PRIME SPECTRA OF QUANTIZED COORDINATE RINGS

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This paper is partly a report on current knowledge concerning the structure of (generic) quantized coordinate rings and their prime spectra, and partly propaganda in support of the conjecture that since these algebras share many common properties, there must be a common basis on which to treat them. The first part of the paper is expository. We survey a number of classes of quantized coordinate rings, as well as some related algebras that share common properties, and we record some of the basic properties known to occur for many of these algebras, culminating in stratifications of the prime spectra by the actions of tori of automorphisms. As our main interest is in the generic case, we assume various parameters are not roots of unity whenever convenient. In the second part of the paper, which is based on [**20**], we offer some support for the conjecture above, in the form of an axiomatic basis for the observed stratifications and their properties. At present, the existence of a suitable supply of normal elements is taken as one of the axioms; the search for better axioms that yield such normal elements is left as an open problem.

I. Quantized Coordinate Rings and Related Algebras

This part of the paper is an expository account of the prime ideal structure of algebras on the "quantized coordinate ring" side of the theory of quantum groups – quantizations of the coordinate rings of affine spaces, matrices, semisimple groups, symplectic and Euclidean spaces, as well as a few related algebras – quantized enveloping algebras of Borel and nilpotent subalgebras of semisimple Lie algebras, and quantized Weyl algebras. These algebras occur widely throughout the quantum groups literature, and different papers often investigate different versions. Thus, we begin by giving definitions of the most general versions of which we are aware; in quoting results from the literature, we will specify which version is under consideration. The

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reader should bear in mind that many authors, when studying one version of a quantized coordinate algebra, use results proved for slightly different versions, on the understanding that the proofs carry over. This is especially prevalent with regard to quantized coordinate rings of semisimple groups; a detailed development covering the most general case would be a welcome addition to the literature.

Fix a base field k throughout. It need not be algebraically closed, and may have arbitrary characteristic.

1. Descriptions

This section is designed to be a reference source for descriptions of the main classes of the standard quantized coordinate rings currently studied, as well as some related algebras with similar properties. (The reader lacking a strong stomach for generators and relations should just skim this section and refer back to it as necessary.) For the sake of uniformity, and to emphasize that these algebras are deformations of classical coordinate rings, we label all quantized coordinate rings using notations of the form $\mathcal{O}_\bullet(\mathcal{C})$, where $\bullet$ records one or more parameters and $\mathcal{C}$ records the name of the classical object. Thus, $\mathcal{O}_q(k^n)$ refers to a one-parameter quantization of the coordinate ring of affine n-space over a field k, while $\mathcal{O}_{\lambda,\boldsymbol{p}}(GL_n(k))$ refers to a multiparameter quantization of the coordinate ring of $GL_n(k)$, and so on. Many different labels are used throughout the literature for these algebras, and we do not attempt to list the alternates here.

1.1. Quantum affine spaces. These are meant to be viewed as deformations of the coordinate rings $\mathcal{O}(k^n)$. Recall that $\mathcal{O}(k^n)$ is a (commutative) polynomial ring $k[x_1, \dots, x_n]$, where x_i is the i-th coordinate function on k^n. In the present case, one deforms $\mathcal{O}(k^n)$ in a rather straightforward manner, by altering the commutativity relations $x_ix_j = x_jx_i$ to "commutativity up to scalars", that is, $x_ix_j = q_{ij}x_jx_i$. In order to prevent degeneracy, one needs some assumptions on the q_{ij}. In particular, they should be nonzero, and q_{ji} should equal q_{ij}^{-1} in order to prevent the pair of relations $x_ix_j = q_{ij}x_jx_i$ and $x_jx_i = q_{ji}x_ix_j$ from implying $x_ix_j = 0$. Thus, one assumes that $\boldsymbol{q} = (q_{ij})$ is a *multiplicatively antisymmetric* $n\times n$ matrix over k, that is, $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i, j. Given $\boldsymbol{q}$, the corresponding *multiparameter coordinate ring of quantum affine n-space over k* is the k-algebra $\mathcal{O}_{\boldsymbol{q}}(k^n)$ generated by elements $x_1, \dots, x_n$ subject only to the relations $x_ix_j = q_{ij}x_jx_i$ for $i, j = 1, \dots, n$. The standard single parameter version occurs when the q_{ij} for $i < j$ are all equal to a fixed nonzero scalar q; in this case, we denote the algebra $\mathcal{O}_q(k^n)$.

Quantum affine spaces occur already in the work of Manin [**37**, Section 1, §2 and Section 4, §5]; a superalgebra version is discussed in [**38**, §1.4].

1.2. Quantum matrices. The set $M_n(k)$ of $n\times n$ matrices over k, viewed as an algebraic variety, is just affine n^2-space, and its coordinate ring $\mathcal{O}(M_n(k))$ is a polynomial ring in n^2 indeterminates. Hence, one might expect to deform this algebra exactly as in (1.1). However, there is more structure in this case, and one seeks to preserve it as far as possible. Namely, $\mathcal{O}(M_n(k))$ is a bialgebra with a comultiplication $\Delta : \mathcal{O}(M_n(k)) \to \mathcal{O}(M_n(k)) \otimes \mathcal{O}(M_n(k))$ which is effectively the transpose of the map $M_n(k) \times M_n(k) \to M_n(k)$ given by matrix multiplication: If X_{ij} denotes the i,j-th coordinate function on $M_n(k)$, then $\Delta(X_{ij}) = \sum_{l=1}^n X_{il} \otimes X_{lj}$. In particular, one would like deformations of $\mathcal{O}(M_n(k))$, with appropriate sets of generators X_{ij}, which are also bialgebras in which the comultiplication of the X_{ij} is given by the equation above.

In addition, multiplication of matrices with row or column vectors from k^n induces on $\mathcal{O}(k^n)$ structures of left and right comodule over $\mathcal{O}(M_n(k))$, and one would like some quantum affine spaces $\mathcal{O}_{\boldsymbol{q}}(k^n)$ to be left and right comodules over the deformation of $\mathcal{O}(M_n(k))$, with comodule structure maps behaving as in the classical case. In the single parameter case, there is a deformation $\mathcal{O}_q(M_n(k))$ over which $\mathcal{O}_q(k^n)$ becomes both a left and a right comodule in the desired manner (see [**38**] and [**52**]). However, if one tries to make a multiparameter quantum affine space $\mathcal{O}_{\boldsymbol{q}}(k^n)$ into a left <u>and</u> right comodule over the desired type of deformation of $\mathcal{O}(M_n(k))$, the latter deformation will usually be degenerate. To obtain nondegenerate deformations, one must use different multiparameter quantum affine spaces as left and right comodules, as discovered in [**2**] and [**57**], to which we refer the reader for a more complete discussion. The resulting algebras can be described as follows.

Let $\boldsymbol{p} = (p_{ij})$ be a multiplicatively antisymmetric $n \times n$ matrix over k, and let λ be a nonzero element of k not equal to -1. The corresponding *multiparameter coordinate ring of quantum $n \times n$ matrices over k* is the k-algebra $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ generated by elements X_{ij} (for $i,j = 1,\dots,n$) subject only to the following relations:

$$X_{\ell m}X_{ij} = \begin{cases} p_{\ell i}p_{jm}X_{ij}X_{\ell m} + (\lambda-1)p_{\ell i}X_{im}X_{\ell j} & (\ell > i,\ m > j) \\ \lambda p_{\ell i}p_{jm}X_{ij}X_{\ell m} & (\ell > i,\ m \le j) \\ p_{jm}X_{ij}X_{\ell m} & (\ell = i,\ m > j). \end{cases}$$

(We assume $\lambda \neq 0$ to avoid obvious degeneracies in the second relation above. The assumption $\lambda \neq -1$ is needed to ensure that $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ has the same Hilbert series as a commutative polynomial algebra in n^2 indeterminates [**2**, Theorem 1].) The standard single parameter version, denoted $\mathcal{O}_q(M_n(k))$, occurs when the p_{ij} for $i > j$ are all equal to a fixed nonzero scalar q, and $\lambda = q^{-2}$. In some references, such as [**49**] and [**56**], the roles of q and q^{-1} are interchanged, and/or q is squared.

Quantized coordinate rings for rectangular matrices are defined as subalgebras of those for square matrices. For instance, if $m < n$ then $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$ is defined to be the k-subalgebra of $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ generated by those X_{ij} with $i \le m$. There is a k-algebra retraction of $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ onto $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$ whose kernel is generated by the X_{ij} with $i > m$; thus $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k)) \cong \mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))/\langle X_{ij} \mid i > m\rangle$. Hence, results for $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ easily extend to the rectangular case; we shall so extend results from the literature without explicit mention.

1.3. Quantum general linear groups. Within the algebraic variety $M_n(k)$, the set $GL_n(k)$ of invertible matrices forms an open subvariety, the complement of the variety defined by the vanishing of the determinant function D. The coordinate ring $\mathcal{O}(GL_n(k))$ thus has the form $\mathcal{O}(M_n(k))[D^{-1}]$. To construct a quantum analog, one inverts a "quantum determinant", call it $D_{\lambda,\boldsymbol{p}}$, in $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$. In order for the inversion process to work smoothly and avoid degeneracies, the powers of $D_{\lambda,\boldsymbol{p}}$ should form an Ore set. This occurs because the chosen $D_{\lambda,\boldsymbol{p}}$ turns out to be a normal element. (Recall that a *normal element* in a ring R is an element c such that $cR = Rc$.)

As in the classical case, the quantum determinant $D_{\lambda,\boldsymbol{p}}$ can be defined as a linear combination of products $X_{1,\pi(1)}X_{2,\pi(2)}\cdots X_{n,\pi(n)}$ as π runs through the symmetric group S_n. However, the coefficients are taken to be certain products of elements $-p_{ij}$ rather than powers of -1. Specifically,

$$D_{\lambda,\boldsymbol{p}} = \sum_{\pi\in S_n} \Bigg(\prod_{\substack{1\le i<j\le n \\ \pi(i)>\pi(j)}} (-p_{\pi(i),\pi(j)}) \Bigg) X_{1,\pi(1)}X_{2,\pi(2)}\cdots X_{n,\pi(n)}.$$

The motivation for this choice of $D_{\lambda,\boldsymbol{p}}$ comes from the construction of a quantized version of the exterior algebra on k^n; the n-th "quantum exterior power" of k^n is 1-dimensional, spanned by $D_{\lambda,\boldsymbol{p}}$ (see [**2**] for details). In the single parameter case, the quantum determinant D_q in $\mathcal{O}_q(M_n(k))$ can be expressed as

$$D_q = \sum_{\pi\in S_n} (-q)^{\ell(\pi)} X_{1,\pi(1)}X_{2,\pi(2)}\cdots X_{n,\pi(n)},$$

where $\ell(\pi)$ denotes the *length* of the permutation π, that is, the minimum length of an expression for π as a product of adjacent transpositions $(i, i+1)$. (Cf. [**49**, Lemma 4.1.1] but interchange q and q^{-1}.)

It has been computed that $D_{\lambda,\boldsymbol{p}}$ is a normal element of $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ [**2**, Theorem 3]; in fact,

$$D_{\lambda,\boldsymbol{p}} X_{ij} = \lambda^{j-i} \left(\prod_{l=1}^{n} p_{jl} p_{li} \right) X_{ij} D_{\lambda,\boldsymbol{p}}$$

for all i, j. In the single parameter case, D_q is central (e.g., [**49**, Theorem 4.6.1]). It follows that the set of nonnegative powers of $D_{\lambda,\boldsymbol{p}}$ is a right and left Ore set in $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$. The *multiparameter coordinate ring of quantum* $GL_n(k)$ is now defined to be the localization $\mathcal{O}_{\lambda,\boldsymbol{p}}(GL_n(k)) = \mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))[D_{\lambda,\boldsymbol{p}}^{-1}]$; in the single parameter case, $\mathcal{O}_q(M_n(k))[D_q^{-1}]$ is denoted $\mathcal{O}_q(GL_n(k))$.

1.4. Quantum special linear groups. Note that the set $SL_n(k)$ of $n \times n$ matrices with determinant 1 forms a closed subvariety of $M_n(k)$, defined by the single equation $D = 1$. The coordinate ring $\mathcal{O}(SL_n(k))$ thus has the form $\mathcal{O}(M_n(k))/(D-1)$. To construct a quantum analog, we would thus factor out from $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ the ideal generated by $D_{\lambda,\boldsymbol{p}} - 1$. However, unless $D_{\lambda,\boldsymbol{p}}$ is central, such a factor would be degenerate. Hence, quantum special linear groups are only defined for special choices of the parameters.

From the normality relations for $D_{\lambda,\boldsymbol{p}}$, we see that $D_{\lambda,\boldsymbol{p}}$ is central if and only if

$$\lambda^i \prod_{l=1}^{n} p_{il} = \lambda^j \prod_{l=1}^{n} p_{jl}$$

for all i, j. In this case, the *multiparameter coordinate ring of quantum* $SL_n(k)$ is defined to be the factor algebra

$$\mathcal{O}_{\lambda,\boldsymbol{p}}(SL_n(k)) = \mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))/\langle D_{\lambda,\boldsymbol{p}} - 1 \rangle.$$

In the single parameter case, where D_q is automatically central, the factor $\mathcal{O}_q(M_n(k))/\langle D_q - 1 \rangle$ is denoted $\mathcal{O}_q(SL_n(k))$.

1.5. Quantum semisimple groups. A systematic development of quantized coordinate rings for simple algebraic groups corresponding to the Dynkin diagrams A_n, B_n, C_n, D_n was given by Reshetikhin, Takhtadzhyan, and Fadeev in [**53**]. These algebras were expressed in terms of the entries of certain "R-matrices" attached to quantizations of the corresponding simple Lie algebras. Calculations of generators and relations for these quantized coordinate rings can be found in, e.g., [**58**]. Quantized coordinate rings for the exceptional groups, however, were not computed, due to the lack of explicit R-matrices in those cases (cf. [**53**, Remark 14, p. 212]; the G_2 case is considered in [**55**]). The now standard approach to these algebras involves a quantization of the classical duality between the coordinate ring of a semisimple algebraic group and the enveloping algebra of its Lie algebra. Thus, one constructs quantized enveloping algebras of semisimple Lie algebras and *defines* the quantized coordinate rings of the corresponding semisimple groups as certain Hopf algebra duals. We outline this procedure.

(a) Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let G be a connected complex semisimple algebraic group with Lie algebra $\mathfrak{g}$. Both $\mathfrak{g}$ and G play symbolic roles; they serve mainly as suggestive labels. The only data needed from these objects are: a (symmetrized) Cartan matrix $(d_i a_{ij})$ for $\mathfrak{g}$; root and weight lattices $Q \subseteq P$ corresponding to choices of Cartan subalgebra $\mathfrak{h}$ and root system for $\mathfrak{g}$; a lattice L lying between Q and P corresponding to the character group of a maximal torus of G with Lie algebra $\mathfrak{h}$; choices of simple roots $\alpha_1, \dots, \alpha_n$ and fundamental weights $\omega_1, \dots, \omega_n$; and the (unique) bilinear pairing $(-,-) : P \times Q \to \mathbb{Z}$ such that $(\omega_i, \alpha_j) = \delta_{ij} d_i$ for all i, j. There is a unique extension of $(-,-)$ to a symmetric bilinear form $P \times P \to \mathbb{Q}$, which we also denote by $(-,-)$.

Different authors base quantized coordinate algebras of G on different quantizations of the enveloping algebra $U(\mathfrak{g})$. Thus we first describe a general one-parameter quantization of $U(\mathfrak{g})$, as in [**11**, §0.3]. Let M be a lattice lying between Q and P (independent of L). Then let $q \in k$ be a nonzero scalar, set $q_i = q^{d_i}$ for $i = 1, \dots, n$, and assume that these $q_i \neq \pm 1$. (In much of the literature on quantized enveloping algebras, k is taken to be $\mathbb{C}$, or a rational function field $\mathbb{C}(q)$ or $\mathbb{Q}(q)$, or the algebraic closure of one of these rational function fields.) Let U^0 be a copy of the group algebra kM, written as the k-algebra with basis $\{k_\lambda \mid \lambda \in M\}$ where $k_0 = 1$ and $k_\lambda k_\mu = k_{\lambda+\mu}$ for $\lambda, \mu \in M$. The *single parameter quantized enveloping algebra of* $\mathfrak{g}$ associated with the above choices is the k-algebra $U_q(\mathfrak{g}, M)$ generated by U^0 and elements $e_1, \dots, e_n$, $f_1, \dots, f_n$ satisfying the following relations for $\lambda \in M$ and $i, j = 1, \dots, n$:

$$k_\lambda e_i k_\lambda^{-1} = q^{(\lambda,\alpha_i)} e_i \qquad \text{and} \qquad k_\lambda f_i k_\lambda^{-1} = q^{-(\lambda,\alpha_i)} f_i$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_{\alpha_i} - k_{\alpha_i}^{-1}}{q_i - q_i^{-1}}$$

$$\sum_{l=1}^{1-a_{ij}} (-1)^l \binom{1-a_{ij}}{l}_{q_i} e_i^{1-a_{ij}-l} e_j e_i^l = 0 \qquad (i \neq j)$$

$$\sum_{l=1}^{1-a_{ij}} (-1)^l \binom{1-a_{ij}}{l}_{q_i} f_i^{1-a_{ij}-l} f_j f_i^l = 0 \qquad (i \neq j),$$

where $\binom{1-a_{ij}}{l}_{q_i}$ is a q_i-binomial coefficient. The most commonly studied case is $U_q(\mathfrak{g}, Q)$, denoted just $U_q(\mathfrak{g})$. This algebra is generated by e_i, f_i, k_i for $i = 1, \dots, n$, where $k_i = k_{\alpha_i}$. At the other extreme, $U_q(\mathfrak{g}, P)$ is often denoted $\check{U}_q(\mathfrak{g})$.

The algebra $U_q(\mathfrak{g}, M)$ is in fact a Hopf algebra, with comultiplication Δ, counit ϵ, and antipode S such that

$$\begin{aligned} \Delta(k_\lambda) &= k_\lambda \otimes k_\lambda & \epsilon(k_\lambda) &= 1 & S(k_\lambda) &= k_\lambda^{-1} \\ \Delta(e_i) &= e_i \otimes 1 + k_{\alpha_i} \otimes e_i & \epsilon(e_i) &= 0 & S(e_i) &= -k_{\alpha_i}^{-1} e_i \\ \Delta(f_i) &= f_i \otimes k_{\alpha_i}^{-1} + 1 \otimes f_i & \epsilon(f_i) &= 0 & S(f_i) &= -f_i k_{\alpha_i}. \end{aligned}$$

Hence, one can define the *finite dual* $U_q(\mathfrak{g}, M)^\circ$ as in, e.g., [**41**, Definition 1.2.3]; this is a k-linear subspace of $U_q(\mathfrak{g}, M)^*$ which becomes a Hopf algebra using the transposes of the multiplication, comultiplication, and antipode of $U_q(\mathfrak{g}, M)$ (e.g., [**41**, Theorem 9.1.3]).

(b) Single-parameter quantizations of $\mathcal{O}(G)$ are defined as subalgebras of $U_q(\mathfrak{g}, M)^\circ$ generated by the "coordinate functions" of certain "highest weight" $U_q(\mathfrak{g}, M)$-modules. It turns out that the resulting algebras are independent of the choice of M. Hence, we first describe quantizations of $\mathcal{O}(G)$ as subalgebras of $U_q(\mathfrak{g})^\circ$; this has the advantage that no roots of q are required. For descriptions in terms of $U_q(\mathfrak{g}, M)^\circ$, see part (c). To avoid problems with certain calculations, one assumes that $\operatorname{char}(k) \neq 2$, and also $\operatorname{char}(k) \neq 3$ in case $\mathfrak{g}$ has a component of type G_2.

Here we only give a quantization of $\mathcal{O}(G)$ for the case that q is not a root of unity. The root of unity case requires defining a suitable algebra over a Laurent polynomial ring $k_0[t^{\pm 1}]$ and then specializing t to q (see, e.g., [**35**, Sections 7,8] or [**12**, Section 4]).

For each $\lambda \in P^+$, there is a finite dimensional simple $U_q(\mathfrak{g})$-module $V(\lambda)$ with *highest weight* λ, that is, $V(\lambda)$ is generated by an element u_λ satisfying $k_\mu u_\lambda = q^{(\mu,\lambda)} u_\lambda$ for all $\mu \in Q$ and $e_i u_\lambda = 0$ for $i = 1, \dots, n$ (see, e.g., [**26**, Theorem 5.10]). For $v \in V(\lambda)$ and $f \in V(\lambda)^*$, let $c^{V(\lambda)}_{f,v} \in U_q(\mathfrak{g})^\circ$ denote the *coordinate function* defined by the rule $c^{V(\lambda)}_{f,v}(u) = f(uv)$. The *single parameter quantized coordinate ring of* G is the k-subalgebra $\mathcal{O}_q(G, L)$ of $U_q(\mathfrak{g})^\circ$ generated by the $c^{V(\lambda)}_{f,v}$ for $\lambda \in L^+$, $f \in V(\lambda)^*$, and $v \in V(\lambda)$. There is some redundancy in the notation $\mathcal{O}_q(G, L)$, since L is determined by G, but we prefer to emphasize L in this way. (Thus, the pair (G, L) is used as a label for that connected semisimple group with Lie algebra $\mathfrak{g}$ and weight lattice L.) The most commonly studied case corresponds to a *simply* *connected* group. This is the case $L = P$, and in this case we write $\mathcal{O}_q(G)$ for $\mathcal{O}_q(G, P)$.

When $G = SL_n(\mathbb{C})$, one obtains the algebra $\mathcal{O}_q(M_n(\mathbb{C}))/\langle D_q - 1\rangle$ as in (1.4) (e.g., [**22**, Theorem 1.4.1]; replace q^2 by q). Thus, there is no ambiguity in the notation $\mathcal{O}_q(SL_n(\mathbb{C}))$.

(c) In order to exhibit $\mathcal{O}_q(G, L)$ as a subalgebra of $U_q(\mathfrak{g}, M)^\circ$, one needs sufficient roots of q in k so that $q^{(M,L)} \subset k$. For $\lambda \in L^+$, there is then a simple finite dimensional $U_q(\mathfrak{g}, M)$-module $V(M, \lambda)$ with highest weight λ, where now the highest weight vector u_λ must satisfy $k_\mu u_\lambda = q^{(\mu,\lambda)} u_\lambda$ for all $\mu \in M$. Let us write $\mathcal{O}_q(G, L, M)$ for the k-subalgebra of $U_q(\mathfrak{g}, M)^\circ$ generated by the coordinate functions of the $V(M, \lambda)$ for $\lambda \in L^+$. Each $V(M, \lambda)$ is also simple as a $U_q(\mathfrak{g})$-module; thus $V(M, \lambda)$ becomes $V(\lambda)$ by restriction, and coordinate functions of $V(M, \lambda)$ map to coordinate functions of $V(\lambda)$ by restriction. Therefore the restriction map $U_q(\mathfrak{g}, M)^\circ \to U_q(\mathfrak{g})^\circ$ induces a k-algebra homomorphism $\mathcal{O}_q(G, L, M) \to \mathcal{O}_q(G, L)$. This map is an isomorphism, because

$$\mathcal{O}_q(G, L, M) = \bigoplus_{\lambda \in L^+} C^{V(M,\lambda)} \qquad \text{and} \qquad \mathcal{O}_q(G, L) = \bigoplus_{\lambda \in L^+} C^{V(\lambda)}$$

where $C^{V(M,\lambda)}$ and $C^{V(\lambda)}$ denote the k-linear spans of the coordinate functions on $V(M, \lambda)$ and $V(\lambda)$, respectively (cf. [**29**, §2.2], [**30**, §1.4.13], [**24**, §3.3]). One also needs the fact that $C^{V(M,\lambda)}$ and $C^{V(\lambda)}$ have the same k-dimension, since both are isomorphic to $V(\lambda) \otimes V(\lambda)^*$.

For the reasons sketched above, we can – and do – choose to work with $\mathcal{O}_q(G, L)$. In [**24**, §3.3], for instance, the corresponding algebra – there denoted $\mathbb{C}_q[G]$ – is defined as $\mathcal{O}_q(G, L, L)$.

(d) Multiparameter versions of $\mathcal{O}_q(G, L)$ are obtained by twisting the multiplication. Let $p : L \times L \to k^\times$ be an *alternating bicharacter*, that is,

$$\begin{aligned} p(\lambda, \lambda) &= 1 \\ p(\mu, \lambda) &= p(\lambda, \mu)^{-1} \\ p(\lambda, \mu + \mu') &= p(\lambda, \mu) p(\lambda, \mu') \end{aligned}$$

for $\lambda, \mu, \mu' \in L$. There is a natural $(L \times L)$-bigrading on $\mathcal{O}_q(G, L)$ (cf. [**24**, §3.3]), and the multiplication on $\mathcal{O}_q(G, L)$ can be twisted via p as in [**24**, §2.1]; the new multiplication, call it $\cdot$, is determined by the rule

$$a \cdot b = p(\lambda, \lambda') p(\mu, \mu')^{-1} ab \qquad \text{for } a \in \mathcal{O}_q(G, L)_{\lambda,\mu} \text{ and } b \in \mathcal{O}_q(G, L)_{\lambda',\mu'}.$$

The vector space $\mathcal{O}_q(G, L)$, equipped with this new multiplication, is called a *multiparameter quantized coordinate ring of* G, denoted $\mathcal{O}_{q,p}(G, L)$. In the case $L = P$, we write simply $\mathcal{O}_{q,p}(G)$.

1.6. Quantized enveloping algebras of Borel and nilpotent subalgebras. Each of the quantized enveloping algebras $U_q(\mathfrak{g}, M)$ contains subalgebras which are viewed as quantizations of the enveloping algebras of Borel or nilpotent subalgebras of $\mathfrak{g}$. Unlike $U_q(\mathfrak{g}, M)$, these algebras behave much like $\mathcal{O}_q(G)$, and they play prominent roles in the study of $\mathcal{O}_q(G)$. Hence, we include them here under the rubric of "algebras similar to quantized coordinate rings", along with quantized Weyl algebras (see the following subsection).

Carry over the objects and notation from (1.5a), in particular the Lie algebra $\mathfrak{g}$, lattices $Q \subseteq M \subseteq P$, the powers $q_i = q^{d_i} \neq \pm 1$, and the group algebra $U^0 \subseteq U_q(\mathfrak{g}, M)$. Let $\mathfrak{b}^+$ and $\mathfrak{n}^+$ denote the positive Borel and nilpotent subalgebras of $\mathfrak{g}$ corresponding to the choices above. The *single parameter quantized enveloping algebra of* $\mathfrak{b}^+$ associated with the above choices is the k-algebra $U_q(\mathfrak{b}^+, M)$ generated by U^0 and $e_1, \dots, e_n$. The corresponding *single parameter quantized enveloping algebra of* $\mathfrak{n}^+$ is the k-subalgebra $U_q(\mathfrak{n}^+)$ of $U_q(\mathfrak{b}^+, M)$ generated by $e_1, \dots, e_n$; this algebra is often denoted $U_q(\mathfrak{g})^+$ (it is, of course, independent of the choice of M).

There is an $(M \times M)$-bigrading on the algebra $U_q(\mathfrak{b}^+, M)$ such that $k_\lambda \in U_q(\mathfrak{b}^+, M)_{-\lambda,\lambda}$ for $\lambda \in M$ and $e_i \in U_q(\mathfrak{b}^+, M)_{-\alpha_i,0}$ for $i = 1, \dots, n$ (see [**24**, Corollary 3.3]). Obviously, $U_q(\mathfrak{n}^+)$ is a bigraded subalgebra. Given an alternating bicharacter $p : M \times M \to k^\times$, we can twist the multiplications in $U_q(\mathfrak{b}^+, M)$ and $U_q(\mathfrak{n}^+)$ via p just as for $\mathcal{O}_q(G, L)$ above. The resulting algebras are the *multiparameter quantized enveloping algebras of* $\mathfrak{b}^+$ *and* $\mathfrak{n}^+$, denoted $U_{q,p}(\mathfrak{b}^+, M)$ and $U_{q,p}(\mathfrak{n}^+)$. As in (1.5a), we omit M from the notation in the case $M = Q$.

There are two cases in which a quantized Borel is known to be a homomorphic image of a quantum semisimple group. Namely, when q is transcendental over $\mathbb{Q}$, the algebra $U_q(\mathfrak{b}^+, P)$ is isomorphic to a factor algebra of $\mathcal{O}_q(G)$ [**30**, Corollary 9.2.12], and for $q \in \mathbb{C}$ not a root of unity, $U_{q,p}(\mathfrak{b}^+, M)$ is a homomorphic image of $\mathcal{O}_{q,p^{-1}}(G, M)^{\mathrm{op}}$ [**24**, Proposition 4.6].

1.7. Quantized Weyl algebras. Recall that the Weyl algebra $A_n(k)$ can (at least in characteristic zero) be viewed as the algebra of polynomial differential operators on affine n-space, i.e., the algebra of all linear partial differential operators with polynomial coefficients on the coordinate ring $\mathcal{O}(k^n)$. Quantized versions of $A_n(k)$ arose in Maltsiniotis' development of "quantum differential calculus" [**36**], and can be viewed as algebras of "partial q-difference operators" on quantum affine spaces.

Let $Q = (q_1, \dots, q_n)$ be a vector in $(k^\times)^n$, and let $\Gamma = (\gamma_{ij})$ be a multiplicatively antisymmetric $n \times n$ matrix over k. The *multiparameter quantized Weyl algebra of degree* n *over* k is the k-algebra $A_n^{Q,\Gamma}(k)$ generated by elements $x_1, y_1, \dots, x_n, y_n$ subject only to the following relations:

$$\begin{aligned} y_i y_j &= \gamma_{ij} y_j y_i && \text{(all } i, j) \\ x_i x_j &= q_i \gamma_{ij} x_j x_i && (i < j) \\ x_i y_j &= \gamma_{ji} y_j x_i && (i < j) \\ x_i y_j &= q_j \gamma_{ji} y_j x_i && (i > j) \\ x_j y_j &= 1 + q_j y_j x_j + \sum_{l<j} (q_l - 1) y_l x_l && \text{(all } j). \end{aligned}$$

See [**28**, §2.9] for a discussion of how to view elements of $A_n^{Q,\Gamma}(k)$ as q-difference operators on $\mathcal{O}_\Gamma(k^n)$.

1.8. Quantum symplectic spaces. The classical linear, symplectic, and orthogonal groups all act on the same affine space, k^n. We have already mentioned above that the relations in quantum matrices are partly chosen so that quantum affine spaces become comodules over quantum matrices, in a "standard" manner. It follows that quantum affine spaces are also comodules over quantum special linear groups. In order to develop an analogous situation for quantum symplectic and orthogonal groups, one needs, as it turns out, different deformations of $\mathcal{O}(k^n)$ than quantum affine spaces. These new algebras are called quantum symplectic and Euclidean spaces; they have (to our knowledge) only been defined in single parameter versions to date.

Let q be a nonzero scalar in k and n a positive integer. Set

$$\begin{aligned} (1, 2, \dots, n, n', (n-1)', \dots, 1') &= (1, 2, \dots, n, n+1, \dots, 2n) \\ (\rho(1), \rho(2), \dots, \rho(2n)) &= (n, n-1, \dots, 1, -1, -2, \dots, -n) \\ (\epsilon_1, \dots, \epsilon_n, \epsilon_{n+1}, \dots, \epsilon_{2n}) &= (1, \dots, 1, -1, \dots, -1). \end{aligned}$$

The one-parameter *coordinate ring of quantum symplectic* $2n$*-space over* k (cf. [**53**, Definition 14] or [**43**, §1.1]) is the k-algebra $\mathcal{O}_q(\mathfrak{sp}\, k^{2n})$ generated by elements $x_1, \dots, x_{2n}$ satisfying the following relations:

$$\begin{aligned} x_i x_j &= q x_j x_i && (i < j;\ j \neq i') \\ x_i x_{i'} &= x_{i'} x_i + (1 - q^2) \sum_{l=1}^{i'-1} q^{\rho(i') - \rho(l)} \epsilon_{i'} \epsilon_l x_l x_{l'} && (i \leq n) \end{aligned}$$

(cf. [**53**, p. 210]). A simpler set of relations was found by Musson [**43**, §1.1] (cf. [**45**, §1.1]):

$$\begin{aligned} x_i x_j &= q x_j x_i && (i < j;\ j \neq i') \\ x_i x_{i'} &= q^2 x_{i'} x_i + (q^2 - 1) \sum_{l=1}^{i-1} q^{l-i} x_l x_{l'} && (i \leq n). \end{aligned}$$

1.9. Quantum Euclidean spaces. Let $n \geq 2$ be an integer and q a nonzero scalar in k, such that q has a square root in k in case n is odd. Set $m = \lfloor n/2 \rfloor$, the integer part of $n/2$, and set $i' = n+1-i$ for $1 \leq i \leq n$. Further, set

$$(\rho(1), \dots, \rho(n)) \\ = \begin{cases} (m - \frac{1}{2}, m - \frac{3}{2}, \dots, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{3}{2}, \dots, -m + \frac{1}{2}) & (n \text{ odd}) \\ (m-1, m-2, \dots, 1, 0, 0, -1, -2, \dots, -m+1) & (n \text{ even}). \end{cases}$$

The one-parameter *coordinate ring of quantum Euclidean n-space over k* (cf. [**53**, Definition 12] or [**43**, §2.1]) is the k-algebra $\mathcal{O}_q(\mathfrak{o}\, k^n)$ generated by elements $x_1, \dots, x_n$ satisfying the following relations:

$$x_i x_j = q x_j x_i \qquad (i < j;\ j \neq i')$$

$$x_i x_{i'} = x_{i'} x_i + \frac{1-q^2}{1+q^{n-2}} \left(q^{n-2} \sum_{l=1}^{i'-1} q^{\rho(i')-\rho(l)} x_l x_{l'} - \sum_{l=i'}^{n} q^{\rho(i')-\rho(l)} x_l x_{l'} \right) \qquad (i \leq m).$$

A simpler set of relations was given in [**43**, §§2.1, 2.2]:

$$x_i x_j = q x_j x_i \qquad (i < j;\ j \neq i')$$

$$x_i x_{i'} = x_{i'} x_i + (1-q^2) \sum_{l=i+1}^{m} q^{l-i-2} x_l x_{l'} \qquad (i \leq m;\ n \text{ even})$$

$$x_i x_{i'} = x_{i'} x_i + (1-q) q^{m-i-(1/2)} x_{m+1}^2 + (1-q^2) \sum_{l=i+1}^{m} q^{l-i-2} x_l x_{l'} \qquad (i \leq m;\ n \text{ odd}).$$

For an alternative presentation of $\mathcal{O}_q(\mathfrak{o}\, k^n)$, see [**46**, Example 5; **47**, Definition 3.1].

2. Some common properties

We record some fundamental properties common to the algebras discussed in Section 1. The cases we list are those for which we have located references in which a result appears, or from which it follows readily. Many cases which we conjecture should be included must be omitted at present, because they have not (to our knowledge) been addressed in the literature. We leave it to the reader to identify appropriate "missing" cases, and to formulate the corresponding conjectures. Perhaps more important than filling in specific cases is the problem of developing general theorems which could verify properties for all these algebras simultaneously.

2.1. Noetherian domains.

Theorem. *All of the following algebras are noetherian domains:* $\mathcal{O}_{\boldsymbol{q}}(k^n)$; $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$; $\mathcal{O}_{\lambda,\boldsymbol{p}}(GL_n(k))$; $\mathcal{O}_{\lambda,\boldsymbol{p}}(SL_n(k))$; $\mathcal{O}_{q,p}(G,L)$ *for* q *transcendental over* $\mathbb{Q}$ *or* $q \in \mathbb{C}$ *not a root of unity;* $U_{q,p}(\mathfrak{b}^+)$; $U_{q,p}(\mathfrak{n}^+)$; $A_n^{Q,\Gamma}(k)$; $\mathcal{O}_q(\mathfrak{sp}\, k^{2n})$; $\mathcal{O}_q(\mathfrak{o}\, k^n)$.

Proof. In the cases $\mathcal{O}_{\boldsymbol{q}}(k^n)$; $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$; $A_n^{Q,\Gamma}(k)$; $\mathcal{O}_q(\mathfrak{sp}\, k^{2n})$; $\mathcal{O}_q(\mathfrak{o}\, k^n)$, one has only to observe that the algebra is an iterated skew polynomial ring over k. This is clear for $\mathcal{O}_{\boldsymbol{q}}(k^n)$; for the other cases, see [**2**, pp. 890-891], [**28**, §§2.1, 2.8], [**43**, §§1.2, 2.3]. It follows that the localization $\mathcal{O}_{\lambda,\boldsymbol{p}}(GL_n(k))$ is a noetherian domain, and that the factor algebra $\mathcal{O}_{\lambda,\boldsymbol{p}}(SL_n(k))$ is noetherian. That the latter is a domain is proved in [**34**, Corollary].

For $\mathcal{O}_q(G)$ in the stated cases, see [**29**, Lemma 3.1, Proposition 4.1] and [**30**, Lemma 9.1.9, Proposition 9.2.2]. (An alternate proof of noetherianity is given in [**4**, Corollary 5.6].) The desired properties of $\mathcal{O}_q(G,L)$ are proved by the same arguments, and they carry over to $\mathcal{O}_{q,p}(G,L)$ by graded ring methods [**24**, Remark, p. 80] or by twisting results [**60**, Propositions 5.1, 5.2].

By [**10**, Corollary 1.8], $U_q(\mathfrak{g})$ is a domain, and so $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{n}^+)$ are domains. There are several ways to see that $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{n}^+)$ are noetherian. For instance, it follows from the existence of a PBW basis for $U_q(\mathfrak{g})$ (e.g., [**26**, Theorem 4.21]) that $U_q(\mathfrak{g})$ is free as a right $U_q(\mathfrak{b}^+)$-module and as a right $U_q(\mathfrak{n}^+)$-module. Hence, the poset of left ideals of either subalgebra embeds in the poset of left ideals of $U_q(\mathfrak{g})$, via $I \mapsto U_q(\mathfrak{g})I$. This shows that $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{n}^+)$ are left noetherian. There exists an antiautomorphism τ on $U_q(\mathfrak{g})$ such that $\tau(e_i) = e_i$ for $i = 1, \dots, n$ and $\tau(k_\lambda) = k_\lambda^{-1}$ for $\lambda \in Q$ (e.g., [**26**, Lemma 4.6]). Since τ obviously stabilizes $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{n}^+)$, these subalgebras must be right as well as left noetherian.

On the other hand, in view of [**10**, Proposition 1.7], given either $A_0 = U_q(\mathfrak{b}^+)$ or $A_0 = U_q(\mathfrak{n}^+)$, there is a sequence of algebras $A_0, A_1, \dots, A_N$ such that each A_{i+1} is the associated graded ring of A_i with respect to some $\mathbb{Z}^+$-filtration, and such that A_N is an iterated skew polynomial ring over either U^0 or k. Since A_N is noetherian, it follows from standard filtered/graded techniques (e.g., [**39**, Theorem 1.6.9]) that A_0 is noetherian.

Finally, the twisting results mentioned above ([**60**, Propositions 5.1, 5.2]) imply that $U_{q,p}(\mathfrak{b}^+)$ and $U_{q,p}(\mathfrak{n}^+)$ are noetherian domains. □

2.2. Complete primeness of prime factors. For any set $\{\alpha_i\}$ of nonzero scalars in k, let $\langle \alpha_i \rangle$ denote the multiplicative subgroup of $k^\times$ generated by the α_i.

Theorem. *Let A be one of the following algebras: $\mathcal{O}_{\boldsymbol{q}}(k^n)$ with $\langle q_{ij}\rangle$ torsionfree; $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$ or $\mathcal{O}_{\lambda,\boldsymbol{p}}(GL_n(k))$ or $\mathcal{O}_{\lambda,\boldsymbol{p}}(SL_n(k))$ with $\langle \lambda, p_{ij}\rangle$ torsionfree; $\mathcal{O}_q(G)$ or $U_q(\mathfrak{b}^+)$ with $q \in \mathbb{C}$ not a root of unity; $U_q(\mathfrak{n}^+)$ with q transcendental over $\mathbb{Q}$; $A_n^{Q,\Gamma}(k)$ with $\langle q_i, \gamma_{ij}\rangle$ torsionfree; $\mathcal{O}_q(\mathfrak{sp}\, k^{2n})$ or $\mathcal{O}_q(\mathfrak{o}\, k^n)$ with q not a root of unity. Then all prime ideals of A are completely prime, i.e., all prime factor rings of A are domains.*

Proof. The cases $\mathcal{O}_{\boldsymbol{q}}(k^n)$; $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$; $A_n^{Q,\Gamma}(k)$; $\mathcal{O}_q(\mathfrak{sp}\, k^{2n})$; $\mathcal{O}_q(\mathfrak{o}\, k^n)$ follow from a general result about prime ideals of certain iterated skew polynomial rings: [**18**, Theorem 2.3]. See [**18**, Theorem 2.1], [**16**, Theorem 5.1], [**8**, Proposition II.1.2], [**43**, Corollary 1.2, Theorem 2.3] for further details. The cases $\mathcal{O}_{\lambda,\boldsymbol{p}}(GL_n(k))$ and $\mathcal{O}_{\lambda,\boldsymbol{p}}(SL_n(k))$ follow immediately. In addition, $U_q(\mathfrak{n}^+)$ is an iterated q-skew polynomial ring over k (see [**54**, Section 5] for the case $k = \mathbb{Q}(q)$ and extend scalars), and so in this case also the result follows from [**18**, Theorem 2.3] – see [**54**, Corollary to Theorem 3]. Finally, the cases $\mathcal{O}_q(G)$ and $U_q(\mathfrak{b}^+)$ are proved in [**29**, Theorems 11.4, 11.5]. □

2.3. Division rings of fractions.

Theorem. *Let A be one of the algebras $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$; $U_q(\mathfrak{sl}_n(k))^+$ with q not a root of unity; $\mathcal{O}_q(G)$ or $U_q(\mathfrak{b}^+, P)$ or $U_q(\mathfrak{n}^+)$ with $q \in \mathbb{C}$ not a root of unity; $A_n^{Q,\Gamma}(k)$; $\mathcal{O}_q(\mathfrak{sp}\, k^{2n})$; $\mathcal{O}_q(\mathfrak{o}\, k^n)$. Then* $\operatorname{Fract} A \cong \operatorname{Fract} \mathcal{O}_{\boldsymbol{q}}(K^t)$ *for some $\boldsymbol{q}$, some field $K \supseteq k$, and some t.*

Now let A be either $\mathcal{O}_q(M_n(k))$ with q not a root of unity, or $A_n^{Q,\Gamma}(k)$ with $\langle q_i, \gamma_{ij}\rangle$ torsionfree. If P is any prime ideal of A, then $\operatorname{Fract}(A/P) \cong \operatorname{Fract} \mathcal{O}_{\boldsymbol{q}}(K^t)$ *for some $\boldsymbol{q}$, some field $K \supseteq k$, and some t.*

Proof. For the first set of cases, see [**42**, Theorem 1.24], [**1**, Théorème 2.15], [**25**, Theorem 3.5], [**7**, Theorems 3.3, 3.1], [**28**, §3.1], [**43**, Theorems 1.3, 2.4]. The case of $\operatorname{Fract} \mathcal{O}_q(M_n(k)) = \operatorname{Fract} \mathcal{O}_q(GL_n(k))$ was treated earlier in [**9**, Proposition 5] and [**48**, Theorem 3.8]. The case of $U_q(\mathfrak{n}^+)$ for q transcendental over $\mathbb{C}$ was also obtained in [**31**]. For the cases of prime factors of $\mathcal{O}_q(M_n(k))$ and $A_n^{Q,\Gamma}(k)$, see [**8**, Théorèmes II.2.1, III.3.2.1]. □

2.4. Catenarity. Recall that (the prime spectrum of) a ring A is *catenary* if for any comparable prime ideals $P \supset Q$ in A, all saturated chains of prime ideals from P to Q have the same length. Our discussion of this property will involve the following concept. The ring A, or its prime spectrum $\operatorname{spec} A$, is said to have *normal separation* provided the following condition holds: For any proper inclusion $P \supset Q$ of prime ideals of A, the factor P/Q contains a nonzero element which is normal in A/Q. In other words, there must exist an element $c \in P \setminus Q$ such that $c + Q$ is normal in A/Q; we refer to the latter condition by saying that c is *normal modulo Q*.

Theorem. *The following algebras are catenary:* $\mathcal{O}_{\boldsymbol{q}}(k^n)$; $\mathcal{O}_q(GL_n(\mathbb{C}))$ *and* $\mathcal{O}_q(SL_n(\mathbb{C}))$ *with* $q \in \mathbb{C}$ *not a root of unity;* $U_q(\mathfrak{n}^+)$ *with* q *transcendental over* $\mathbb{Q}$; $A_n^{Q,\Gamma}(k)$ *with no* q_i *a root of unity;* $\mathcal{O}_q(\mathfrak{sp}\, k^{2n})$ *and* $\mathcal{O}_q(\mathfrak{o}\, k^n)$ *with* q *not a root of unity.*

Proof. These cases all follow from a general theorem based on Gabber's methods: If A is an affine, noetherian, Auslander-Gorenstein, Cohen-Macaulay algebra with finite Gelfand-Kirillov dimension, and if $\operatorname{spec} A$ is normally separated, then A is catenary [**17**, Theorem 1.6]. See [**17**, Theorems 2.6, 3.13, 4.5, 4.8] and [**46**, Corollaries 12, 13] for the individual cases listed. □

2.5. Normal separation. As indicated in (2.4), normal separation plays a key role in proving catenarity. Normal separation in a ring A also implies Jategaonkar's strong second layer condition (cf. [**27**, Theorem 3.3.16, Proposition 8.1.7, discussion p. 225]), from which it follows, in particular, that every module has a finite filtration such that associated primes in adjacent layers are linked in $\operatorname{spec} A$ (cf. [**27**, Theorem 9.1.2]). Other influences of the second layer condition on the representation theory of A are discussed in [**5**] and [**27**]. Finally, we mention that normal separation, in conjunction with several other properties typical of quantum coordinate rings, leads to a description of links and cliques in terms of lattices of automorphisms [**3**, Section 3].

Theorem. *The following algebras have normal separation:* $\mathcal{O}_{\boldsymbol{q}}(k^n)$; $\mathcal{O}_q(G)$ *with* q *transcendental over* $\mathbb{Q}$; $\mathcal{O}_{q,p}(G,L)$ *with* $q \in \mathbb{C}$ *not a root of unity;* $U_q(\mathfrak{b}^+, P)$ *and* $U_q(\mathfrak{n}^+)$ *over a rational function field* $\mathbb{C}(q)$; $U_q(\mathfrak{b}^+, P)$ *with* q *transcendental over* $\mathbb{Q}$; $U_{q,p}(\mathfrak{b}^+, M)$ *with* $q \in \mathbb{C}$ *not a root of unity;* $A_n^{Q,\Gamma}(k)$ *with no* q_i *a root of unity;* $\mathcal{O}_q(\mathfrak{sp}\, k^{2n})$ *and* $\mathcal{O}_q(\mathfrak{o}\, k^n)$ *with* q *not a root of unity.*

Proof. For $\mathcal{O}_{\boldsymbol{q}}(k^n)$, $A_n^{Q,\Gamma}(k)$, $\mathcal{O}_q(\mathfrak{sp}\, k^{2n})$ and $\mathcal{O}_q(\mathfrak{o}\, k^n)$, see [**17**, Corollary 2.4, Theorem 3.12] and [**46**, Theorem 10]. The algebras $U_q(\mathfrak{n}^+)$ and $U_q(\mathfrak{b}^+, P)$ over $\mathbb{C}(q)$ are actually *polynormal*, i.e., every ideal has a normalizing sequence of generators [**6**, Corollaires 3.2, 3.3].

Normal separation for $A = \mathcal{O}_q(G)$ or $A = \mathcal{O}_{q,p}(G,L)$ is a consequence of results in [**30**] and [**24**], as follows (cf. [**3**, Theorem 5.8]). In these cases, there is a partition

$$\operatorname{spec} A = \bigsqcup_{w \in W \times W} \operatorname{spec}_w A,$$

where W is the Weyl group of G, described in [**30**, Corollary 9.3.9] and [**24**, Corollary 4.5]. For each w, there are an ideal I_w of A and an Ore set $\mathcal{E}_w$ of nonzero normal elements in A/I_w such that

$$\operatorname{spec}_w A = \{P \in \operatorname{spec} A \mid P \supseteq I_w \text{ and } (P/I_w) \cap \mathcal{E}_w = \varnothing\}$$

[**30**, Proposition 10.3.2], [**24**, Theorem 4.4]. In particular, normal separation follows for primes $P \supset Q$ such that $Q \in \operatorname{spec}_w A$ but $P \notin \operatorname{spec}_w A$. Further, the localization $(A/I_w)[\mathcal{E}_w^{-1}]$ has *central separation*: for any primes $P' \supset Q'$, there exists a central element in $P' \setminus Q'$. In one of our cases, this follows from a sequence of results in [**30**], as shown in [**3**, §5.7]; in the other case, it is known that all ideals of $(A/I_w)[\mathcal{E}_w^{-1}]$ are centrally generated [**24**, Theorem 4.15]. Central separation in $(A/I_w)[\mathcal{E}_w^{-1}]$ together with normality of the elements of $\mathcal{E}_w$ yields normal separation for primes $P \supset Q$ in the same $\operatorname{spec}_w A$ [**3**, Proposition 1.7]. An alternate proof of normal separation for $\mathcal{O}_q(G)$, based on Hopf algebra technology, is given in [**33**, Proposition 2.4].

Finally, the second $U_q(\mathfrak{b}^+, P)$ case and the $U_{q,p}(\mathfrak{b}^+, M)$ case follow from the cases of $\mathcal{O}_q(G)$ and $\mathcal{O}_{q,p}(G, L)$ (see the last paragraph of (1.6)). □

2.6. The Dixmier-Moeglin equivalence. Recall that a prime ideal P of a noetherian k-algebra A is *rational* provided the center of $\operatorname{Fract}(A/P)$ is algebraic over k. Recall also that P is *locally closed* in $\operatorname{spec} A$ provided the singleton $\{P\}$ is closed in some Zariski-neighborhood, i.e., the intersection of the primes properly containing P is larger than P. One says that the algebra A satisfies the *Dixmier-Moeglin equivalence* if the sets of rational prime ideals, locally closed prime ideals, and primitive ideals all coincide (cf. [**51**]). The advantage of this equivalence is that when it holds, the primitive ideals of A can be identified without constructing any irreducible representations.

Theorem. *The Dixmier-Moeglin equivalence holds in the following algebras:* $\mathcal{O}_{\boldsymbol{q}}(k^n)$; $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$; $\mathcal{O}_{\lambda,\boldsymbol{p}}(GL_n(k))$; $\mathcal{O}_{\lambda,\boldsymbol{p}}(SL_n(k))$; $\mathcal{O}_q(G)$ *with q transcendental over $\mathbb{Q}$ and k algebraically closed;* $\mathcal{O}_{q,p}(G, L)$ *with $q \in \mathbb{C}$ not a root of unity;* $U_q(\mathfrak{b}^+, P)$ *with q transcendental over $\mathbb{Q}$;* $U_{q,p}(\mathfrak{b}^+, M)$ *with $q \in \mathbb{C}$ not a root of unity;* $A_n^{Q,\Gamma}(k)$ *with the q_i not roots of unity;* $\mathcal{O}_q(\mathfrak{sp}\, k^{2n})$ *and* $\mathcal{O}_q(\mathfrak{o}\, k^n)$ *with q not a root of unity.*

Proof. For $\mathcal{O}_{\boldsymbol{q}}(k^n)$, $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$, $A_n^{Q,\Gamma}(k)$, $\mathcal{O}_q(\mathfrak{sp}\, k^{2n})$ and $\mathcal{O}_q(\mathfrak{o}\, k^n)$, see [**19**, Corollary 2.5, Theorem 3.2] and [**20**, Theorems 5.3, 5.5, 5.8, 5.11]. The cases $\mathcal{O}_{\lambda,\boldsymbol{p}}(GL_n(k))$ and $\mathcal{O}_{\lambda,\boldsymbol{p}}(SL_n(k))$ follow directly. The case $\mathcal{O}_{q,p}(G, L)$ follows from the results of [**24**] exactly as in the case $\mathcal{O}_q(SL_n(\mathbb{C}))$, which was given explicitly in [**23**, Theorem 4.2]. The case $\mathcal{O}_q(G)$ follows from results in [**30**], as noted in [**20**, §2.4]. Finally, the cases $U_q(\mathfrak{b}^+, P)$ and $U_{q,p}(\mathfrak{b}^+, M)$ follow from the previous cases. □

3. Stratified spectra

In this section, we give a more extensive discussion of another feature common to many quantized coordinate algebras – a stratification of the prime spectrum in which each stratum is a classical scheme, in fact the prime spectrum of a (commutative) Laurent polynomial algebra. Such a stratification was first discovered in spec $\mathcal{O}_q(SL_3(\mathbb{C}))$ by Hodges and Levasseur [**22**]; they soon extended it to $\mathcal{O}_q(SL_n(\mathbb{C}))$ [**23**]. This was generalized by Joseph to $\mathcal{O}_q(G)$ [**29**, **30**], and finally extended to $\mathcal{O}_{q,p}(G,L)$ by Hodges, Levasseur, and Toro [**24**]. In all these cases, the result followed from a long sequence of involved calculations specific to the algebra at hand. Later, it was noticed that some features of these stratifications could be tied to the action of a torus of automorphisms (cf. [**3**, Section 5 and Proposition 1.9]), and similar stratifications were observed in other quantized algebras (e.g., [**3**, Section 4], [**19**, §2.1, Theorem 2.3]). A general development of this type of stratification was begun in [**20**].

We begin with an outline of the main features of the stratification of spec $\mathcal{O}_{q,p}(G,L)$, followed by a discussion of the axiomatic framework into which this stratification nicely fits. Then we exhibit appropriate tori of automorphisms for the algebras from Section 1, and indicate the cases in which the details of the stratification have been worked out.

3.1. Let A be either the algebra $\mathcal{O}_q(G)$ with q transcendental over $\mathbb{Q}$, or $\mathcal{O}_{q,p}(G,L)$ with $q \in \mathbb{C}$ not a root of unity. As noted in (2.5), there is a partition

$$\operatorname{spec} A = \bigsqcup_{w \in W \times W} \operatorname{spec}_w A,$$

where W is the Weyl group of G, described in [**30**, Corollary 9.3.9] and [**24**, Corollary 4.5]. For each w, there are an ideal I_w of A and an Ore set $\mathcal{E}_w$ of nonzero normal elements in A/I_w such that

$$\operatorname{spec}_w A = \{P \in \operatorname{spec} A \mid P \supseteq I_w \text{ and } (P/I_w) \cap \mathcal{E}_w = \varnothing\}.$$

In fact, $\mathcal{E}_w$ is finitely generated, and so if e_w is the product (in some order) of a finite set of generators for $\mathcal{E}_w$, we can write

$$\operatorname{spec}_w A = \{P \in \operatorname{spec} A \mid P \supseteq I_w\} \cap \{P \in \operatorname{spec} A \mid e_w \notin P\},$$

an intersection of a closed and an open set. Thus $\operatorname{spec}_w A$ is a locally closed subset of $\operatorname{spec} A$. Let $A_w = (A/I_w)[\mathcal{E}_w^{-1}]$. In the case of $\mathcal{O}_{q,p}(G,L)$ over $\mathbb{C}$, the center $Z(A_w)$ is a Laurent polynomial ring over k [**24**, Theorem 4.14], the ideals of A_w are centrally generated [**24**, Theorem 4.15], and it follows that $\operatorname{spec}_w A$ is homeomorphic to $\operatorname{spec} Z(A_w)$. That such a homeomorphism also exists in the case of $\mathcal{O}_q(G)$ over $k \supseteq \mathbb{Q}(q)$ is not proved in [**30**] (a Laurent polynomial subalgebra $Z_w \subseteq Z(A_w)$ is studied instead [**30**, §10.3.3]), but this can be deduced from the results there together with what we shall prove in Part II.

If $\operatorname{prim} A$ denotes the *primitive spectrum* of A (the set of primitive ideals, equipped with the Zariski topology), we obtain a corresponding partition

$$\operatorname{prim} A = \bigsqcup_{w \in W \times W} \operatorname{prim}_w A,$$

where each $\operatorname{prim}_w A = \operatorname{prim} A \cap \operatorname{spec}_w A$ is a locally closed subset of $\operatorname{prim} A$. Further, $\operatorname{prim}_w A$ is precisely the set of maximal elements of $\operatorname{spec}_w A$ (this is given explicitly in [**30**, Theorem 10.3.7] for the first case, and is implicit in [**24**, Section 4] for the second; see also [**3**, Corollary 1.5]). Over an algebraically closed base field, a maximal torus H of G acts naturally as automorphisms of A [**30**, §10.3.8], [**24**, §3.3], and the sets $\operatorname{prim}_w A$ are precisely the H-orbits in $\operatorname{prim} A$ [**30**, Theorem 10.3.8], [**24**, Theorem 4.16].

3.2. The picture of $\operatorname{spec} A$ outlined in (3.1) can be conveniently organized in terms of the action of the group H. There is a unique minimal element J_w in $\operatorname{spec}_w A$ for each w ([**30**, Proposition 10.3.5]; implicit in [**24**, Section 4]), and it is easily checked that J_w is H-stable (cf. [**3**, §5.4]). Over an algebraically closed base field, it then follows that the J_w are precisely the H-stable prime ideals of A, and that each $\operatorname{spec}_w A$ consists precisely of those prime ideals P such that the intersection of the H-orbit of P equals J_w [**3**, Proposition 1.9].

These observations provide a framework that can be set up with respect to any group of automorphisms of any ring, as follows. As we shall see later, the special properties of this framework in the cases discussed in (3.1) appear in many other algebras as well.

3.3. ***H*-stratifications.** Let H be a group acting as automorphisms on a ring A. Recall that an *H-prime ideal* of A is any proper H-stable ideal J of A such that a product of H-stable ideals is contained in J only when one of the factors is contained in J. We shall write $H\text{-}\operatorname{spec} A$ for the set of H-prime ideals of A. For any ideal I of A, let $(I : H)$ denote the intersection of the H-orbit of I, that is,

$$(I : H) = \bigcap_{h \in H} h(I).$$

Alternatively, $(I : H)$ is the largest H-stable ideal of A contained in I.

Observe that if P is a prime ideal of A, then $(P : H)$ is an H-prime ideal. On the other hand, if A is noetherian and I is an H-prime ideal, it is easily checked that I is semiprime and that the prime ideals minimal over I form a single H-orbit (cf. [**15**, Remarks 4*,5*, p. 338]). In this case, $I = (P : H)$ for any prime P minimal over I.

The *H-stratum* of any H-prime ideal J is the set

$$\operatorname{spec}_J A = \{P \in \operatorname{spec} A \mid (P : H) = J\}.$$

One defines H-strata in any subset of $\operatorname{spec} A$ by intersection with these strata. In particular, the H-stratum of J in $\operatorname{prim} A$ is the set

$$\operatorname{prim}_J A = \operatorname{prim} A \cap \operatorname{spec}_J A.$$

Since $(P : H) \in H\text{-}\operatorname{spec} A$ for each $P \in \operatorname{spec} A$, the H-strata partition $\operatorname{spec} A$, that is,

$$\operatorname{spec} A = \bigsqcup_{J \in H\text{-}\operatorname{spec} A} \operatorname{spec}_J A.$$

We refer to this partition as *the H-stratification of* $\operatorname{spec} A$. Similarly, the partition of $\operatorname{prim} A$ into H-strata $\operatorname{prim}_J A$ is called the H-stratification of $\operatorname{prim} A$.

3.4. Finite H-stratifications enjoy the minimal topological properties required of stratifications in algebraic and differential geometry, as follows. A *stratification* of an algebraic variety X is defined in [**32**, p. 56] to be a partition of X into a disjoint union of finitely many locally closed subsets (called *strata*) such that the closure of each stratum is a union of strata.

Lemma. *Let H be a group acting as automorphisms on a ring A, and assume that H-*$\operatorname{spec} A$ *is finite.*

(a) *Each H-stratum in* $\operatorname{spec} A$ *is locally closed.*

(b) *The closure of each H-stratum in* $\operatorname{spec} A$ *is a union of H-strata.*

(c) *For each nonnegative integer d, the union of the H-strata corresponding to the elements of height at most d in H-*$\operatorname{spec} A$ *is open in* $\operatorname{spec} A$.

Proof. (a) Let $J \in H\text{-}\operatorname{spec} A$, and let J' be the intersection of the H-primes properly containing J (take $J' = A$ if J is maximal in $H\text{-}\operatorname{spec} A$). Since J is H-prime and $H\text{-}\operatorname{spec} A$ is finite, $J' \supsetneq J$. Thus

$$\operatorname{spec}_J A = \{P \in \operatorname{spec} A \mid P \supseteq J\} \cap \{P \in \operatorname{spec} A \mid P \not\supseteq J'\}.$$

(b) If $J \in H\text{-}\operatorname{spec} A$, the closure of $\operatorname{spec}_J A$ is just the set $V(J)$ of those prime ideals of A containing J, and $V(J)$ equals the union of the H-strata $\operatorname{spec}_K A$ as K runs over those H-primes containing J.

(c) Let Y be the set of elements of $H\text{-}\operatorname{spec} A$ of height greater than d, that is, those H-primes K from which there descends a chain $K = K_0 \supsetneq K_1 \supsetneq \cdots \supsetneq K_{d+1}$ in $H\text{-}\operatorname{spec} A$. The union of the H-strata $\operatorname{spec}_J A$ for J of height at most d in $H\text{-}\operatorname{spec} A$ is the set

$$\{P \in \operatorname{spec} A \mid P \not\supseteq \cap Y\},$$

which is open in $\operatorname{spec} A$. $\square$

3.5. Tori of automorphisms. Whether or not our base field k is algebraically closed, we shall refer to any finite direct product of copies of the multiplicative group $k^\times$ as an *(algebraic) torus*. For each of the algebras A discussed in Section 1, there is a naturally occurring torus $\mathcal{H}$ acting as k-algebra automorphisms on A, as follows. The action of an element $h \in \mathcal{H}$ on an element $a \in A$ will be denoted $h.a$; we leave it to the reader to verify in each case that there exist k-algebra automorphisms $h.(-)$ as described, and that the rule $h \mapsto h.(-)$ is a group homomorphism from $\mathcal{H}$ to $\operatorname{Aut} A$.

Many of these tori arise as groups of 'winding automorphisms' of a bialgebra or Hopf algebra, as follows. Suppose that A is a bialgebra over k, and let $A^\wedge$ be the set of linear characters on A, that is, k-algebra homomorphisms $A \to k$. This set forms a monoid under convolution, the counit of A being the identity element of $A^\wedge$. For $\chi \in A^\wedge$, there are k-algebra endomorphisms θ^l_χ and θ^r_χ on A given by the rules

$$\theta^l_\chi(a) = \sum_{(a)} a_1\chi(a_2) \qquad \text{and} \qquad \theta^r_\chi(a) = \sum_{(a)} \chi(a_1)a_2.$$

When χ is invertible in $A^\wedge$, say with inverse χ', the endomorphisms θ^l_χ and θ^r_χ are automorphisms of A, with inverses $\theta^l_{\chi'}$ and $\theta^r_{\chi'}$ respectively, and we refer to them as the left and right *winding automorphisms* of A associated with χ. The maps $\chi \mapsto \theta^l_\chi$ and $\chi \mapsto \theta^r_\chi$ are a homomorphism and an anti-homomorphism, respectively, from the group of units of $A^\wedge$ to $\operatorname{Aut} A$. The facts outlined above are well known when A is a Hopf algebra, in which case $A^\wedge$ is a group under convolution (see, e.g., [**30**, §1.3.4]).

If C is a right comodule algebra over the bialgebra A (e.g., see [**41**, Definition 4.1.2]), with structure map $C \to C \otimes A$ written as $c \mapsto \sum_{(c)} c_0 \otimes c_1$, then each (invertible) character $\chi \in A^\wedge$ induces a k-algebra endomorphism (automorphism) of C according to the rule $c \mapsto \sum_{(c)} c_0\chi(c_1)$. The analogous statements hold for a left comodule algebra over A.

1. Quantum affine spaces. For $A = \mathcal{O}_{\boldsymbol{q}}(k^n)$, take $\mathcal{H} = (k^\times)^n$ and

$$(\alpha_1, \dots, \alpha_n).x_i = \alpha_i x_i$$

for $i = 1, \dots, n$. These automorphisms can be viewed as winding automorphisms arising when A is made into a right comodule algebra over the bialgebra $\mathcal{O}_{\lambda,\boldsymbol{q}}(M_n(k))$, or into a left comodule algebra over $\mathcal{O}_{\lambda,\lambda^{-1}\boldsymbol{q}}(M_n(k))$.

2. Quantum matrices. For the algebra $A = \mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$, take $\mathcal{H} = (k^\times)^m \times (k^\times)^n$ and

$$(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n).X_{ij} = \alpha_i\beta_j X_{ij}$$

for $i = 1, \dots, m$ and $j = 1, \dots, n$. In the case $m = n$, the automorphisms above are compositions of left and right winding automorphisms. Namely, for $\gamma = (\gamma_1, \dots, \gamma_n) \in (k^\times)^n$ there is a character $\chi_\gamma \in A^\wedge$ such that $\chi_\gamma(X_{ij}) = \delta_{ij}\gamma_i X_{ij}$ for all i, j, and the automorphism displayed above can be expressed as $\theta^l_{\chi_\beta}\theta^r_{\chi_\alpha}$ where $\beta = (\beta_1, \dots, \beta_n)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$. The map $\gamma \mapsto \chi_\gamma$ embeds the torus $(k^\times)^n$ in the group of units of $A^\wedge$; it is an isomorphism when λ and $\boldsymbol{p}$ are sufficiently generic.

3. Quantum general linear groups. For $A = \mathcal{O}_{\lambda,\boldsymbol{p}}(GL_n(k))$, we can take the torus $\mathcal{H} = (k^\times)^n \times (k^\times)^n$ acting on $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ as above, and extend the given automorphisms from $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ to A, because $D_{\lambda,\boldsymbol{p}}$ is an $\mathcal{H}$-eigenvector.

4. Quantum special linear groups. For $A = \mathcal{O}_{\lambda,\boldsymbol{p}}(SL_n(k))$ (assuming a suitable choice of $\lambda, \boldsymbol{p}$ so that $D_{\lambda,\boldsymbol{p}}$ is central), we must take a subgroup of the torus used for $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$, namely the stabilizer of $D_{\lambda,\boldsymbol{p}}$:

$$\mathcal{H} = \{(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) \in (k^\times)^n \times (k^\times)^n \mid \alpha_1\alpha_2\cdots\alpha_n\beta_1\beta_2\cdots\beta_n = 1\}.$$

The automorphisms of $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ corresponding to elements of $\mathcal{H}$ all fix $D_{\lambda,\boldsymbol{p}} - 1$ and therefore induce automorphisms of A.

5. Quantum semisimple groups. Let $A = \mathcal{O}_{q,p}(G, L)$ as described in (1.5). In the case $k = \mathbb{C}$, one works with a maximal torus of G with Lie algebra $\mathfrak{h}$ and character group L [**24**, §3.1]. In the general case, it is simplest to take the set $\mathcal{H}$ of group homomorphisms $L \to k^\times$; this is a group under pointwise multiplication, and it is a torus because L is a lattice. An action of $\mathcal{H}$ on A derives from an embedding of $\mathcal{H}$ into the character group $A^\wedge$. Namely, there is a monic group homomorphism $\iota : \mathcal{H} \to A^\wedge$ such that

$$\iota(h)(c^{V(\lambda)}_{f,v}) = h(\mu)f(v)$$

for $\lambda \in P^+$, $\mu \in P$, $v \in V(\lambda)_\mu$, $f \in V(\lambda)^*$ (see [**24**, §3.3]).

Therefore there is a group homomorphism $\mathcal{H} \to \operatorname{Aut} A$ such that $h \mapsto \theta^l_{\iota(h)}$, which provides an action of $\mathcal{H}$ on A. This is the action used in [**30**] and [**24**]. For some purposes, however, an action of $\mathcal{H} \times \mathcal{H}$ is needed, given by $(h_1, h_2) \mapsto \theta^l_{\iota(h_1)}\theta^r_{\iota(h_2)}$ (see [**3**, §5.2]).

6. Quantized enveloping algebras of Borel and nilpotent subalgebras. For $A = U_{q,p}(\mathfrak{n}^+)$, take $\mathcal{H} = (k^\times)^n$ and

$$(\alpha_1, \dots, \alpha_n).e_i = \alpha_i e_i$$

for $i = 1, \dots, n$. For $A = U_{q,p}(\mathfrak{b}^+, M)$, let $\mathcal{H}_M$ be the group of group homomorphisms $M \to k^\times$ under pointwise multiplication and take $\mathcal{H} =$

$\mathcal{H}_M \times (k^\times)^n$, with

$$(h, \alpha_1, \dots, \alpha_n).k_\mu = h(\mu)k_\mu$$
$$(h, \alpha_1, \dots, \alpha_n).e_i = \alpha_i e_i$$

for $\mu \in M$ and $i = 1, \dots, n$.

7. Quantized Weyl algebras. For $A = A_n^{Q,\Gamma}(k)$, take $\mathcal{H} = (k^\times)^n$ and

$$(\alpha_1, \dots, \alpha_n).x_i = \alpha_i x_i$$
$$(\alpha_1, \dots, \alpha_n).y_i = \alpha_i^{-1} y_i$$

for $i = 1, \dots, n$.

8. Quantum symplectic spaces. For $A = \mathcal{O}_q(\mathfrak{sp}\, k^{2n})$, take $\mathcal{H} = (k^\times)^n$ and

$$(\alpha_1, \dots, \alpha_n).x_i = \alpha_i x_i$$
$$(\alpha_1, \dots, \alpha_n).x_{i'} = \alpha_i^{-1} x_{i'}$$

for $i = 1, \dots, n$.

9. Quantum Euclidean spaces. For $A = \mathcal{O}_q(\mathfrak{o}\, k^n)$, set $m = \lfloor n/2 \rfloor$ and take $\mathcal{H} = (k^\times)^m$, with

$$\begin{aligned}
(\alpha_1, \dots, \alpha_m).x_i &= \alpha_i x_i && (i = 1, \dots, m)\\
(\alpha_1, \dots, \alpha_m).x_{i'} &= \alpha_i^{-1} x_{i'} && (i = 1, \dots, m)\\
(\alpha_1, \dots, \alpha_m).x_{m+1} &= x_{m+1} && (\text{if } n \text{ is odd})
\end{aligned}$$

3.6. We close this part of the paper by recording cases in the literature where the prime spectrum of an algebra A from Section 1, stratified by a torus $\mathcal{H}$ from (3.5), fits into the pattern discussed in (3.1) and (3.2). Although the results quoted from [**20**] do not provide Ore sets of normal elements, they can be modified to do so in all but one of the cases of interest here, as we shall show in Part II.

Theorem. *Let A be one of the following algebras: $\mathcal{O}_{\boldsymbol{q}}(k^n)$; $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$ or $\mathcal{O}_{\lambda,\boldsymbol{p}}(GL_n(k))$ or $\mathcal{O}_{\lambda,\boldsymbol{p}}(SL_n(k))$ with λ not a root of unity; $\mathcal{O}_q(G)$ with q transcendental over $\mathbb{Q}$ and k algebraically closed; $\mathcal{O}_{q,p}(G, L)$ with $q \in \mathbb{C}$ not a root of unity; $A_n^{Q,\Gamma}(k)$ with the q_i not roots of unity; $\mathcal{O}_q(\mathfrak{sp}\, k^{2n})$ or $\mathcal{O}_q(\mathfrak{o}\, k^n)$ with q not a root of unity. Let $\mathcal{H}$ be the corresponding torus acting on A as in* (3.5)*. Then $\mathcal{H}$-spec A is finite, and for each $J \in \mathcal{H}$-spec A the following hold:*

(a) *There exists an Ore set* $\mathcal{E}_J$ *of regular elements in* A/J *such that* $\operatorname{spec}_J A$ *is homeomorphic to* $\operatorname{spec} Z((A/J)[\mathcal{E}_J^{-1}])$ *via localization and contraction.*

(b) $Z((A/J)[\mathcal{E}_J^{-1}])$ *is a commutative Laurent polynomial algebra over an extension field of* k.

(c) $\operatorname{prim}_J A$ *equals the set of maximal elements of* $\operatorname{spec}_J A$.

(d) *If* k *is algebraically closed, then* $\operatorname{prim}_J A$ *consists of a single* $\mathcal{H}$*-orbit.*

Proof. For $\mathcal{O}_{\boldsymbol{q}}(k^n)$, see [**19**, Theorem 2.3, Proposition 2.11, Corollary 1.5]. For $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$, $A_n^{Q,\Gamma}(k)$, $\mathcal{O}_q(\mathfrak{sp}\, k^{2n})$ and $\mathcal{O}_q(\mathfrak{o}\, k^n)$, see [**20**, Theorems 5.3, 5.5, 5.8, 5.11, 6.6, 6.8]. The cases $\mathcal{O}_{\lambda,\boldsymbol{p}}(GL_n(k))$ and $\mathcal{O}_{\lambda,\boldsymbol{p}}(SL_n(k))$ follow directly. For $\mathcal{O}_q(G)$ and $\mathcal{O}_{q,p}(G,L)$, see (3.1) and (3.2). $\square$

II. An Axiomatic Development of Stratification Properties

This part of the paper is a variation on a theme developed in joint work with E. S. Letzter [**20**]. The goal is to show that, in the case of a torus $\mathcal{H}$ acting *rationally* on a noetherian k-algebra A, the $\mathcal{H}$-stratification of $\operatorname{spec} A$ enjoys analogs of all the properties of the stratification presented in Theorem 3.6, under relatively mild hypotheses on A and $\mathcal{H}$. Both developments (here and in [**20**]) include – at present – one hypothesis which requires substantial work to verify in the examples of interest. In [**20**, Section 6], that hypothesis is the complete primeness of the $\mathcal{H}$-prime ideals of A. Here, we rely on a type of normal separation in $\mathcal{H}$-$\operatorname{spec} A$. One disadvantage is the "loss" of one example, namely $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$, for which normal separation has not been verified. (It is conjectured to hold, and is easily checked for the case $m = n = 2$.) However, the development here provides a better fit with the work of Hodges-Levasseur, Joseph, and Hodges-Levasseur-Toro on $\mathcal{O}_q(G)$ and $\mathcal{O}_{q,p}(G,L)$ in which the stratified picture of $\operatorname{spec} A$ first arose. For example, the localizations $(A/J)[\mathcal{E}_J^{-1}]$ in the present development coincide with the ones obtained in [**22**, **23**, **29**, **30**, **24**] in the appropriate cases, whereas the localizations used in [**20**, Section 6] are larger. Here we also obtain a few additional properties, namely that $(A/J)[\mathcal{E}_J^{-1}]$ is an affine k-algebra in appropriate circumstances, and that normal separation in $\mathcal{H}$-$\operatorname{spec} A$ by $\mathcal{H}$-eigenvectors implies normal separation in $\operatorname{spec} A$.

The development of our theme is based on the equivalence between rational actions of $(k^\times)^r$ and $\mathbb{Z}^r$-gradings (see (5.1) for details). We begin by working out the basic properties of a stratification relative to graded-prime ideals for $\mathbb{Z}^r$-graded rings satisfying an appropriate normal separation condition. This may have some independent interest in the context of group-graded rings. Since it is at this stage where our variation differs in certain aspects from that in [**20**, Section 6], we provide full details. This is the content of Section 4. In Section 5, we translate these results into the context of $(k^\times)^r$-actions. That process is essentially the same as in [**20**, Section 6].

4. Graded-normal separation

Here we build a stratification in the context of rings graded by a free abelian group, with a normal separation hypothesis. This hypothesis will allow us to construct certain localizations which are graded-simple (i.e., have no nontrivial homogeneous ideals). Hence, we begin by analyzing the prime spectra of some graded-simple rings.

Lemma 4.1. *Let R be a graded-simple ring graded by an abelian group G.*

(a) *The center $Z(R)$ is a homogeneous subring of R, strongly graded by the subgroup $G_Z = \{x \in G \mid Z(R)_x \neq 0\}$ of G.*

(b) *Every nonzero homogeneous element of $Z(R)$ is invertible.*

(c) *As $Z(R)$-modules, $Z(R)$ is a direct summand of R.*

(d) *Suppose that G_Z is a free abelian group of finite rank. Choose a basis $\{g_1, \dots, g_n\}$ for G_Z, and choose a nonzero element $z_j \in Z(R)_{g_j}$ for each j. Then $Z(R) = Z(R)_1[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$, a Laurent polynomial ring over the field $Z(R)_1$.*

Proof. It is obvious that $Z(R)$ is a homogeneous subring of R, graded by the set G_Z. Part (b) follows from the graded-simplicity of R, and then it is clear that G_Z is a subgroup of G and that $Z(R)$ is strongly G_Z-graded. This proves part (a). Part (d) is routine (see the proof of [**20**, Lemma 6.3(c)]).

Note that $S = \bigoplus_{x \in G_Z} R_x$ is a homogeneous subring of R containing $Z(R)$. Further, S is a left S-module direct summand of R, with complement $M = \bigoplus_{x \in G \setminus G_Z} R_x$. On the other hand, as in [**20**, Lemma 6.3(c)], S is a free $Z(R)$-module with a basis including 1, whence $Z(R)$ is a $Z(R)$-module direct summand of S. Part (c) follows. $\square$

Proposition 4.2. *Let R be a graded-simple ring graded by an abelian group G. Then there exist bijections between the sets of ideals of R and $Z(R)$, given by contraction and extension, that is,*

$$I \mapsto I \cap Z(R) \qquad \text{and} \qquad J \mapsto JR.$$

Proof. It is clear from Lemma 4.1(c) that $JR \cap Z(R) = J$ for every ideal J of $Z(R)$. It remains to show that $(I \cap Z(R))R = I$ for any ideal I of R. Set $J = I \cap Z(R)$, and suppose that $I \neq JR$. Pick an element $x \in I \setminus JR$ of minimal length, say length n. Then $x = x_1 + \cdots + x_n$ for some nonzero elements $x_i \in R_{g_i}$, where $g_1, \dots, g_n$ are distinct elements of G.

Now Rx_1R is a nonzero homogeneous ideal of R, so $Rx_1R = R$ by graded-simplicity and $\sum_j a_jx_1b_j = 1$ for some $a_j, b_j \in R$. Express each a_j, b_j as a sum of homogeneous elements, and substitute these expressions in the terms $a_jx_1b_j$. This expands the sum $\sum_j a_jx_1b_j$ in the form $\sum_t c_tx_1d_t$ with all c_t, d_t homogeneous. Hence, after relabelling, we may assume that the a_j and b_j are all homogeneous, say of degrees e_j and f_j, respectively.

Comparing identity components in the equation $\sum_j a_jx_1b_j = 1$, we obtain

$$\sum_{e_jg_1f_j=1} a_jx_1b_j = 1.$$

Thus, after deleting all other $a_jx_1b_j$ terms, we may assume that $e_jg_1f_j = 1$ for all j. Since G is abelian, it follows that $e_jf_j = g_1^{-1}$ for all j.

Set $x' = \sum_j a_jxb_j$, and note that $x' \in I$. Further,

$$x' = \sum_{i=1}^{n} \sum_j a_jx_ib_j$$

where $\sum_j a_jx_ib_j \in R_{g_1^{-1}g_i}$ for each i. As a result, x' is an element whose support is contained in $\{1, g_1^{-1}g_2, \dots, g_1^{-1}g_n\}$ and whose identity component is 1. Hence, x_1x' is an element of I whose support is contained in $\{g_1, \dots, g_n\}$ and whose g_1-component is x_1. Comparing this element with x, we see that $x - x_1x'$ is an element of I whose support is contained in $\{g_2, \dots, g_n\}$. By the minimality of n, we must have $x - x_1x' \in JR$, and so $x' \notin JR$. Therefore, after replacing x by x', there is no loss of generality in assuming that $g_1 = 1$ and $x_1 = 1$.

Claim: There do not exist $g \in G$ and a nonzero element $y \in I$ whose support is properly contained in the set $\{gg_1, \dots, gg_n\}$.

Suppose there do exist such g and y. Say the gg_s-component of y is nonzero. As above, there exists an element of the form $y' = \sum_j a_jyb_j$ whose support is properly contained in $\{g_s^{-1}g_1, \dots, g_s^{-1}g_n\}$ and whose identity component is 1. Then $y' \in I$, and $x - x_sy'$ is an element of I whose support is properly contained in $\{g_1, \dots, g_n\}$. Since y' and $x - x_sy'$ are elements of I of length less than n, they must lie in JR, by the minimality of n. But then $x \in JR$, contradicting our assumptions. Therefore the claim is proved.

Finally, consider any homogeneous element $r \in R$, say $r \in R_g$. Then $rx - xr$ is an element of I with support contained in $\{gg_2, \dots, gg_n\}$, and so $rx - xr = 0$ by the claim. It follows that $x \in Z(R)$ and so $x \in J$, contradicting our assumption that $x \notin JR$.

Therefore $I = JR$. $\square$

Corollary 4.3. *If R is a graded-simple ring graded by an abelian group, then contraction and extension provide mutually inverse homeomorphisms between* $\operatorname{spec} R$ *and* $\operatorname{spec} Z(R)$. $\square$

4.4. Graded-normal separation. Let R be a group-graded ring. Recall that a *graded-prime* ideal of R is any proper homogeneous ideal P such that whenever I, J are homogeneous ideals of R with $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$. We say that R has *graded-normal separation* provided that for any proper inclusion $P \supset Q$ of graded-prime ideals of R, there exists a homogeneous element $c \in P \setminus Q$ which is normal modulo Q.

Theorem 4.5. *Let R be a right noetherian ring graded by an abelian group G, and assume that R has graded-normal separation.*

Let J be a graded-prime ideal of R, set

$$\mathcal{S}_J = \{P \in \operatorname{spec} R \mid J \textit{ is the largest homogeneous ideal contained in } P\},$$

and let $\mathcal{E}_J$ be the multiplicative set of all the nonzero homogeneous normal elements in the ring R/J.

(a) *$\mathcal{E}_J$ is a right and left denominator set of regular elements in R/J.*

(b) *The localization map $R \to R/J \to R_J = (R/J)[\mathcal{E}_J^{-1}]$ induces a homeomorphism of $\mathcal{S}_J$ onto* $\operatorname{spec} R_J$.

(c) *Contraction and extension induce mutually inverse homeomorphisms between* $\operatorname{spec} R_J$ *and* $\operatorname{spec} Z(R_J)$.

(d) *If G is free abelian of rank $r < \infty$, then $Z(R_J)$ is a commutative Laurent polynomial ring over the field $Z(R_J)_1$ (the identity component of $Z(R_J)$ in the induced G-grading), in r or fewer indeterminates.*

Proof. Without loss of generality, $J = 0$. Consequently, R is now a graded-prime ring, i.e., 0 is a graded-prime ideal of R.

(a) For $x \in \mathcal{E}_J$, observe that $\operatorname{r.ann}_R(x)$ is a homogeneous ideal such that $(xR)\operatorname{r.ann}_R(x) = 0$. Since R is graded-prime and $x \neq 0$, we must have $\operatorname{r.ann}_R(x) = 0$. Similarly, $\operatorname{l.ann}_R(x)=0$; thus all elements of $\mathcal{E}_J$ are regular. The Ore condition follows directly from normality.

(b) If $P \in \operatorname{spec} R$ and P_0 is the largest homogeneous ideal contained in P, then P_0 is a graded-prime ideal. Because of graded-normal separation, P_0 is nonzero precisely when $P_0 \cap \mathcal{E}_J$ is nonempty. Hence, $\mathcal{S}_J$ consists precisely of those prime ideals of R disjoint from $\mathcal{E}_J$. Standard localization theory (e.g., [**21**, Theorem 9.22]) now yields part (b).

(c) Since the elements of $\mathcal{E}_J$ are homogeneous, the G-grading on R extends (uniquely) to a G-grading on R_J. Because of the noetherian hypothesis on R, two-sided ideals I in R induce two-sided ideals IR_J in R_J (e.g., [**21**, Theorem

9.20(a)]). The standard arguments concerning contractions of prime ideals (e.g., [**21**, Theorem 9.20(c)]) now show that any graded-prime ideal of R_J must contract to a graded-prime ideal of R. Since every nonzero graded-prime ideal of R meets $\mathcal{E}_J$, we thus see that R_J has no nonzero graded-prime ideals. On the other hand, any maximal proper homogeneous ideal of R_J is graded-prime. Therefore R_J must be graded-simple. Part (c) now follows from Corollary 4.3.

(d) This follows from Lemma 4.1(d). $\square$

Corollary 4.6. *Let R be a right noetherian ring graded by an abelian group G. If R has graded-normal separation, then R also has normal separation.*

Proof. Let $P \supset Q$ be a proper inclusion of prime ideals of R. Let P_0 and Q_0 be the largest homogeneous ideals contained in P and Q, respectively, and note that P_0 and Q_0 are graded-prime ideals of R. Obviously $P_0 \supseteq Q_0$.

If $P_0 \neq Q_0$, then by assumption there exists a homogeneous element $c \in P_0 \setminus Q_0$ which is normal modulo Q_0. In particular, $c \in P$ and c is normal modulo Q. Since RcR is a homogeneous ideal of R, not contained in Q_0, we see that $RcR \not\subseteq Q$. Thus in this case we are done.

Now assume that $P_0 = Q_0$. It is harmless to pass to R/Q_0, and hence there is no loss of generality in assuming that $P_0 = Q_0 = 0$. Consequently, 0 is now a graded-prime ideal of R.

Let S denote the localization $R_0 = R[\mathcal{E}_0^{-1}]$, in the notation of Theorem 4.5. By part (b) of the theorem, $PS \supset QS$ is a proper inclusion of prime ideals of S. Part (c) then implies that there exists a central element $z \in PS \setminus QS$. Write $z = cx^{-1}$ for some $c \in P$ and $x \in \mathcal{E}_0$, and note that $c \notin Q$. Since $xR = Rx$, we have $zxR = zRx$ in S. But z commutes with R, and so $zxR = Rzx$, that is, $cR = Rc$. Therefore c is a normal element of R, and normal separation is proved. $\square$

5. Stratification under rational torus actions

We now turn to rational actions by tori as automorphisms of noetherian algebras. Under suitable normal separation hypotheses, the results of the previous section apply, yielding a picture of the corresponding stratifications that incorporates the features of the examples discussed in Section 3.

Throughout this section, we assume that our base field k is infinite. In the examples of interest, this is automatic due to the presence of non-roots of unity.

5.1. Rational torus actions. Let A be a k-algebra, and let $\mathcal{H}$ be a torus over k, say $\mathcal{H} = (k^\times)^r$. An action of $\mathcal{H}$ on A by k-algebra automorphisms is said to be *rational* provided A is a directed union of finite dimensional

$\mathcal{H}$-stable subspaces V_i such that the induced group homomorphisms $\mathcal{H} \to GL(V_i)$ are morphisms of algebraic varieties. Let $\widehat{\mathcal{H}}$ denote the set of *rational characters* of $\mathcal{H}$, that is, algebraic group morphisms $\mathcal{H} \to k^\times$. Then $\widehat{\mathcal{H}}$ is an abelian group under pointwise multiplication. Since k is infinite, $\widehat{\mathcal{H}}$ is a lattice of rank r, with a basis given by the component projections $(k^\times)^r \to k^\times$.

The rationality of an $\mathcal{H}$-action implies that A is spanned by $\mathcal{H}$-eigenvectors [**44**, Chapter 5, Corollary to Theorem 36], and that the eigenvalues of these eigenvectors are rational. This yields a k-vector space decomposition $A = \bigoplus_{x \in \widehat{\mathcal{H}}} A_x$, where A_x denotes the x-eigenspace of A. (Up to this point, everything holds for a rational action by k-linear transformations.) Since $\mathcal{H}$ acts by automorphisms, $A_x A_y \subseteq A_{xy}$ for all $x, y \in \widehat{\mathcal{H}}$. Thus we have a grading of A by the free abelian group $\widehat{\mathcal{H}}$, such that the homogeneous elements are the $\mathcal{H}$-eigenvectors in A.

Conversely, any grading of A by $\mathbb{Z}^r$ arises in this fashion, as noted in [**50**, p. 784]. To see this, identify $\mathbb{Z}^r$ with $\widehat{\mathcal{H}}$, and let $A = \bigoplus_{x \in \widehat{\mathcal{H}}} A_x$ be an $\widehat{\mathcal{H}}$-grading. Let each $h \in \mathcal{H}$ act on A as the k-linear transformation with eigenvalue $x(h)$ on A_x for all x, that is, $h.a = x(h)a$ for $a \in A_x$. This yields a rational action of $\mathcal{H}$ on A by k-algebra automorphisms, such that each A_x is the x-eigenspace. The reader may take this as the definition of a rational $\mathcal{H}$-action if so desired.

In case k is algebraically closed and A is noetherian, a remarkable theorem of Moeglin-Rentschler and Vonessen [**40**, Théorème 2.12(ii)], [**59**, Theorem 2.2] states that $\mathcal{H}$ acts transitively on the $\mathcal{H}$-strata of rational ideals in A, that is, each $\mathcal{H}$-stratum of rational ideals is a single $\mathcal{H}$-orbit. This holds, in fact, for any algebraic group $\mathcal{H}$, not just tori. If, further, the Dixmier-Moeglin equivalence holds, then each $\mathcal{H}$-stratum in $\operatorname{prim} A$ is a single $\mathcal{H}$-orbit. For the very special case where $\mathcal{H}$ is a torus and all $\mathcal{H}$-prime ideals of A are completely prime, a direct proof of this transitivity result is given in [**20**, Theorem 6.8]. Here we shall derive such a result using normal separation in place of complete primeness – see Theorem 5.5(c).

5.2. Normal separation in $\mathcal{H}$-spec. Assume that we have a torus $\mathcal{H}$ acting rationally on a k-algebra A by k-algebra automorphisms. In this situation, we shall assume that A has been equipped with the natural $\widehat{\mathcal{H}}$-grading as in (5.1). With respect to this grading, the homogeneous ideals of A are precisely the $\mathcal{H}$-stable ideals, and so the graded-prime ideals coincide with the $\mathcal{H}$-prime ideals. We say that $\mathcal{H}\text{-spec}\, A$ has *normal $\mathcal{H}$-separation* provided that for any proper inclusion $P \supset Q$ of $\mathcal{H}$-prime ideals, there exists an $\mathcal{H}$-eigenvector $c \in P \setminus Q$ which is normal modulo Q. This condition is just graded-normal separation with respect to the $\widehat{\mathcal{H}}$-grading as defined in

(4.4).

The term '$\mathcal{H}$-normal separation' we would reserve for a slightly stronger condition involving '$\mathcal{H}$-normal' elements. If Q is an $\mathcal{H}$-stable ideal of A, an element $c \in A$ is said to be *$\mathcal{H}$-normal modulo Q* in case there exists $h \in \mathcal{H}$ such that $ca - h(a)c \in Q$ for all $a \in A$. It is easily checked that each of the homogeneous components of such an element c is also $\mathcal{H}$-normal modulo Q. Now $\mathcal{H}\text{-spec}\, A$ has *$\mathcal{H}$-normal separation* provided that for any proper inclusion $P \supset Q$ of $\mathcal{H}$-prime ideals of A, there exists an element $c \in P \setminus Q$ which is $\mathcal{H}$-normal modulo Q. This strengthening of normal $\mathcal{H}$-separation is not needed in our proofs, but it does hold in almost all the examples discussed in Sections 2 and 3, and is conjectured to hold in the remaining one, namely $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$.

We note also that since the homogeneous ideals of A coincide with the $\mathcal{H}$-stable ideals, the largest homogeneous ideal of A contained in a given ideal I is just $(I : \mathcal{H})$. Hence, the strata of $\operatorname{spec} A$ with respect to graded-prime ideals, as in Theorem 4.5, coincide with the $\mathcal{H}$-strata. Thus Theorem 4.5 and Corollary 4.6 yield the following results.

Theorem 5.3. *Let A be a right noetherian k-algebra, and let $\mathcal{H}$ be a torus of rank r, acting rationally on A by k-algebra automorphisms. Assume that $\mathcal{H}\text{-spec}\, A$ has normal $\mathcal{H}$-separation. Then $\operatorname{spec} A$ has normal separation.*

Now let J be an $\mathcal{H}$-prime ideal of A, let $\mathcal{E}_J$ be the multiplicative set of all nonzero normal $\mathcal{H}$-eigenvectors in A/J, and set $A_J = (A/J)[\mathcal{E}_J^{-1}]$.

(a) *The localization map $A \to A/J \to A_J$ induces a homeomorphism of $\operatorname{spec}_J A$ onto $\operatorname{spec} A_J$.*

(b) *Contraction and extension induce mutually inverse homeomorphisms between $\operatorname{spec} A_J$ and $\operatorname{spec} Z(A_J)$.*

(c) *The ring $Z(A_J)$ is a commutative Laurent polynomial ring over the field $Z(\operatorname{Fract} A/J)^{\mathcal{H}}$ (the fixed subfield of $Z(\operatorname{Fract} A/J)$ under the induced action of $\mathcal{H}$), in r or fewer indeterminates.*

Proof. These statements follow immediately from Theorem 4.5 and Corollary 4.6 except for the description of the coefficient field in part (c). According to Theorem 4.5(d), this field equals $Z(A_J)_1$. By definition of the $\widehat{\mathcal{H}}$-grading, $Z(A_J)_1 = Z(A_J)^{\mathcal{H}}$, which is clearly contained in $Z(\operatorname{Fract} A/J)^{\mathcal{H}}$. Thus, it only remains to prove the reverse inequality.

As in the proof of Theorem 4.5(c), A_J is graded-simple with respect to the $\widehat{\mathcal{H}}$-grading, and so it is $\mathcal{H}$-simple. Given any $u \in Z(\operatorname{Fract} A/J)^{\mathcal{H}}$, observe that the set $I = \{a \in A_J \mid au \in A_J\}$ is a nonzero $\mathcal{H}$-stable ideal of A_J. By $\mathcal{H}$-simplicity, $I = A_J$, whence $u \in A_J$. Therefore $u \in Z(A_J)^{\mathcal{H}}$, as desired. $\square$

5.4. To compare Theorem 5.3 with [**20**, Theorem 6.6], note that the latter theorem only applies to completely prime $\mathcal{H}$-prime ideals of A. That restriction is due to the lack of any known graded Goldie theorem for graded-prime rings (cf. [**20**, §6.1]), whereas one can easily prove a graded Ore theorem [**20**, Lemma 6.2]. We have finessed the graded Goldie problem here by assuming a suitable supply of normal elements. Other advantages of this assumption include the following result.

Proposition. *Let A, $\mathcal{H}$, J, $\mathcal{E}_J$, A_J be as in Theorem 5.3. If $\mathcal{H}\text{-spec}\,A$ is finite and A is an affine k-algebra, then A_J is an affine k-algebra.*

Proof. After passing to A/J, we may assume that $J = 0$. If $\mathcal{E}_J$ consists of units, then $A_J = A$ and we are done. Otherwise, A contains some proper nonzero $\mathcal{H}$-stable ideals (generated by nonzero normal $\mathcal{H}$-eigenvectors which are not units). Hence, the maximal proper $\mathcal{H}$-stable ideals of A are nonzero, and these are, in particular, $\mathcal{H}$-prime. In other words, 0 is not the only $\mathcal{H}$-prime ideal of A.

Let $J_1, \dots, J_n$ be the nonzero $\mathcal{H}$-prime ideals of A. By our normal $\mathcal{H}$-separation assumption, each J_i contains a nonzero normal $\mathcal{H}$-eigenvector c_i. Set $c = c_1 c_2 \cdots c_n$, a normal $\mathcal{H}$-eigenvector contained in all nonzero $\mathcal{H}$-primes of A. Since $J = 0$ is $\mathcal{H}$-prime, $c \neq 0$.

The action of $\mathcal{H}$ on A extends naturally to an action by k-algebra automorphisms on the localization $B = A[c^{-1}]$. Since any $\mathcal{H}$-prime of $A[c^{-1}]$ contracts to an $\mathcal{H}$-prime of A, we see that 0 is the only $\mathcal{H}$-prime of B. Thus B is $\mathcal{H}$-simple.

If $e \in \mathcal{E}_J$, then since eA is an ideal of A and A is right noetherian, eB is an ideal of B. But eB is nonzero and $\mathcal{H}$-stable, whence $eB = B$. Thus e is right invertible in B, and hence invertible. Therefore $A_J = B$, which is clearly affine. $\square$

5.5. Recall that a noetherian k-algebra A satisfies the *Nullstellensatz (over k)* [**39**, Chapter 9] provided that A is a Jacobson ring and the endomorphism rings of all simple A-modules are algebraic over k. In that case, it follows from the Jacobson condition that every locally closed prime of A is primitive, and from [**14**, Lemma 4.1.6] that every primitive ideal of A is rational. As in [**20**, Section 6], the presence of the Nullstellensatz yields the following refinements to Theorem 5.3.

Theorem. *Let A be a noetherian k-algebra satisfying the Nullstellensatz, and let $\mathcal{H}$ be a torus acting rationally on A by k-algebra automorphisms. Assume that $\mathcal{H}\text{-spec}\,A$ is finite and satisfies normal $\mathcal{H}$-separation.*

(a) *A satisfies the Dixmier-Moeglin equivalence, and the primitive ideals of A are precisely the maximal elements of the $\mathcal{H}$-strata in* $\operatorname{spec} A$.

(b) *The fields* $Z(\operatorname{Fract} A/J)^{\mathcal{H}}$ *occurring in Theorem* 5.3(c) *are all algebraic over* k*.*

(c) *If* k *is algebraically closed, each* $\mathcal{H}$*-stratum in* $\operatorname{prim} A$ *consists of a single* $\mathcal{H}$*-orbit.*

Proof. Except for minor modifications, the proofs are the same as in [**20**, Theorem 6.8, Corollary 6.9].

Since $\mathcal{H}$-spec A is finite, maximal elements of $\mathcal{H}$-strata are locally closed in spec A [**20**, §2.2(ii)]. Hence, to prove part (a) it remains to show that rational ideals are maximal within their $\mathcal{H}$-strata in spec A.

Let $J \in \mathcal{H}$-spec A, and define $\mathcal{E}_J$ and A_J as in Theorem 5.3. Then $Z(A_J)$ is a Laurent polynomial ring of the form $k_J[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$, where $k_J = Z(\operatorname{Fract} A/J)^{\mathcal{H}}$. By the previous paragraph, any maximal element of $\operatorname{spec}_J A$ is locally closed, and such primes are rational because of the Nullstellensatz. Hence, there exist rational ideals in $\operatorname{spec}_J A$.

Now let P be any rational ideal in $\operatorname{spec}_J A$, and set $Q = PA_J \cap Z(A_J)$. Then $Z(A_J)/Q$ embeds in $Z(\operatorname{Fract} A/P)$, which is algebraic over k by assumption. Hence, k_J is algebraic over k (thus establishing part (b)), and Q is a maximal ideal of $Z(A_J)$. It follows that P is maximal in $\operatorname{spec}_J A$, which completes part (a).

If k is algebraically closed, one checks that the induced action of $\mathcal{H}$ on $Z(A_J)$ incorporates all automorphisms such that $z_1 \mapsto \beta_1 z_1, \dots, z_n \mapsto \beta_n z_n$ for arbitrary $\beta_i \in k^\times$. Hence, $\mathcal{H}$ acts transitively on $\max Z(A_J)$, and part (c) follows. $\square$

5.6. Theorems 5.3 and 5.5 can be used to derive all cases of Theorem 3.6 except that of $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$. All the necessary hypotheses except for normal separation can be verified fairly readily from the basic descriptions of the algebras. Further, these hypotheses are also known to hold in $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_{m,n}(k))$, where normal separation is conjectured. Thus, we conclude by focusing on normal separation as a key problem:

Problem. *Find general hypotheses on a noetherian* k*-algebra* A *equipped with a rational action of a torus* $\mathcal{H}$ *by* k*-algebra automorphisms which*

(a) *imply that* $\mathcal{H}$-spec A *is finite and satisfies normal* $\mathcal{H}$*-separation;*

(b) *are readily proved for the algebras described in Section 1.*

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