B}_{m,n}$ be the set of all products $[T|T']$ where (T, T') runs over all preferred bitableaux in which the entries of T lie in $\{1, \ldots, m\}$ while the entries of T' lie in $\{1, \ldots, n\}$. Then $\mathcal{B}_{m,n}$ is a basis for $\mathcal{O}_q(M_{m,n}(k))$.*

Proof. We prove only the case $(m < n)$; the other case is identical. By Corollary 1.10, the set $\mathcal{B}_{n,n}$ is a basis for $\mathcal{O}_q(M_n(k))$. On one hand, $\mathcal{B}_{m,n} \subseteq \mathcal{B}_{n,n}$ and so $\mathcal{B}_{m,n}$ is linearly independent. On the other hand, $\pi(\mathcal{B}_{n,n}) = \mathcal{B}_{m,n} \cup \{0\}$, and therefore $\mathcal{B}_{m,n}$ spans $\mathcal{O}_q(M_{m,n}(k))$. $\square$

II. One-parameter quantum determinantal ideals

In this section, we prove that quantum determinantal ideals in $\mathcal{O}_q(M_{m,n}(k))$ are completely prime. The case of the ideal generated by all 2×2 quantum minors has been proved by Rigal [**14**], using different methods.

2.1. As in the previous section, we fix $n \geq 2$, a field k, a scalar $q \in k^\times$, and set $\mathcal{A} = \mathcal{O}_q(M_n(k))$. Fix $t \in \{1, \ldots, n-1\}$, and let $\mathcal{I}_t = I_q^{[t]}(M_n(k))$ denote the ideal of $\mathcal{A}$ generated by all $(t+1) \times (t+1)$ quantum minors. Again, it is convenient to remain with this case until the main result is proved, and to derive the corresponding result for $\mathcal{O}_q(M_{m,n}(k))$ as an easy corollary. We proceed by establishing a quantized version of the theorem stating that, in the classical case, $\mathcal{I}_t$ equals the kernel of the k-algebra homomorphism

$$\mu_* : \mathcal{O}(M_n(k)) \to \mathcal{O}(M_{n,t}(k) \times M_{t,n}(k)) = \mathcal{O}(M_{n,t}(k)) \otimes \mathcal{O}(M_{t,n}(k))$$

discussed in the introduction.

First, some labels: set

$$\begin{aligned} \mathcal{A}_{nt} &= \mathcal{O}_q(M_{n,t}(k)) = k\langle X_{ij} \mid j \leq t \rangle \subseteq \mathcal{A} \\ \mathcal{A}_{tn} &= \mathcal{O}_q(M_{t,n}(k)) = k\langle X_{ij} \mid i \leq t \rangle \subseteq \mathcal{A}. \end{aligned}$$

For $\tau = nt$ or tn, let $\pi_\tau : \mathcal{A} \to \mathcal{A}_\tau$ denote the natural k-algebra retraction. Thus

$$\pi_{nt}(X_{ij}) = \begin{cases} X_{ij} & (j \leq t) \\ 0 & (j > t); \end{cases} \qquad \pi_{tn}(X_{ij}) = \begin{cases} X_{ij} & (i \leq t) \\ 0 & (i > t). \end{cases}$$

The kernels of these homomorphisms are the ideals $\langle X_{ij} \mid j > t \rangle$ and $\langle X_{ij} \mid i > t \rangle$, respectively. Finally, define the k-algebra homomorphism

$$\theta_t : \mathcal{A} \xrightarrow{\Delta} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\pi_{nt} \otimes \pi_{tn}} \mathcal{A}_{nt} \otimes \mathcal{A}_{tn},$$

where Δ denotes the comultiplication on the bialgebra $\mathcal{A}$.

By [**12**, (1.9)], comultiplication of quantum minors is given by the rule

$$\Delta[I|J] = \sum_{|K|=|I|} [I|K] \otimes [K|J].$$

Since all $(t+1) \times (t+1)$ quantum minors are killed by π_{nt}, we see that $\mathcal{I}_t \subseteq \ker \theta_t$. We shall prove equality in Proposition 2.4.

Note that any product $[T|T']$ for which the shape of T has more than t columns lies in $\mathcal{I}_t$. Hence, $\mathcal{A}/\mathcal{I}_t$ is spanned by the images of those products $[T|T']$ indexed by preferred bitableaux (T, T') with shapes having at most t columns.

2.2. Consider an allowable bitableau (T, T'). For $l = 1, \ldots, n$, let $\rho_l(T)$ be the number of rows of T of length $\geq l$, and set $\overline{\rho}(T) = (\rho_1(T), \rho_2(T), \ldots, \rho_n(T))$. Let $\mu(T)$ and $\mu'(T)$ denote the tableaux with the same shape as T and entries as follows: each row of length l is filled $1, 2, \ldots, l$ in $\mu(T)$ and is filled $l, l-1, \ldots, 1$ in $\mu'(T)$. If (T, T') is preferred, then $(T, \mu(T))$ and $(\mu'(T), T')$ are preferred bitableaux.

For any homogeneous element $x \in \mathcal{A}$, label the bidegree of x as

$$(\overline{r}(x), \overline{c}(x)) = (r_1(x), r_2(x), \ldots, r_n(x), c_1(x), c_2(x), \ldots, c_n(x)).$$

Thus with respect to the usual PBW basis of ordered monomials, $r_l(x)$ records the number of $X_{l?}$ factors in each monomial in x, and $c_l(x)$ the number of $X_{?l}$ factors. For $[T|T']$ as in the previous paragraph, $r_l[T|T']$ is the number of l's in T and $c_l[T|T']$ is the number of l's in T'. Note that $c_l[T|\mu(T)] = r_l[\mu'(T)|T'] = \rho_l(T)$.

We shall write $<_{\mathrm{rlex}}$ for the reverse lexicographic order on n-tuples of integers.

2.3. Lemma. *Let (T,T') be an allowable bitableau with a shape having at most t columns. Then*
$$\theta_t[T|T'] = [T|\mu(T)] \otimes [\mu'(T)|T'] + \sum_i X_i \otimes Y_i$$
where the X_i and Y_i are homogeneous with $\overline{c}(X_i) = \overline{r}(Y_i) >_{\text{rlex}} \overline{\rho}(T)$.

Proof. Write
$$(T,T') = \begin{pmatrix} I_1 & J_1 \\ I_2 & J_2 \\ \vdots & \vdots \\ I_s & J_s \end{pmatrix}$$
where the (I_j, J_j) are index pairs. Then
$$\theta_t[I_j|J_j] = \sum_{\substack{|K|=|I_j| \\ K\subseteq\{1,\dots,t\}}} [I_j|K] \otimes [K|J_j]$$
for each j. Hence, $\theta_t[T|T']$ is the sum of all possible terms
$$X_i \otimes Y_i = [I_1|K_1][I_2|K_2]\cdots[I_s|K_s] \otimes [K_1|J_1][K_2|J_2]\cdots[K_s|J_s]$$
where each $K_j \subseteq \{1,\dots,t\}$ and $|K_j| = |I_j|$. Obviously X_i and Y_i are homogeneous.

Let $i = i_0$ label the special case where $K_j = \{1,2,\dots,|I_j|\}$ for all j. This yields the term $X_{i_0} \otimes Y_{i_0} = [T|\mu(T)] \otimes [\mu'(T)|T']$. Now assume that $i \neq i_0$. Obviously $\overline{c}(X_i) = \overline{r}(Y_i)$, so it remains to show that $\overline{c}(X_i) >_{\text{rlex}} \overline{\rho}(T)$. Note that $c_l(X_i) = \rho_l(T) = 0$ for $l > t$.

We claim that if $c_l(X_i) = \rho_l(T)$ for $l = n, n-1, \dots, h+1$, then $K_j = \{1,2,\dots,|I_j|\}$ for all j such that $|I_j| \geq h$. This is vacuously true for $h > t$. Now suppose that $h \leq t$ and that $K_j = \{1,2,\dots,|I_j|\}$ whenever $|I_j| > h$. For $l > h$, there are $\rho_l(T)$ indices j for which $|I_j| \geq l$, and $l \in K_j$ for each such j. Since $c_l(X_i) = \rho_l(T)$, this uses up all the available column l's in X_i, and so $l \notin K_j$ for any j with $|I_j| < l$. Thus $K_j \subseteq \{1,2,\dots,h\}$ for all j with $|I_j| \leq h$. In particular $K_j = \{1,2,\dots,h\}$ for all j with $|I_j| = h$, verifying the induction step. This establishes the claim.

Since we are in the case $i \neq i_0$, we cannot have $K_j = \{1,2,\dots,|I_j|\}$ for all j, and so the claim shows that we cannot have $c_l(X_i) = \rho_l(T)$ for all l. Hence, there is an index $g \geq 1$ such that $c_g(X_i) \neq \rho_g(T)$ while $c_l(X_i) = \rho_l(T)$ for all $l > g$. By the claim, $K_j = \{1,2,\dots,|I_j|\}$ for all j such that $|I_j| \geq g$. Hence, $g \in K_j$ for all j with $|I_j| \geq g$, and so $c_g(X_i) \geq \rho_g(T)$. By our choice of g, we thus must have $c_g(X_i) > \rho_g(T)$. Therefore $\overline{c}(X_i) >_{\text{rlex}} \overline{\rho}(T)$, as required. □

2.4. Proposition. $\mathcal{I}_t = \ker\theta_t$.

Proof. If $\ker\theta_t$ properly contains $\mathcal{I}_t$, then $\ker\theta_t$ contains a nonzero element of the form
$$x = \sum_{i=1}^m \alpha_i [T_i|T_i']$$

where the α_i are nonzero scalars and the (T_i, T_i') are distinct preferred bitableaux with shapes having at most t columns. Let $\overline{\rho}$ be the minimum of the n-tuples $\overline{\rho}(T_i)$ under reverse lexicographic order. Without loss of generality, there exists m' such that $\overline{\rho}(T_i) = \overline{\rho}$ for $i \leq m'$ and $\overline{\rho}(T_i) >_{\text{rlex}} \overline{\rho}$ for $i > m'$.

Applying Lemma 2.3 to each $\theta_t[T_i|T_i']$ and collecting terms, we see that

$$0 = \theta_t(x) = \sum_{i=1}^{m'} \alpha_i [T_i|\mu(T_i)] \otimes [\mu'(T_i)|T_i'] + \sum_j X_j \otimes Y_j$$

where the X_j and Y_j are homogeneous with $\overline{c}(X_j) = \overline{r}(Y_j) >_{\text{rlex}} \overline{\rho}$. Since $\overline{c}[T_i|\mu(T_i)] = \overline{\rho}$ for $i \leq m'$, all of the X_j belong to different homogeneous components than the $[T_i|\mu(T_i)]$ for $i \leq m'$. Consequently,

$$\sum_{i=1}^{m'} \alpha_i [T_i|\mu(T_i)] \otimes [\mu'(T_i)|T_i'] = 0.$$

For $1 \leq i < j \leq m'$, either $T_i \neq T_j$ or $T_i' \neq T_j'$, so $(T_i, \mu(T_i)) \neq (T_j, \mu(T_j))$ or $(\mu'(T_i), T_i') \neq (\mu'(T_j), T_j')$. It thus follows from the linear independence of the preferred products $[\bullet|\bullet]$ in $\mathcal{A}_{nt}$ and $\mathcal{A}_{tn}$ (Corollary 1.11) that the terms $[T_i|\mu(T_i)] \otimes [\mu'(T_i)|T_i']$ are linearly independent. But then $\alpha_i = 0$ for $i = 1, \dots, m'$, contradicting our assumptions. $\square$

2.5. Theorem. *$\mathcal{A}/\mathcal{I}_t = \mathcal{O}_q(M_n(k))/I_q^{[t]}(M_n(k))$ is an integral domain.*

Proof. By Proposition 2.4, $\mathcal{A}/\mathcal{I}_t$ embeds in $\mathcal{A}_{nt} \otimes \mathcal{A}_{tn}$. Now $\mathcal{A}_{nt}$ and $\mathcal{A}_{tn}$ are iterated Ore extensions of k, with respect to k-algebra automorphisms and k-linear skew derivations. In particular, both of these algebras are domains. Further, $\mathcal{A}_{nt} \otimes \mathcal{A}_{tn}$ is an iterated Ore extension of $\mathcal{A}_{nt}$, and so it too is a domain. Therefore $\mathcal{A}/\mathcal{I}_t$ is a domain. $\square$

2.6. Corollary. *Let m, n, t be any positive integers such that $t < \min\{m, n\}$, and let $I_q^{[t]}(M_{m,n}(k))$ be the ideal of $\mathcal{O}_q(M_{m,n}(k))$ generated by all $(t+1) \times (t+1)$ quantum minors. Then $\mathcal{O}_q(M_{m,n}(k))/I_q^{[t]}(M_{m,n}(k))$ is an integral domain.*

Proof. Consider the case $(m < n)$, and put $I = I_q^{[t]}(M_{m,n}(k))$ and $J = I_q^{[t]}(M_n(k))$. Obviously $I \subseteq J \cap \mathcal{O}_q(M_{m,n}(k))$. For the reverse inclusion, we use the retraction $\pi : \mathcal{O}_q(M_n(k)) \to \mathcal{O}_q(M_{m,n}(k))$ discussed in (1.11). Note that the image of any quantum minor $[X|Y]$ under π is either $[X|Y]$ or 0, and hence $\pi(J) \subseteq I$. Since π is the identity on $\mathcal{O}_q(M_{m,n}(k))$, it follows that $I = J \cap \mathcal{O}_q(M_{m,n}(k))$. Therefore the corollary follows from Theorem 2.5. $\square$

III. Twisting

Artin, Schelter, and Tate showed in [**1**] that multiparameter quantum matrix algebras $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ can be obtained from the one-parameter versions by a process of twisting by 2-cocycles. In this section, we recall some details of this process and determine its effect on quantum minors.

3.1. Let k be a field, and let $\boldsymbol{p} = (p_{ij})$ be a multiplicatively antisymmetric matrix over $k^\times$, that is, $p_{ii} = 1$ and $p_{ji} = p_{ij}^{-1}$ for all i, j. Let $\lambda \in k^\times \setminus \{-1\}$. The algebra $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ is the k-algebra with generators Y_{ij} for $i, j = 1, \dots, n$ and relations

$$Y_{lm}Y_{ij} = \begin{cases} p_{li}p_{jm}Y_{ij}Y_{lm} + (\lambda - 1)p_{li}Y_{im}Y_{lj} & \text{when } l > i \text{ and } m > j \\ \lambda p_{li}p_{jm}Y_{ij}Y_{lm} & \text{when } l > i \text{ and } m \le j \\ p_{jm}Y_{ij}Y_{lm} & \text{when } l = i \text{ and } m > j. \end{cases}$$

We shall denote this algebra $\mathcal{A}_{\lambda,\boldsymbol{p}}$ for short.

3.2. The *quantum exterior algebra* $\Lambda_{\boldsymbol{p}} = \Lambda_{\boldsymbol{p}}(k^n)$ is the k algebra with generators $\eta_1, \dots, \eta_n$ and relations

$$\eta_i^2 = 0, \qquad\qquad \eta_i\eta_j = -p_{ij}\eta_j\eta_i$$

for all i, j. For any subset $I \subseteq \{1, \dots, n\}$, write the elements of I in ascending order, say $I = \{i_1 < i_2 < \cdots < i_r\}$, and set $\eta_I = \eta_{i_1}\eta_{i_2}\cdots\eta_{i_r}$. By convention, $\eta_\varnothing = 1$. The elements η_I form a k-basis for $\Lambda_{\boldsymbol{p}}$.

As is well known (and easily checked), there is a k-algebra homomorphism

$$L_{\lambda,\boldsymbol{p}} : \Lambda_{\boldsymbol{p}} \longrightarrow \mathcal{A}_{\lambda,\boldsymbol{p}} \otimes \Lambda_{\boldsymbol{p}}$$

such that $L_{\lambda,\boldsymbol{p}}(\eta_i) = \sum_j Y_{ij} \otimes \eta_j$ for all i (cf. [**11**, Chapter 6, Theorem 3], [**1**, (11), (12)]). For any nonempty subset $I \subseteq \{1, \dots, n\}$, the image of η_I under $L_{\lambda,\boldsymbol{p}}$ has the form

$$L_{\lambda,\boldsymbol{p}}(\eta_I) = \sum_{|J|=|I|} U_{IJ} \otimes \eta_J$$

[**1**, Lemma 1]. The elements $U_{IJ} \in \mathcal{A}_{\lambda,\boldsymbol{p}}$ are unique due to the linear independence of the η_J. Each U_{IJ} is the quantum minor corresponding to the rows $i \in I$ and columns $j \in J$; we shall use the notation

$$D_{\lambda,\boldsymbol{p}}(I, J) = U_{IJ}$$

to indicate the dependence on the parameters λ, p_{ij}. Explicit formulas for $D_{\lambda,\boldsymbol{p}}(I, J)$ are given in [**1**, Lemma 1].

3.3. The quantum minors in $\mathcal{A}_q = \mathcal{O}_q(M_n(k))$ can be obtained as in (3.2), of course. Since we must consider both settings simultaneously, let us use ξ_i for the generators of the quantum exterior algebra in this case. Thus, $\Lambda_q = \Lambda_q(k^n)$ is the k-algebra with generators $\xi_1, \dots, \xi_n$ and relations

$$\xi_i^2 = 0 \quad (\text{all } i); \qquad\qquad \xi_j\xi_i = -q\xi_i\xi_j \quad (i < j).$$

There is a basis consisting of elements $\xi_I = \xi_{i_1}\xi_{i_2}\cdots\xi_{i_r}$ where $I = \{i_1 < i_2 < \cdots < i_r\}$ runs through all subsets of $\{1, \dots, n\}$.

There is a k-algebra homomorphism $L_q : \Lambda_q \to \mathcal{A}_q \otimes \Lambda_q$ such that $L_q(\xi_i) = \sum_j X_{ij} \otimes \xi_j$ for all i. The quantum minors in $\mathcal{A}_q$, which we shall now denote $D_q(I, J)$ to indicate the dependence on q, arise in the formulas

$$L_q(\xi_I) = \sum_{|J|=|I|} D_q(I, J) \otimes \xi_J$$

(cf. [**13**, Lemma 4.4.2]; [**12**, Remark, p. 36]).

3.4. As observed in [**1**, p. 889], the algebra $\mathcal{A}_{\lambda,\boldsymbol{p}}$ can be obtained as a cocycle twist of $\mathcal{A}_q$ provided $\lambda = q^{-2}$ (we take q^{-2} rather than q^2 to account for the difference $q \leftrightarrow q^{-1}$ between [**1**, (43)] and our choice of relations for $\mathcal{A}_q$). We thus carry out the twisting process under the assumption that λ has a square root in k; the general cases of our results require a passage to $\overline{k}$. Since we must simultaneously work with twists of $\mathcal{A}_q$, Λ_q, and a subalgebra of $\mathcal{A}_q \otimes \Lambda_q$, it is helpful to give the appropriate cocycle explicitly.

For the remainder of this section, $\boldsymbol{p} = (p_{ij})$ will be an arbitrary multiplicatively antisymmetric matrix over $k^\times$, and we will take $\lambda = q^{-2}$ for some (fixed) $q \in k^\times$. Define a map $c : \mathbb{Z}^n \times \mathbb{Z}^n \to k^\times$ by the rule

$$c((a_1,\dots,a_n),(b_1,\dots,b_n)) = \prod_{i>j} (qp_{ji})^{a_i b_j}.$$

Then c is a multiplicative bicharacter on $\mathbb{Z}^n$ (that is, $c(a+a',b) = c(a,b)c(a',b)$, and similarly in the second variable), and hence also a 2-cocycle. Note that

$$c(\epsilon_i,\epsilon_j) = \begin{cases} qp_{ji} & (i>j) \\ 1 & (i \le j) \end{cases}$$

where $\epsilon_1,\dots,\epsilon_n$ denotes the standard basis for $\mathbb{Z}^n$.

3.5. Recall the $\mathbb{Z}^n \times \mathbb{Z}^n$-bigrading on $\mathcal{A}_q$ from (1.5). Following [**1**, Theorem 4], we simultaneously twist $\mathcal{A}_q$ on the left by c^{-1} and on the right by c. This results in a new algebra, denoted $\mathcal{A}'_q$, as follows. As a graded vector space, $\mathcal{A}_q$ is isomorphic to $\mathcal{A}'_q$ via an isomorphism $a \mapsto a'$. The multiplication in $\mathcal{A}'_q$ is given by

$$a'b' = c(u_1,v_1)^{-1}c(u_2,v_2)(ab)'$$

for homogeneous elements $a, b \in \mathcal{A}_q$ of bidegrees (u_1,u_2) and (v_1,v_2). In particular,

$$X'_{ij}X'_{lm} = \begin{cases} p_{il}p_{mj}(X_{ij}X_{lm})' & (i>l,\ j>m) \\ q^{-1}p_{il}(X_{ij}X_{lm})' & (i>l,\ j\le m) \\ qp_{mj}(X_{ij}X_{lm})' & (i\le l,\ j>m) \\ (X_{ij}X_{lm})' & (i\le l,\ j\le m). \end{cases}$$

Observe that Λ_q has a natural $\mathbb{Z}^n$-grading, where ξ_i has degree ϵ_i. We twist Λ_q by c^{-1} to obtain a new algebra Λ'_q. Note that

$$\xi'_i\xi'_j = \begin{cases} q^{-1}p_{ij}(\xi_i\xi_j)' & (i>j) \\ (\xi_i\xi_j)' & (i\le j). \end{cases}$$

3.6. Lemma. *There are k-algebra isomorphisms*

$$\phi : \mathcal{A}_{\lambda,\boldsymbol{p}} \longrightarrow \mathcal{A}'_q \qquad \text{and} \qquad \psi : \Lambda_{\boldsymbol{p}} \longrightarrow \Lambda'_q$$

such that $\phi(Y_{ij}) = X'_{ij}$ and $\psi(\eta_i) = \xi'_i$ for all i, j.

Proof. The existence of ϕ follows from [**1**, Theorem 4], and the existence of ψ is proved in the same manner. One first checks that the elements $X'_{ij} \in \mathcal{A}'_q$ and $\xi'_i \in \Lambda'_q$ satisfy the same relations as the Y_{ij} and the η_i. For instance, for $i < j$ we have $\xi_j\xi_i = -q\xi_i\xi_j$ and so

$$\xi'_j\xi'_i = q^{-1}p_{ji}(\xi_j\xi_i)' = -p_{ji}(\xi_i\xi_j)' = -p_{ji}\xi'_i\xi'_j.$$

Hence, there exist k-algebra homomorphisms ϕ and ψ sending $Y_{ij} \mapsto X'_{ij}$ and $\eta_i \mapsto \xi'_i$. Since $\mathcal{A}_q$ has a basis of ordered monomials $X_{i_1j_1} \cdots X_{i_tj_t}$, and since each $(X_{i_1j_1} \cdots X_{i_tj_t})'$ is a nonzero scalar multiple of $X'_{i_1j_1} \cdots X'_{i_tj_t}$, we see that $\mathcal{A}'_q$ has a basis of ordered monomials $X'_{i_1j_1} \cdots X'_{i_tj_t}$. In addition, $\mathcal{A}_{\lambda,\boldsymbol{p}}$ has a basis of ordered monomials $Y_{i_1j_1} \cdots Y_{i_tj_t}$, which ϕ maps to the $X'_{i_1j_1} \cdots X'_{i_tj_t}$. Therefore ϕ is an isomorphism, and similarly so is ψ. $\square$

3.7. There is a $\mathbb{Z}^n$-graded subalgebra $\mathcal{B}_q \subseteq \mathcal{A}_q \otimes \Lambda_q$ where

$$(\mathcal{B}_q)_u = \bigoplus_{v \in \mathbb{Z}^n} (\mathcal{A}_q)_{uv} \otimes (\Lambda_q)_v$$

for $u \in \mathbb{Z}^n$. Using this grading, we twist $\mathcal{B}_q$ by c^{-1} to obtain a new algebra $\mathcal{B}'_q$. Note that there is a vector space embedding $\mathcal{B}'_q \to \mathcal{A}'_q \otimes \Lambda'_q$ where $(a \otimes b)' \mapsto a' \otimes b'$ for $a \in (\mathcal{A}_q)_{uv}$ and $b \in (\Lambda_q)_v$. We shall identify $\mathcal{B}'_q$ with its image in $\mathcal{A}'_q \otimes \Lambda'_q$ via this embedding.

Lemma. *Under the above identification, $\mathcal{B}'_q$ is a k-subalgebra of $\mathcal{A}'_q \otimes \Lambda'_q$.*

Proof. It suffices to show that the product of any two homogeneous elements from $\mathcal{B}'_q$ is the same in both algebras. Given $x \in (\mathcal{B}_q)_{u_1}$ and $y \in (\mathcal{B}_q)_{u_2}$, we can write $x = \sum_i x_i$ and $y = \sum_j y_j$ where each $x_i = a_i \otimes b_i \in (\mathcal{A}_q)_{u_1v_i} \otimes (\Lambda_q)_{v_i}$ for some $v_i \in \mathbb{Z}^n$ and $y_j = d_j \otimes e_j \in (\mathcal{A}_q)_{u_2w_j} \otimes (\Lambda_q)_{w_j}$ for some $w_j \in \mathbb{Z}^n$. It is enough to compare the products of any x_i with any y_j. Hence, there is no loss of generality in assuming that $x = a \otimes b \in (\mathcal{A}_q)_{u_1v_1} \otimes (\Lambda_q)_{v_1}$ and $y = d \otimes e \in (\mathcal{A}_q)_{u_2v_2} \otimes (\Lambda_q)_{v_2}$.

Under the product in $\mathcal{B}'_q$, we have $x'y' = c(u_1, u_2)^{-1}(xy)'$. On the other hand, under the product in $\mathcal{A}'_q \otimes \Lambda'_q$, we have

$$\begin{aligned} x'y' &= (a' \otimes b')(d' \otimes e') = a'd' \otimes b'e' \\ &= \big[c(u_1, u_2)^{-1}c(v_1, v_2)(ad)'\big] \otimes \big[c(v_1, v_2)^{-1}(be)'\big] \\ &= c(u_1, u_2)^{-1}(ad)' \otimes (be)' = c(u_1, u_2)^{-1}(xy)'. \end{aligned}$$

Therefore the two products do coincide, as required. $\square$

3.8. Observe that the k-algebra homomorphism $L_q : \Lambda_q \to \mathcal{A}_q \otimes \Lambda_q$ actually maps Λ_q to $\mathcal{B}_q$. Viewed as a map from Λ_q to $\mathcal{B}_q$, the homomorphism L_q is homogeneous of degree 0 with respect to the $\mathbb{Z}^n$-gradings on these algebras. Since we have twisted both algebras by the same cocycle, namely c^{-1}, we see that L_q induces a k-algebra homomorphism $L'_q : \Lambda'_q \to \mathcal{B}'_q$, where $L'_q(a') = L_q(a)'$ for $a \in \Lambda_q$.

The various k-algebra homomorphisms we have been discussing fit into the following diagram:

$$\begin{tikzcd}
\Lambda_{\boldsymbol{p}} \arrow[rr, "L_{\lambda,\boldsymbol{p}}"] \arrow[d, "\psi"', "\cong"] & & \mathcal{A}_{\lambda,\boldsymbol{p}} \otimes \Lambda_{\boldsymbol{p}} \arrow[d, "\cong"', "\phi\otimes\psi"] \\
\Lambda'_q \arrow[r, "L'_q"] & \mathcal{B}'_q \arrow[r, "\subseteq"] & \mathcal{A}'_q \otimes \Lambda'_q
\end{tikzcd}$$

This diagram commutes, since

$$(\phi \otimes \psi)L_{\lambda,\boldsymbol{p}}(\eta_i) = \sum_j \phi(Y_{ij}) \otimes \psi(\eta_j) = \sum_j X'_{ij} \otimes \xi'_j$$

$$L'_q\psi(\eta_i) = L_q(\xi_i)' = \sum_j X'_{ij} \otimes \xi'_j$$

for all i.

3.9. Proposition. *$\phi(D_{\lambda,\boldsymbol{p}}(I,J)) = D_q(I,J)'$ for all I, J.*

Proof. We first show that $\psi(\eta_H) = \xi'_H$ for all $H \subseteq \{1,\dots,n\}$. This is clear for $|H| \le 1$. If $H = \{h_1 < h_2 < \cdots < h_r\} = \{h_1\} \sqcup J$ for some $r \ge 2$, we may assume by induction that $\psi(\eta_J) = \xi'_J$. Hence,

$$\psi(\eta_H) = \psi(\eta_{h_1}\eta_J) = \xi'_{h_1}\xi'_J = c(\epsilon_{h_1},\, \epsilon_{h_2} + \cdots + \epsilon_{h_r})^{-1}(\xi_{h_1}\xi_J)'.$$

But $c(\epsilon_{h_1},\, \epsilon_{h_2}+\cdots+\epsilon_{h_r}) = c(\epsilon_{h_1},\epsilon_{h_2})c(\epsilon_{h_1},\epsilon_{h_3})\cdots c(\epsilon_{h_1},\epsilon_{h_r}) = 1$ because $h_1 < h_2 < \cdots < h_r$, and so $\psi(\eta_H) = (\xi_{h_1}\xi_J)' = \xi'_H$. This establishes the induction step for our claim.

Now let I be an arbitrary nonempty subset of $\{1,\dots,n\}$. In view of the commutativity of the diagram in (3.8),

$$\begin{aligned}\sum_{|J|=|I|} \phi(D_{\lambda,\boldsymbol{p}}(I,J)) \otimes \xi'_J &= (\phi \otimes \psi)L_{\lambda,\boldsymbol{p}}(\eta_I) = L'_q\psi(\eta_I) \\ &= L_q(\xi_I)' = \sum_{|J|=|I|} D_q(I,J)' \otimes \xi'_J.\end{aligned}$$

Since the ξ'_J are linearly independent, the proposition follows. $\square$

IV. Multiparameter quantum determinantal ideals

Using the twisting method discussed in the previous section, we extend our main result from quantum determinantal ideals in one-parameter quantum matrix algebras $\mathcal{O}_q(M_{m,n}(k))$ to those in multiparameter quantum matrix algebras $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$.

4.1. Let $\mathcal{A}_{\lambda,\boldsymbol{p}} = \mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ be an arbitrary multiparameter quantum matrix algebra over an arbitrary base field k, as in (3.1). Fix $t \in \{1,\dots,n-1\}$, and let $\mathcal{I}_{\lambda,\boldsymbol{p}} = I^{[t]}_{\lambda,\boldsymbol{p}}(M_n(k))$ denote the ideal of $\mathcal{A}_{\lambda,\boldsymbol{p}}$ generated by all $(t+1)\times(t+1)$ quantum minors, i.e., all $D_{\lambda,\boldsymbol{p}}(I,J)$ with $|I| = |J| = t+1$.

Theorem. *$\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))/I^{[t]}_{\lambda,\boldsymbol{p}}(M_n(k))$ is an integral domain.*

Proof. First set $\overline{\mathcal{A}}_{\lambda,\boldsymbol{p}} = \mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(\overline{k}))$ and $\overline{\mathcal{I}}_{\lambda,\boldsymbol{p}} = I^{[t]}_{\lambda,\boldsymbol{p}}(M_n(\overline{k}))$. We identify $\overline{\mathcal{A}}_{\lambda,\boldsymbol{p}}$ with $\mathcal{A}_{\lambda,\boldsymbol{p}} \otimes \overline{k}$. Since the quantum minors in $\mathcal{A}_{\lambda,\boldsymbol{p}}$ and $\overline{\mathcal{A}}_{\lambda,\boldsymbol{p}}$ are the same, $\overline{\mathcal{I}}_{\lambda,\boldsymbol{p}} = \mathcal{I}_{\lambda,\boldsymbol{p}} \otimes \overline{k}$. As a result, $\overline{\mathcal{A}}_{\lambda,\boldsymbol{p}}/\overline{\mathcal{I}}_{\lambda,\boldsymbol{p}} \cong (\mathcal{A}_{\lambda,\boldsymbol{p}}/\mathcal{I}_{\lambda,\boldsymbol{p}}) \otimes \overline{k}$; in particular, $\mathcal{A}_{\lambda,\boldsymbol{p}}/\mathcal{I}_{\lambda,\boldsymbol{p}}$ embeds in $\overline{\mathcal{A}}_{\lambda,\boldsymbol{p}}/\overline{\mathcal{I}}_{\lambda,\boldsymbol{p}}$. Thus it suffices to show that latter algebra is a domain, and hence we may pass to the case where k is algebraically closed.

Now there exists $q \in k^\times$ such that $q^{-2} = \lambda$. Let c be the 2-cocycle defined in (3.4), set $\mathcal{A}_q = \mathcal{O}_q(M_n(k))$, and twist $\mathcal{A}_q$ on the left by c^{-1} and on the right by c as in (3.5). In view of Lemma 3.6 and Proposition 3.9, there is a k-algebra isomorphism $\phi : \mathcal{A}_{\lambda,\boldsymbol{p}} \to \mathcal{A}'_q$ such that $\phi(\mathcal{I}_{\lambda,\boldsymbol{p}}) = \mathcal{I}'_q$, where $\mathcal{I}_q = I^{[t]}_q(M_n(k))$. Thus $\mathcal{A}_{\lambda,\boldsymbol{p}}/\mathcal{I}_{\lambda,\boldsymbol{p}} \cong (\mathcal{A}_q/\mathcal{I}_q)'$, a twist of $\mathcal{A}_q/\mathcal{I}_q$. Since $\mathcal{A}_q/\mathcal{I}_q$ is a domain by Theorem 2.5, it only remains to check that the property of being a domain is preserved in the twist $(\mathcal{A}_q/\mathcal{I}_q)'$.

We may view $\mathcal{A}_q/\mathcal{I}_q$ as graded by $\mathbb{Z}^{2n}$, which can be made into a totally ordered group; then $(\mathcal{A}_q/\mathcal{I}_q)'$ is graded by the same totally ordered group. To see that the product of any two nonero elements of $(\mathcal{A}_q/\mathcal{I}_q)'$ is nonzero, it suffices to show that the product of their highest terms is nonzero. Hence, we just need to show that the product of any two nonzero homogeneous elements $a', b' \in (\mathcal{A}_q/\mathcal{I}_q)'$ is nonzero. But that is clear since $a'b'$ is a nonzero scalar multiple of $(ab)'$, while $ab \neq 0$ because $\mathcal{A}_q/\mathcal{I}_q$ is a domain. Therefore $(\mathcal{A}_q/\mathcal{I}_q)'$ is a domain, as required. $\square$

4.2. Just as in Corollary 2.6, the rectangular case follows directly from Theorem 4.1:

Corollary. *Let m, n, t be positive integers with $t < \min\{m,n\}$, and let $I^{[t]}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$ be the ideal of $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$ generated by all $(t+1)\times(t+1)$ quantum minors. Then $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))/I^{[t]}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$ is an integral domain.* $\square$

4.3. The method of proof used above can also be applied to the other results of Sections I and II. In particular, we obtain a basis of products of quantum minors for $\mathcal{A}_{\lambda,\boldsymbol{p}}$ as follows.

Define preferred bitableaux as in (1.3). For any preferred bitableau

$$(T,T') = \begin{pmatrix} I_1 & J_1 \\ I_2 & J_2 \\ \vdots & \vdots \\ I_l & J_l \end{pmatrix},$$

where $(I_1,J_1) \le (I_2,J_2) \le \cdots \le (I_l,J_l)$ are index pairs, define

$$[T|T']_{\lambda,\boldsymbol{p}} = D_{\lambda,\boldsymbol{p}}(I_1,J_1)D_{\lambda,\boldsymbol{p}}(I_2,J_2)\cdots D_{\lambda,\boldsymbol{p}}(I_l,J_l).$$

Theorem. *The products $[T|T']_{\lambda,\boldsymbol{p}}$, as (T,T') runs over all preferred bitableaux, form a basis for $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$.*

Proof. First note that the symbols $[T|T']_{\lambda,\boldsymbol{p}}$ stand for the same elements in the algebras $\mathcal{A}_{\lambda,\boldsymbol{p}}$ and $\overline{\mathcal{A}}_{\lambda,\boldsymbol{p}} = \mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(\overline{k})) = \mathcal{A}_{\lambda,\boldsymbol{p}} \otimes \overline{k}$. If these elements form a $\overline{k}$-basis for $\overline{\mathcal{A}}_{\lambda,\boldsymbol{p}}$, then they must also form a k-basis for $\mathcal{A}_{\lambda,\boldsymbol{p}}$. Hence, there is no loss of generality in assuming that k is algebraically closed.

Now choose $q \in k^\times$ such that $q^{-2} = \lambda$, and twist $\mathcal{A}_q$ as in (3.5). In view of Lemma 3.6 and Proposition 3.9, there is a k-algebra isomorphism $\phi : \mathcal{A}_{\lambda,\boldsymbol{p}} \to \mathcal{A}'_q$ such that $\phi([T|T']_{\lambda,\boldsymbol{p}})$ is a nonzero scalar multiple of $[T|T']'$ for all preferred bitableaux (T,T'). Since the products $[T|T']'$ form a basis for $\mathcal{A}'_q$ by Corollary 1.10, the theorem follows. $\square$

V. Appendix: Commutation Relations

We derive some commutation relations for quantum minors in $\mathcal{O}_q(M_n(k))$, expressed using the notation and conventions of (1.1)–(1.4).

5.1. We begin by restating some identities from [**13**], given there for generators and maximal minors, in a form that applies to minors of arbitrary size. Note the difference between our choice of relations for $\mathcal{O}_q(M_n(k))$ (see (1.1)) and that in [**13**, p. 37]. Because of this, we must interchange q and q^{-1} whenever carrying over a formula from [**13**].

Lemma. *Let $r,c \in \{1,\dots,n\}$ and $I,J \subseteq \{1,\dots,n\}$ with $|I| = |J| \geq 1$.*

(a) *If $r \in I$ and $c \in J$, then $X_{rc}[I|J] = [I|J]X_{rc}$.*

(b) *If $r \in I$ and $c \notin J$, then*

$$X_{rc}[I|J] - q^{-1}[I|J]X_{rc} = (q^{-1}-q)\sum_{\substack{j\in J\\ j>c}} (-q)^{-|J\cap[c,j]|}[I|J\cup\{c\}\setminus\{j\}]X_{rj}.$$

(c) *If $r \notin I$ and $c \in J$, then*

$$X_{rc}[I|J] - q[I|J]X_{rc} = (q-q^{-1})\sum_{\substack{i\in I\\ i<r}} (-q)^{|I\cap[i,r]|}[I\cup\{r\}\setminus\{i\}|J]X_{ic}.$$

(d) *If $r \notin I$ and $c \notin J$, then*

$$X_{rc}[I|J] - [I|J]X_{rc} = (1-q^{-2})\left(\sum_{\substack{i\in I\\ i<r}} (-q)^{|I\cap[i,r]|}[I\cup\{r\}\setminus\{i\}|J]X_{ic} - \sum_{\substack{j\in J\\ j>c}} (-q)^{|J\cap[c,j]|}X_{rj}[I|J\cup\{c\}\setminus\{j\}]\right).$$

Proof. Let $t = |I| = |J|$.

(a) There is a k-algebra isomorphism

$$\mathcal{O}_q(M_t(k)) \xrightarrow{\cong} k\langle X_{ij} \mid i \in I,\ j \in J\rangle \subseteq \mathcal{O}_q(M_n(k))$$

which sends the quantum determinant of $\mathcal{O}_q(M_t(k))$ to $[I|J]$. Since the quantum determinant is central in $\mathcal{O}_q(M_t(k))$, part (a) follows.

(b) Pick $r_0 \in \{1,\dots,n\} \setminus I$. Set $\overline{I} = I \cup \{r_0\}$ and $\overline{J} = J \cup \{c\}$, and label the elements of these sets in ascending order, say

$$\overline{I} = \{i_1 < i_2 < \cdots < i_{t+1}\} \qquad \text{and} \qquad \overline{J} = \{j_1 < j_2 < \cdots < j_{t+1}\}.$$

There exists a k-algebra embedding $\phi : \mathcal{O}_q(M_{t+1}(k)) \to \mathcal{O}_q(M_n(k))$ such that $\phi(X_{ab}) = X_{i_a j_b}$ for $a,b = 1,\dots,t+1$. Let ρ,γ,σ be the indices such that $i_\rho = r$, $j_\gamma = c$, and $i_\sigma = r_0$. Then $\phi(X_{\rho\gamma}) = X_{rc}$ and $\phi(A(\sigma\ \gamma)) = [I|J]$, where $A(\sigma\ \gamma)$ is (in the notation of [**13**]) the $t \times t$ quantum minor in $\mathcal{O}_q(M_{t+1}(k))$ obtained by deleting the σ-th row and γ-th column.

By the second part of [**13**, 4.5.1(2)],

$$X_{\rho\gamma} A(\sigma\ \gamma) - q^{-1} A(\sigma\ \gamma) X_{\rho\gamma} = (q^{-1} - q) \sum_{\delta > \gamma} (-q)^{\gamma-\delta} A(\sigma\ \delta) X_{\rho\delta}. \tag{†}$$

Note that $\delta - \gamma = |\overline{J} \cap (c, j_\delta]| = |J \cap [c, j_\delta]|$ for $\delta = \gamma+1,\dots,t+1$. Thus, part (b) results from applying ϕ to (†).

(c)(d) These follow, in the same manner, from the first part of [**13**, 4.5.1(4)] and the first part of [**13**, 5.1.2], respectively. $\square$

5.2. Corollary. *Let $r,c \in \{1,\dots,n\}$ and $I,J \subseteq \{1,\dots,n\}$ with $|I| = |J|$. Then the term*

$$Y := X_{rc}[I|J] - q^{\delta(c,J)-\delta(r,I)}[I|J]X_{rc}$$

is a linear combination of terms $[I'|J']X_{ij}$, with the same bidegree as $X_{rc}[I|J]$, such that $|I'| = |I|$ and $(I',J') < (I,J)$.

Proof. We allow the trivial case $I = J = \varnothing$ for completeness. Now assume that $|I| = |J| \geq 1$. The cases in which $r \in I$ or $c \in J$ (or both) are clear from the first three parts of Lemma 5.1. Hence, we may assume that $r \notin I$ and $c \notin J$.

If the corollary fails, we may suppose that we have a counterexample in which J is minimal with respect to $\leq_c$. By Lemma 5.1(d), Y is a linear combination of

(i) Terms $[I'|J]X_{ic}$ of the desired form, and

(ii) Terms $X_{rj}[I|J \cup \{c\} \setminus \{j\}]$ with $j \in J$ and $j > c$.

Note that the terms in (ii) have the form $X_{rj}[I|J']$ with the same bidegree as $X_{rc}[I|J]$, and with $J' <_c J$. By the minimality of J, each term in (ii) equals $[I|J']X_{rj}$ plus a linear combination of terms $[I''|J'']X_{st}$, with the same bidegree as $X_{rj}[I|J']$, such that $|I''| = |I|$ and $(I'',J'') < (I,J') < (I,J)$. But if these expressions for the terms in (ii) are substituted in our initial expression for Y, we have written Y in the desired form, contradicting the assumption of a counterexample.

Therefore the corollary holds. $\square$

5.3. Proposition. *Let $R, C, I, J \subseteq \{1, \dots, n\}$ with $|R| = |C|$ and $|I| = |J|$. If M is any element of $\mathcal{O}_q(M_n(k))$ of bidegree (χ_R, χ_C), then the term*

$$Z := M[I|J] - q^{|C\cap J|-|R\cap I|}[I|J]M$$

is a linear combination of terms $[I'|J']M'$ such that
(a) *$[I'|J']M'$ has the same bidegree as $M[I|J]$;*
(b) *M' is a monomial of length $|R|$;*
(c) *$|I'| = |I|$ and $(I', J') < (I, J)$.*
In particular, this holds for $M = [R|C]$.

Proof. The proposition holds trivially if either R, C or I, J are empty. Now assume that R, C, I, J are all nonempty. Write M as a linear combination of monomials M_l with length $|R|$ and bidegree (χ_R, χ_C). If each of the terms

$$Z_l := M_l[I|J] - q^{|C\cap J|-|R\cap I|}[I|J]M_l$$

is a linear combination of terms $[I'|J']M'$ satisfying (a),(b),(c), then so is Z. Thus, we may assume that M is a monomial.

We now induct on the length of M, namely $|R|$. The case $|R| = 1$ is given by Corollary 5.2.

If $|R| > 1$, write $M = X_{rc}N$ for some r, c and some monomial N of length $|R|-1$. Note that N has bidegree (χ_Q, χ_B) where $Q = R \setminus \{r\}$ and $B = C \setminus \{c\}$. By induction,

$$N[I|J] = q^{|B\cap J|-|Q\cap I|}[I|J]N + \text{lin. comb. of terms } [I_1|J_1]N_1 \tag{1}$$

such that
- $[I_1|J_1]N_1$ has the same bidegree as $N[I|J]$;
- N_1 is a monomial of length $|R| - 1$;
- $|I_1| = |I|$ and $(I_1, J_1) < (I, J)$.

Multiplying (1) on the left by X_{rc}, we obtain

$$M[I|J] = q^{|B\cap J|-|Q\cap I|}X_{rc}[I|J]N + \text{lin. comb. of terms } X_{rc}[I_1|J_1]N_1. \tag{*}$$

Next, apply Corollary 5.2 to both $X_{rc}[I|J]$ and $X_{rc}[I_1|J_1]$. In the first case,

$$X_{rc}[I|J] = q^{\delta(c,J)-\delta(r,I)}[I|J]X_{rc} + \text{lin. comb. of terms } [I_2|J_2]X_{ij} \tag{2}$$

such that $[I_2|J_2]X_{ij}$ has the same bidegree as $X_{rc}[I|J]$, while $|I_2| = |I|$ and $(I_2, J_2) < (I, J)$. Since

$$|B \cap J| - |Q \cap I| + \delta(c, J) - \delta(r, I) = |C \cap J| - |R \cap I|,$$

it follows that

$$q^{|B\cap J|-|Q\cap I|}X_{rc}[I|J]N = q^{|C\cap J|-|R\cap I|}[I|J]M + \text{lin. comb. of terms } [I_2|J_2]X_{ij}N. \tag{\dagger}$$

In the second case, for each term $X_{rc}[I_1|J_1]$ we have an expression of the following type, where we incorporate the $[I_1|J_1]X_{rc}$ term with the remaining terms:

$$X_{rc}[I_1|J_1] = \text{lin. comb. of terms } [I_3|J_3]X_{st} \tag{3}$$

such that $[I_3|J_3]X_{st}$ has the same bidegree as $X_{rc}[I_1|J_1]$, while $|I_3| = |I_1| = |I|$ and $(I_3, J_3) \leq (I_1, J_1) < (I, J)$. Consequently,

$$X_{rc}[I_1|J_1]N_1 = \text{lin. comb. of terms } [I_3|J_3]X_{st}N_1. \tag{‡}$$

Finally, substitute (†) and (‡) in (*), which yields

$$M[I|J] = q^{|C\cap J|-|R\cap I|}[I|J]M + \text{lin. comb. of terms } [I_2|J_2]X_{ij}N \text{ and } [I_3|J_3]X_{st}N_1. \tag{**}$$

Observe that the terms $[I_2|J_2]X_{ij}N$ and $[I_3|J_3]X_{st}N_1$ have the same bidegree as $M[I|J]$, and that the terms $X_{ij}N$ and $X_{st}N_1$ are monomials of length $|R|$. We already have $|I_2| = |I_3| = |I|$ while $(I_2, J_2) < (I, J)$ and $(I_3, J_3) < (I, J)$. Therefore (**) gives us the desired relation. □

VI. Appendix: Laplace and exchange relations

We adapt some of the relations derived in [**12**]. (Although the base field is taken to be $\mathbb{C}$ in that paper, the arguments are valid over any field.) For subsets $I, J \subseteq \{1, \dots, n\}$, set

$$\ell(I; J) = |\{(i, j) \in I \times J \mid i > j\}|.$$

In the following formulas, we use $\sqcup$ to denote disjoint unions. Notation and conventions from (1.1)–(1.4) are again in force.

6.1. Lemma. (Laplace expansions) *Let $I, J \subseteq \{1, \dots, n\}$ with $|I| = |J|$.*

(a) *If $J = J_1 \sqcup J_2$, then*

$$[I|J] = (-q)^{-\ell(J_1;J_2)} \sum_{\substack{I_1 \sqcup I_2 = I \\ |I_1|=|J_1|}} (-q)^{\ell(I_1;I_2)}[I_1|J_1][I_2|J_2].$$

(b) *If $I = I_1 \sqcup I_2$, then*

$$[I|J] = (-q)^{-\ell(I_1;I_2)} \sum_{\substack{J_1 \sqcup J_2 = J \\ |J_1|=|I_1|}} (-q)^{\ell(J_1;J_2)}[I_1|J_1][I_2|J_2].$$

Proof. The nontrivial cases ($J_1, J_2 \neq \varnothing$ in (a), and $I_1, I_2 \neq \varnothing$ in (b)) are given in [**12**, Proposition 1.1]. The trivial cases are clear. □

6.2. The sums in the next formulas run over certain partitions of index sets; we take these sums to run over only those partitions for which the terms in the formulas are defined. For instance, in part (a) the only allowable partitions $K' \sqcup K'' = K$ are those for which $J_1 \cap K' = J_2 \cap K'' = \varnothing$ while $|J_1| + |K'| = |I_1|$ and $|K''| + |J_2| = |I_2|$.

Observe that in each formula, all terms on both sides of the equation have the same bidegree.

Proposition. (Exchange formulas) *Let $I_1, I_2, J_1, J_2, K \subseteq \{1, \dots, n\}$.*

(a) *If $|J_\nu| \le |I_\nu|$ and $|K| = |I_1| + |I_2| - |J_1| - |J_2|$, then*

$$(*) \qquad \sum_{K' \sqcup K'' = K} (-q)^{\ell(J_1;K') + \ell(K';K'') + \ell(K'';J_2)} [I_1 | J_1 \sqcup K'][I_2 | K'' \sqcup J_2]$$
$$= \sum_{I'_\nu \sqcup I''_\nu = I_\nu} (-q)^{\ell(I'_1;I''_1) + \ell(I''_1;I''_2) + \ell(I''_2;I'_2)} [I'_1 | J_1][I''_1 \sqcup I''_2 | K][I'_2 | J_2].$$

(b) *If $|I_\nu| \le |J_\nu|$ and $|K| = |J_1| + |J_2| - |I_1| - |I_2|$, then*

$$\sum_{K' \sqcup K'' = K} (-q)^{\ell(I_1;K') + \ell(K';K'') + \ell(K'';I_2)} [I_1 \sqcup K' | J_1][K'' \sqcup I_2 | J_2]$$
$$= \sum_{J'_\nu \sqcup J''_\nu = J_\nu} (-q)^{\ell(J'_1;J''_1) + \ell(J''_1;J''_2) + \ell(J''_2;J'_2)} [I_1 | J'_1][K | J''_1 \sqcup J''_2][I_2 | J'_2].$$

Proof. (a) The case in which $1 \le |J_\nu| < |I_\nu|$ is given in the proof of [**12**, Proposition 1.2]; our version of this case includes only the terms with nonzero coefficients. The same proof yields the general case, as follows. First expand the left hand side of (*) by applying Lemma 6.1(a) to both $[I_1 | J_1 \sqcup K']$ and $[I_2 | K'' \sqcup J_2]$. This yields

$$(\dagger) \qquad \sum_{\substack{K' \sqcup K'' = K \\ I'_1 \sqcup I''_1 = I_1 \\ I''_2 \sqcup I'_2 = I_2}} (-q)^{\ell(I'_1;I''_1) + \ell(I''_2;I'_2) + \ell(K';K'')} [I'_1 | J_1][I''_1 | K'][I''_2 | K''][I'_2 | J_2].$$

We can also expand the right hand side of (*) by applying Lemma 6.1(b) to $[I''_1 \sqcup I''_2 | K]$. Since this also yields (†), part (a) is proved.

(b) This is proved in the same manner. □

6.3. Note that if $|I_1 \cup I_2| < |K|$ in Proposition 6.2, there do not exist disjoint subsets $I''_\nu \subseteq I_\nu$ such that $|I''_1 \sqcup I''_2| = |K|$, and so the right hand side of formula (a) is zero. Similarly, if $|J_1 \cup J_2| < |K|$, the right hand side of (b) is zero. These cases are called *generalized Plücker relations* [**12**, Proposition 1.2].

Acknowledgment

We thank James Zhang for several very helpful conversations, and Jacques Alev and Laurent Rigal for useful comments.

References

1. M. Artin, W. Schelter, and J. Tate, *Quantum deformations of GL_n*, Communic. Pure Appl. Math. **44** (1991), 879-895.

2. K. A. Brown and K. R. Goodearl, *Prime spectra of quantum semisimple groups*, Trans. Amer. Math. Soc **348** (1996), 2465-2502.

3. W. Bruns and U. Vetter, *Determinantal Rings*, Lecture Notes in Math. 1327, Springer-Verlag, Berlin, 1988.

4. C. DeConcini, D. Eisenbud, and C. Procesi, *Young diagrams and determinantal varieties*, Inventiones Math. **56** (1980), 129-165.

5. W. Fulton, *Young Tableaux*, Cambridge Univ. Press, Cambridge, 1997.

6. T. J. Hodges and T. Levasseur, *Primitive ideals of $\mathbb{C}_q[SL(n)]$*, J. Algebra **168** (1994), 455-468.

7. R. Q. Huang and J. J. Zhang, *Standard basis theorems for quantum linear groups*, Advances in Math. **102** (1993), 202-229.

8. D. A. Jordan, Notes, 1993.

9. A. Joseph, *Quantum Groups and Their Primitive Ideals*, Springer-Verlag, Berlin, 1995.

10. T. Levasseur and J. T. Stafford, *The quantum coordinate ring of the special linear group*, J. Pure Appl. Algebra **86** (1993), 181-186.

11. Yu. I. Manin, *Quantum Groups and Noncommutative Geometry*, Publ. Centre de Recherches Math., Univ. de Montreal, 1988.

12. M. Noumi, H. Yamada, and K. Mimachi, *Finite dimensional representations of the quantum group $GL_q(n;\mathbb{C})$ and the zonal spherical functions on $U_q(n-1)\backslash U_q(n)$*, Japanese J. Math. **19** (1993), 31-80.

13. B. Parshall and J.-P. Wang, *Quantum linear groups*, Memoirs Amer. Math. Soc. **89** (1991).

14. L. Rigal, *Normalité de certains anneaux déterminantiels quantiques*, Proc. Edinburgh Math. Soc. (to appear).

Department of Mathematics, University of California, Santa Barbara, CA 93106, USA
E-mail address: goodearl@math.ucsb.edu

Department of Mathematics, J.C.M.B., Kings Buildings, Mayfield Road, Edinburgh EH9 3JZ, Scotland
E-mail address: tom@maths.ed.ac.uk