REPRESENTATIONS OF DISTRIBUTIVE SEMILATTICES IN IDEAL LATTICES OF VARIOUS ALGEBRAIC STRUCTURES

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Abstract. We study the relationships among existing results about representations of distributive semilattices by ideals in dimension groups, von Neumann regular rings, C*-algebras, and complemented modular lattices. We prove additional representation results which exhibit further connections with the scattered literature on these different topics.

Introduction

Many algebraic theories afford a notion of *ideal*, and the collection of all ideals of a given object typically forms a complete lattice with respect to inclusion. It is natural to ask which lattices can be represented as a lattice of ideals for a given type of object. Often, the lattice of ideals of an object is algebraic, in which case this lattice is isomorphic to the lattice of ideals of the (join-) subsemilattice of compact elements. For instance, this holds for lattices of ideals of rings, monoids, and partially ordered abelian groups. Hence, lattice representation problems often reduce to corresponding representation problems for (join-) semilattices. For example, to prove that a given algebraic lattice L occurs as the lattice of ideals of a ring of some type, it suffices to show that the semilattice of compact elements of L occurs as the semilattice of finitely generated ideals of a suitable ring.

We shall be concerned here with representation problems for *distributive* algebraic lattices, which correspond to representation problems for distributive semilattices. The contexts we discuss include congruence lattices, complemented modular lattices, (von Neumann) regular rings, dimension groups, and approximately finite dimensional C*-algebras. All these contexts are interconnected, and a main goal of our paper is to develop these interconnections sufficiently to allow representation theorems for distributive semilattices in one context to be transferred to other contexts.

Since readers familiar with one of our contexts may not be fully at home in others, we try to provide full details and all relevant definitions in the appropriate sections of the paper. While the reader may encounter some undefined concepts in this introduction, we hope that the flavor of the results discussed will come through nonetheless on a first reading. All the required concepts will be made precise later in the paper.

Typical representation results for distributive semilattices include the following:

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Schmidt's Theorem. *Every finite distributive lattice is isomorphic to the semilattice of compact congruences of some complemented modular lattice.* □

This is a result of E.T. Schmidt [30]. It is probably the earliest representation result of distributive semilattices by complemented modular lattices. Further lattice-theoretical representation results are discussed in [18], mainly in relation with the *Congruence Lattice Problem*, that asks whether every distributive algebraic lattice is isomorphic to the congruence lattice of a lattice.

A stronger version of Schmidt's Theorem follows from a result of G.M. Bergman [2]:

Bergman's Theorem. *Let L be a distributive algebraic lattice with only countably many compact elements, and K any field. Then there exists a locally matricial K-algebra R of countable dimension whose lattice of two-sided ideals is isomorphic to L. If, in addition, the greatest element of L is compact, then one can choose R unital.* □

According to the abovementioned correspondence between semilattices and algebraic lattices, this can also be formulated as follows: *Every countable distributive 0-semilattice S is isomorphic to the semilattice of finitely generated two-sided ideals in some locally matricial algebra R of countable dimension. If, in addition, S has a largest element, then one can choose R unital.*

Locally matricial algebras are, in particular, regular rings, and the finitely generated right ideals of any regular ring R form a sectionally complemented modular lattice, $\mathcal{L}(R)$. Further, the semilattice of finitely generated two-sided ideals of R turns out to be isomorphic to the semilattice of compact congruences of $\mathcal{L}(R)$, see Proposition 7.3 (*cf.* [37, Corollary 4.4]). Hence, Bergman's Theorem yields the following result:

Corollary. *Any countable distributive 0-semilattice is isomorphic to the semilattice of compact congruences of some sectionally complemented modular lattice.* □

The $\aleph_1$ version of Bergman's Theorem is still open (see the discussion around Problem 2 in Section 10). The second author has shown that the $\aleph_2$ version has a *negative* answer (see [37]). A precursor to Bergman's Theorem was obtained by K.H. Kim and F.W. Roush, who proved that any finite distributive lattice is isomorphic to the lattice of two-sided ideals of some unital locally matricial algebra of countable dimension, see [22, Corollary to Theorem 4]. In view of the connections discussed above, this result is already sufficient to yield Schmidt's Theorem.

An interesting representation result for distributive semilattices of arbitrary size was proved by P. Pudlák, see [27, Fact 4, p. 100]:

Pudlák's Lemma. *Every distributive semilattice is the direct union of all its finite distributive subsemilattices.* □

These results are similar in spirit to representation results in other fields of mathematics, that were proved completely independently.

We start with G.A. Elliott, who classified countable direct limits of locally matricial algebras by an invariant equivalent to their ordered K_0 groups, see [10, Theorem 4.3] (*cf.* [12, Theorem 15.26]). Elliott's initial result towards the question of which ordered groups appear in this classification [10, Theorem 5.5] can be phrased as follows:

Elliott's Lemma. *Let G be the direct limit of a countable sequence of simplicial groups, and let K be a field. Then there exists a locally matricial K-algebra R of* *countable dimension such that $K_0(R) \cong G$. If, in addition, G has an order-unit, then one can choose R unital.* □

Direct limits of countable sequences of simplicial groups, or, more generally, of arbitrary directed families of simplicial groups, were characterized by E.G. Effros, D.E. Handelman and C.-L. Shen as (countable) dimension groups, see [9]. However, a very similar result was proved four years earlier by P.A. Grillet [19], using a categorical result of R.T. Shannon [31]. We refer to Section 3 for details. This characterization, together with Elliott's Lemma, allows one to conclude that any countable dimension group is isomorphic to K_0 of a locally matricial algebra of countable dimension. That representation result was extended by Handelman and the first author [15] to dimension groups of size $\aleph_1$.

The basic aim of this paper is to bring all these results together. For instance, we prove, in Theorem 6.6, the following analogue of the Grillet and Effros-Handelman-Shen theorems: *every distributive semilattice is a direct limit of finite Boolean semilattices.* This gives, in Section 8, a second proof of Bergman's Theorem.

A third proof of Bergman's Theorem, also given in Section 8, involves the relationship between dimension groups and distributive semilattices. More specifically, we prove in Theorem 5.2 that every countable distributive 0-semilattice is isomorphic to the maximal semilattice quotient of some countable dimension group, and then we apply the Effros-Handelman-Shen Theorem and Elliott's Lemma. The machinery that allows us to conclude is, in fact, disseminated in the literature, and it is recalled in Section 7.

A parallel to Bergman's Theorem, in which any distributive algebraic lattice with countably many compact elements is represented as the lattice of closed ideals of an approximately finite-dimensional C*-algebra, is developed in Section 9. As an application, we use this result to provide a normal form for certain C*-algebras recently classified by H. Lin in [23].

Thus the present paper is, at the same time, a survey about many intricately interwoven results in the theories of dimension groups, semilattices, regular rings, C*-algebras, and complemented modular lattices, which have been evolving with various degrees of mutual independence for decades.

1. Basic concepts

We denote by ω the set of all natural numbers. A natural number n is identified with the finite set $\{0, 1, \dots, n-1\}$.

If $f\colon X \to Y$ is a map, $\ker(f)$, the *kernel* of f, denotes the equivalence relation associated with f, that is,

$$\ker(f) = \{\langle u, v\rangle \in X \times X \colon f(u) = f(v)\}.$$

We write commutative monoids additively, and we endow every commutative monoid with its *algebraic preordering* $\leq$, defined by

$$x \leq y \text{ if and only if there exists } z \text{ such that } x + z = y.$$

An *ideal* (sometimes called an *o-ideal*) of a commutative monoid M is a nonempty subset I of M such that for all $x, y \in M$, $x + y \in I$ if and only if $x \in I$ and $y \in I$. (Note that this is a different concept than the notion of 'ideal' as used in semigroup theory.) Write $\operatorname{Id} M$ for the set of ideals of M, ordered by inclusion, and observe that $\operatorname{Id} M$ is a complete lattice (with infima given by intersections).

The *refinement property* is the semigroup-theoretical axiom stating that for all positive integers m and n, all elements a_i $(i < m)$ and b_j $(j < n)$ of M such that $\sum_{i<m} a_i = \sum_{j<n} b_j$, there are elements c_{ij} $(i < m,\ j < n)$ of M such that

$$a_i = \sum_{j<n} c_{ij} \quad \text{for all } i < m, \qquad \text{and} \qquad b_j = \sum_{i<m} c_{ij} \quad \text{for all } j < n.$$

A *refinement monoid* (*e.g.*, [8], [35]) is a commutative monoid which satisfies the refinement property; equivalently, the condition above is satisfied for $m = n = 2$. It is to be noted that in [8], every refinement monoid is, in addition, required to satisfy the axiom $x + y = 0 \Rightarrow x = y = 0$ (*conicality*), while this is not the case for most other authors (*e.g.*, [1], [35]).

A *semilattice* is a commutative semigroup S in which every element x is *idempotent*, that is, $x + x = x$. The algebraic preordering on S is then an ordering, given by $x \leq y$ if and only if $x + y = y$, hence all our semilattices are join-semilattices. We will usually denote by $\vee$, rather than $+$, the addition of a semilattice. An *ideal* (or *order-ideal*) of S is defined by the same axiom used to define an ideal of a monoid. In order-theoretic terms, an ideal of S is any nonempty *lower subset* I (*i.e.*, $(\forall x \in S)(\forall y \in I)(x \leq y \implies x \in I)$) which is closed under $\vee$. A 0-*semilattice* is a semilattice which is also a monoid, or, equivalently, a semilattice which has a least element. Similarly, a 0-*lattice* is a lattice with a least element.

An element a of a lattice L is *compact* if, for every subset X of L such that $\bigvee X$ exists, if $a \leq \bigvee X$, then there exists a finite subset Y of X such that $a \leq \bigvee Y$. Note that the set of compact elements of L forms a subsemilattice of L. A lattice L is *algebraic* if L is complete and every element of L is a supremum of compact elements.

If S is a semilattice, denote by $\operatorname{Id} S$ the set of ideals of S, ordered under inclusion. The *canonical embedding* from S into $\operatorname{Id} S$ is defined by

$$s \mapsto {\downarrow}s = \{x \in S \colon x \leq s\}.$$

Observe that $\operatorname{Id} S$ is a lattice if and only if S is downward directed, and is a complete lattice if and only if S has a least element. In the latter case, $\operatorname{Id} S$ is an algebraic lattice. Conversely, for every algebraic lattice L, the set of all compact elements of L is a 0-semilattice. The following classical result (*cf.* [3, Theorem VIII.8]) expresses the categorical equivalence between algebraic lattices and join 0-semilattices.

Proposition 1.1. *Let L be an algebraic lattice, and let S be the semilattice of all compact elements of L. Then the correspondence*

$$x \mapsto \{s \in S \colon s \leq x\}$$

defines an isomorphism from L onto $\operatorname{Id} S$. □

This can be extended without difficulty to define a categorical equivalence between 0-semilattices and $\{\vee, 0\}$-homomorphisms, and algebraic lattices with a suitable notion of homomorphism.

A semilattice S is *distributive* (see [16, p. 117]) if for all a, b_0, b_1 in S such that $a \leq b_0 \vee b_1$, there are elements a_0 and a_1 of S such that $a = a_0 \vee a_1$ and $a_i \leq b_i$ for all $i < 2$. This is equivalent to saying that S is downward directed and $\operatorname{Id} S$ is a distributive lattice, *cf.* [16, Lemma 11.1(iii)] or [17, Lemma II.5.1]. Together with Proposition 1.1, this shows that if L is a distributive algebraic lattice, then the semilattice of all compact elements of L is a distributive semilattice.

For every lattice L, we denote by $\operatorname{Con} L$ the lattice of all congruences of L. It is a well known theorem of N. Funayama and T. Nakayama (see [16, Corollary 9.16] or [17, II.3]) that $\operatorname{Con} L$ is a distributive algebraic lattice.

We denote by $\operatorname{Con}_{\mathrm{c}} L$ the semilattice of *compact* congruences of L; by the previous paragraph, $\operatorname{Con}_{\mathrm{c}} L$ is a distributive 0-semilattice. The elements of $\operatorname{Con}_{\mathrm{c}} L$ are exactly the finitely generated congruences of L.

For every partially ordered abelian group G, we denote by G^+ the positive cone of G, that is, the set of $x \in G$ such that $x \geq 0$. An *order-unit* of G is any element u of G^+ such that for every $x \in G$, there exists a positive integer n such that $x \leq nu$. We put $\mathbb{N} = \mathbb{Z}^+ \setminus \{0\}$.

Let G and H be partially ordered abelian groups. A *positive homomorphism* from G to H is a homomorphism of partially ordered abelian groups from G to H, that is, a group homomorphism $f\colon G \to H$ such that $f(G^+) \subseteq H^+$. We denote by f^+ the restriction of f from G^+ to H^+.

All the rings that we will consider are associative, but not necessarily unital.

2. Refinement monoids, dimension groups and distributive semilattices

Let M be a commutative monoid. There exists a least monoid congruence $\asymp$ on M such that $M/{\asymp}$ is a semilattice. It is convenient to define $\asymp$ in terms of the preordering $\propto$ defined by

$$x \propto y \text{ if and only if } (\exists n \in \mathbb{N})(x \leq ny);$$

then, $x \asymp y$ if and only if $x \propto y$ and $y \propto x$. The *maximal semilattice quotient* of M is the natural projection from M to $M/{\asymp}$, often identified with the semilattice $M/{\asymp}$ itself. We refer to [5] for the details.

This defines a *functor* from the category of commutative monoids, with monoid homomorphisms, to the category of 0-semilattices, with 0-semilattice homomorphisms. We will denote this functor by ∇. The proof of the following lemma is straightforward.

Lemma 2.1. *The functor ∇ preserves direct limits.* $\square$

Now let us go to refinement monoids:

Lemma 2.2. *Every refinement monoid M satisfies the* Riesz decomposition property*, that is, for all elements a, b_0 and b_1 of M such that $a \leq b_0 + b_1$, there are $a_0 \leq b_0$ and $a_1 \leq b_1$ in M such that $a = a_0 + a_1$.* $\square$

For semilattices, it is well known (and also easy to verify directly) that the converse of Lemma 2.2 is true:

Lemma 2.3. *Let S be a semilattice. Then S is distributive if and only if S satisfies the refinement property.* $\square$

We will be interested in the effect of ∇ on refinement monoids:

Lemma 2.4. *Let M be a commutative monoid. If M satisfies the Riesz decomposition property, then $\nabla(M)$ is a distributive semilattice.*

Proof. For every element x of M, denote by $[x]$ the image of x in $\nabla(M)$. Let a, b_0 and b_1 be elements of M such that $[a] \leq [b_0] \vee [b_1]$. By definition, there exists $n \in \mathbb{N}$ such that $a \leq nb_0 + nb_1$. Since M satisfies the Riesz decomposition property, there are $a_0 \leq nb_0$ and $a_1 \leq nb_1$ such that $a = a_0 + a_1$. Therefore, $[a] = [a_0] \vee [a_1]$, and $[a_i] \leq [b_i]$ for $i < 2$. □

Say that a partially ordered set $\langle P, \leq \rangle$ satisfies the *interpolation property* if, for all a_0, a_1, b_0 and b_1 in P such that $a_i \leq b_j$ for all i, $j < 2$, there exists $x \in P$ such that $a_i \leq x \leq b_j$ for all i, $j < 2$. An *interpolation group* is a partially ordered abelian group satisfying the interpolation property.

Lemma 2.5 (see [14, Proposition 2.1]). *Let G be a partially ordered abelian group. Then G is an interpolation group if and only if its positive cone G^+ is a refinement monoid.* □

Say that a partially ordered abelian group G is *directed* if it is directed as a partially ordered set; equivalently, $G = G^+ + (-G^+)$. Say that G is *unperforated* if for all $m \in \mathbb{N}$ and all $x \in G$, $mx \geq 0$ implies that $x \geq 0$. A *dimension group* is a directed, unperforated interpolation group.

For example, define a *dimension vector space* (over $\mathbb{Q}$) to be a directed interpolation group endowed with a structure of vector space over the field $\mathbb{Q}$ of the rational numbers, for which multiplication by positive rational scalars is order-preserving. Then it is obvious that every dimension vector space is a dimension group ($mx \geq 0$ implies $(1/m)mx \geq 0$, that is, $x \geq 0$).

By Lemma 2.4, the maximal semilattice quotient of the positive cone of a dimension group is a distributive 0-semilattice. The converse is an open problem (see Problem 1 in Section 10).

In Sections 4 and 5, we will solve positively two particular cases of this problem: the case where S is a *lattice* (Theorem 4.4), and the case where S is *countable* (Theorem 5.2).

An *ideal* of a partially ordered abelian group G is a subgroup I of G which is both *directed* and *convex* with respect to the ordering on G, the latter condition meaning that whenever $x \leq y \leq z$ with x, $z \in I$ and $y \in G$, then $y \in I$. We denote by $\operatorname{Id} G$ the set of ideals of G, ordered under inclusion; by [14, Corollary 1.10], $\operatorname{Id} G$ is a complete lattice. Let $\operatorname{Id_c} G$ denote the subsemilattice of compact elements in $\operatorname{Id} G$. It is an easy exercise to see that an ideal I of G lies in $\operatorname{Id_c} G$ if and only if I has an order-unit (when I is viewed as a partially ordered abelian group in its own right).

Similarly, for any commutative monoid M we write $\operatorname{Id_c} M$ for the semilattice of compact elements of $\operatorname{Id} M$, and we observe that the members of $\operatorname{Id_c} M$ are precisely those ideals of M which have order-units.

Proposition 2.6. *Let M be a commutative monoid and G a partially ordered abelian group.*

(i) $\operatorname{Id} G \cong \operatorname{Id} G^+$.
(ii) $\operatorname{Id} M$ *and* $\operatorname{Id} G$ *are algebraic lattices.*
(iii) *If M satisfies the Riesz decomposition property, then* $\operatorname{Id} M$ *is distributive and* $\operatorname{Id_c} M \cong \nabla(M)$. *Hence,* $\operatorname{Id} M \cong \operatorname{Id} \nabla(M)$.
(iv) *If G is an interpolation group, then* $\operatorname{Id} G$ *is distributive and* $\operatorname{Id_c} G \cong \nabla(G^+)$. *Hence,* $\operatorname{Id} G \cong \operatorname{Id} \nabla(G^+)$.

Proof. (i) Inverse isomorphisms are given as follows: map each ideal I of G to $I \cap G^+$, and map each ideal J of G^+ to $J + (-J)$.

(ii) We already know that $\operatorname{Id} M$ and $\operatorname{Id} G$ are complete lattices. Any ideal I of M is the supremum of the *principal* ideals $\{y \in M : y \propto x\}$ for $x \in I$, and each of these principal ideals is in $\operatorname{Id}_c M$. This shows that $\operatorname{Id} M$ is algebraic. One can argue similarly that $\operatorname{Id} G$ is algebraic, or just apply part (i).

(iii) It follows directly from Riesz decomposition that for any two ideals I and J of M, the sum $I + J$ is again an ideal. Hence, finite suprema in $\operatorname{Id} M$ are given by sums. It is clear that $(I + J) \cap K = (I \cap K) + (J \cap K)$ for all I, J, $K \in \operatorname{Id} M$, and therefore $\operatorname{Id} M$ is distributive.

Observe that elements x, $y \in M$ satisfying $x \asymp y$ generate the same principal ideal of M. Hence, there is a map $\theta \colon \nabla(M) \to \operatorname{Id}_c M$ such that

$$\theta([x]) = \{z \in M : z \propto x\}$$

for all $x \in M$. Observe that $\theta([x]) \subseteq \theta([y])$ if and only if $x \propto y$, if and only if $[x] \leq [y]$. Hence, θ is an order embedding. Any ideal $I \in \operatorname{Id}_c M$ has an order-unit, say x, and so $I = \theta([x])$. Therefore θ is an order-isomorphism of $\nabla(M)$ onto $\operatorname{Id}_c M$, hence also a semilattice isomorphism.

That $\operatorname{Id} M \cong \operatorname{Id} \nabla(M)$ now follows from Proposition 1.1.

(iv) By Lemma 2.5 and parts (i), (iii) above, we have that $\operatorname{Id} G \cong \operatorname{Id} G^+$ is distributive (*cf.* [14, Propositions 2.4, 2.5]) and $\operatorname{Id}_c G \cong \operatorname{Id}_c G^+ \cong \nabla(G^+)$. Now $\operatorname{Id} G \cong \operatorname{Id} \nabla(G^+)$ by Proposition 1.1. □

3. Direct limit representation of dimension groups; Triangle Lemma

Here we discuss the Effros-Handelman-Shen Theorem and separate its proof into two parts: a "Triangle Lemma" concerning positive homomorphisms from simplicial groups to dimension groups, and a "direct limit representation lemma" which provides sufficient conditions for objects of a quasivariety to be represented as direct limits of objects from a given subclass. The latter lemma we prove in detail, as it will yield our direct limit representation theorem for distributive semilattices (Theorem 6.6) once we establish a suitable Triangle Lemma in that setting (Corollary 6.5).

A *simplicial group* is a partially ordered abelian group that is isomorphic to some $\mathbb{Z}^n$, equipped with the direct product ordering, for a nonnegative integer n. Obviously, every simplicial group is a dimension group. Conversely, it turns out that simplicial groups are "building blocks" of dimension groups, *via* direct limits. The earliest result of this type is due to P.A. Grillet [19, Theorem 2.1]. Say that a commutative monoid S has the *strong Riesz interpolation property* (strong RIP) if for every positive integer n and for all elements a, b, c and d of S, if $na+b = nc+d$, then there are elements u, v, w and z of S such that $a = u + v$, $b = nw + z$, $c = u + w$, and $d = nv + z$.

Theorem 3.1 (see [19, Theorem 2.1]). *Let S be a commutative monoid. Then the following are equivalent:*

(i) *S is a direct limit of (finitely generated) free commutative monoids.*
(ii) *S is cancellative and S satisfies the strong RIP.* □

The passage from the strong RIP to the direct limit representation is achieved by using a general categorical result, due to R.T. Shannon [31], which gives a characterization of directed colimits of free objects in algebraic categories.

Remark 3.2. Although the fact is absent from [19], it is not difficult, although not trivial, to verify directly that a directed partially ordered abelian group G is a dimension group if and only if G^+ satisfies the strong RIP. To establish the nontrivial implication, one starts by proving directly that any dimension group G satisfies Proposition 3.23 of [14], that is, for all $n \in \mathbb{N}$ and all $a \in G$, the set of $x \in G$ such that $a \leq nx$ is downward directed. This can be done by induction on n; here is an outline of a proof.

Let x_0, $x_1 \in G$ such that $a \leq nx_i$ for $i < 2$, with $n \geq 2$. For all i, $j < 2$, we have $na = a + (n-1)a \leq nx_i + (n-1)nx_j$ and so, by n-unperforation, $a \leq x_i + (n-1)x_j$. Apply interpolation to the relations $a - x_i \leq (n-1)x_j$ to obtain $y \in G$ such that $a - x_i \leq y \leq (n-1)x_j$ for all i, j. By the induction hypothesis, there exists $z \in G$ such that $y \leq (n-1)z$ and $z \leq x_j$ for $j < 2$. Since also $a - y \leq x_j$ for all j, another interpolation yields $x \in G$ such that $z\,, a - y \leq x \leq x_0\,, x_1$. Therefore $a \leq x + y \leq x + (n-1)z \leq nx$, completing the induction step.

Now to prove the strong RIP, let $n \in \mathbb{N}$ and a, b, c and d in G^+ such that $na + b = nc + d$. Put $e = na - d = nc - b$, and note that $e \leq na, nc$. Thus there exists u such that $e \leq nu$ and $u \leq a, c$. Put $v = a - u$, $w = c - u$, and $z = d - nv$.

Therefore, this exercise is an easy proof that Grillet's Theorem (Theorem 3.1) implies the later Effros-Handelman-Shen Theorem (Theorem 3.3) described in the next paragraph.

The direct limit representation result for dimension groups was proved by E.G. Effros, D.E. Handelman and C.-L. Shen:

Theorem 3.3 (see [9, Theorem 2.2])**.** *A partially ordered abelian group is a direct limit of simplicial groups if and only if it is a dimension group.* □

A proof of this result is also presented in [14, Theorem 3.19]. The hard core of the proof consists in what we shall call the *Triangle Lemma*: For every simplicial group S, every dimension group G and every positive homomorphism $f\colon S \to G$, there exist a simplicial group T and positive homomorphisms $\varphi\colon S \to T$ and $g\colon T \to G$ such that $f = g \circ \varphi$ and $\ker(f) = \ker(\varphi)$. Once this step is established, the argument follows a general, categorical pattern. There are, in fact, general categorical results which allow one to go directly from the Triangle Lemma above to the direct limit representation. For example, the main result of R.T. Shannon [31] is quite short to state (modulo numerous necessary definitions), but we did not find it convenient to translate it, for example, to the language of partially ordered abelian groups for the purpose of finding a shorter proof of the Effros-Handelman-Shen Theorem. On the other hand, it seems almost unavoidable that writing down the most general categorical statement that leads from the Triangle Lemma to the direct limit representation would involve a substantial number of extremely unwieldy statements.

To solve this dilemma, we will put ourselves at a medium level of generality, which will be sufficient to deal with current first-order theories (such as commutative monoids, or semilattices). While Shannon's result is stated in a categorical context, we will choose a universal algebraic context. This way, the reader can at least choose, according to his affinities, between a categorical statement and a universal algebraic statement.

We assume familiarity with only the very rudiments of universal algebra, and we refer to [24] for the details. We will fix a language $\mathcal{L}$ of *algebras*, that is, a first-order language with only symbols of operations and constants (no relation symbols). Say that a *quasi-identity* is a first-order sentence of the form

$$(\forall \vec{\mathsf{x}})\big[\varphi(\vec{\mathsf{x}}) \Rightarrow \psi(\vec{\mathsf{x}})\big],$$

where φ is a finite (possibly empty) conjunction of *equations* (= atomic formulas) and ψ is an equation. A *quasivariety* (see [24, Chapter V]) is the class of models of a set of quasi-identities. It is well-known that in any quasivariety $\mathbf{V}$, there are arbitrary colimits. In particular, for every set X, there exists a free object of $\mathbf{V}$ over X.

Lemma 3.4. *Let $\mathbf{V}$ be a quasivariety of algebras of $\mathcal{L}$, and let $M \in \mathbf{V}$. Let $\mathbf{F}$ be a subclass of $\mathbf{V}$ with the following properties:*

(i) *For each $m \in M$, there exist $F \in \mathbf{F}$ and a homomorphism $f\colon F \to M$ such that $m \in f(F)$;*
(ii) *For each coproduct F of finitely many elements of $\mathbf{F}$ and each homomorphism $f\colon F \to M$, there exist $G \in \mathbf{F}$ and homomorphisms $\varphi\colon F \to G$ and $g\colon G \to M$ such that $f = g \circ \varphi$ and $\ker(f) = \ker(\varphi)$.*

Then M is a direct limit of objects from $\mathbf{F}$.

Proof. We mimic the proof presented in [14, Theorem 3.19]. Put $I = M \times \omega$. (This is just to ensure that we base our indexing on an infinite set, to cover the possibility that M might be finite.) Put $P = [I]^{<\omega} \setminus \{\varnothing\}$, the set of all nonempty finite subsets of I, ordered under inclusion. We construct inductively objects $F_p \in \mathbf{F}$ and homomorphisms $f_p\colon F_p \to M$ for $p \in P$, and transition homomorphisms $f_{pq}\colon F_p \to F_q$ for $p \subset q$ in P (where $\subset$ denotes *strict* inclusion). If $p = \{\langle a, n\rangle\}$, where $\langle a, n\rangle \in M \times \omega$, choose, by hypothesis (i), an $F_p \in \mathbf{F}$ and a homomorphism $f_p\colon F_p \to M$ such that $a \in f_p(F_p)$.

Now the induction step. Suppose that $p \in P$ has at least two elements, and suppose that we have constructed objects $F_q \in \mathbf{F}$ for $q \subset p$ in P, homomorphisms $f_q\colon F_q \to M$ for $q \subset p$ in P, and $f_{qr}\colon F_q \to F_r$ for $q \subset r \subset p$ in P, satisfying the following conditions:

(i) If $p_0 \subset p_1 \subset p_2 \subset p$ in P, then $f_{p_0p_2} = f_{p_1p_2} \circ f_{p_0p_1}$.
(ii) If $p_0 \subset p_1 \subset p$ in P, then $f_{p_0} = f_{p_1} \circ f_{p_0p_1}$.
(iii) If $p_0 \subset p_1 \subset p$ in P, then $\ker(f_{p_0}) = \ker(f_{p_0p_1})$.

Put $F^p = \coprod_{q \subset p} F_q$, where $\amalg$ denotes the coproduct in $\mathbf{V}$. For all $q \subset p$ in P, denote by e_{qp} the canonical homomorphism from F_q to F^p. By the universal property of the coproduct, there exists a unique homomorphism $f^p\colon F^p \to M$ such that $f^p \circ e_{qp} = f_q$ for all $q \subset p$ in P. By assumption, there exist an object $F_p \in \mathbf{F}$ and homomorphisms $\varphi_p\colon F^p \to F_p$ and $f_p\colon F_p \to M$ such that $f_p \circ \varphi_p = f^p$ and $\ker(f^p) = \ker(\varphi_p)$. For all $q \subset p$ in P, define $f_{qp} = \varphi_p \circ e_{qp}$. The construction may be described by the commutative diagram below:

$$\begin{tikzcd}
& M & \\
F_q \arrow[ur, "f_q"] \arrow[r, "e_{qp}"'] & F^p \arrow[u, "f^p"'] \arrow[r, "\varphi_p"'] & F_p \arrow[ul, "f_p"']
\end{tikzcd}$$

We verify points (i) to (iii) listed above for the larger set of all $q \in P$ such that $q \subseteq p$.

(i) It suffices to verify that, for $p_0 \subset p_1 \subset p$, we have $f_{p_0p} = f_{p_1p} \circ f_{p_0p_1}$, that is, $\varphi_p \circ e_{p_0p} = \varphi_p \circ e_{p_1p} \circ f_{p_0p_1}$. Since $\ker(\varphi_p) = \ker(f^p)$, it suffices to prove that $f^p \circ e_{p_0p} = f^p \circ e_{p_1p} \circ f_{p_0p_1}$, that is, $f_{p_0} = f_{p_1} \circ f_{p_0p_1}$, which is indeed the case by the induction hypothesis (ii).

(ii) It suffices to verify that, for $q \subset p$ in P, we have $f_q = f_p \circ f_{qp}$. This is a direct calculation:

$$f_p \circ f_{qp} = f_p \circ \varphi_p \circ e_{qp} = f^p \circ e_{qp} = f_q.$$

(iii) It suffices to verify that, for $q \subset p$ in P, we have $\ker(f_q) = \ker(f_{qp})$. Let $x, y \in F_q$. Then $f_{qp}(x) = f_{qp}(y)$ if and only if $\varphi_p \circ e_{qp}(x) = \varphi_p \circ e_{qp}(y)$. Since $\ker(\varphi_p) = \ker(f^p)$, this is equivalent to $f^p \circ e_{qp}(x) = f^p \circ e_{qp}(y)$, that is, $f_q(x) = f_q(y)$.

Therefore, we have constructed a direct system

$$\mathcal{S} = \langle \langle F_p, f_{pq} \rangle : p \subset q \text{ in } P \rangle,$$

and homomorphisms $f_p \colon F_p \to M$ such that $f_p = f_q \circ f_{pq}$ and $\ker(f_p) = \ker(f_{pq})$ for all $p \subset q$ in P. Further, for each $a \in M$ we have, for $p = \{\langle a, 0 \rangle\}$, that $p \in P$ and $a \in f_p(F_p)$. Now if S, together with limiting maps $\eta_p \colon F_p \to S$, is the direct limit of the system $\mathcal{S}$ in $\mathbf{V}$, there exists a unique homomorphism $f \colon S \to M$ such that $f \circ \eta_p = f_p$ for all $p \in P$, and f is surjective. To see that f is injective, let $\langle s_0, s_1 \rangle \in \ker(f)$. Then there exist $p \in P$ and $x_0,\ x_1 \in F_p$ such that $\eta_p(x_i) = s_i$ for $i < 2$, and $\langle x_0, x_1 \rangle \in \ker(f_p)$. Since I is infinite, there exists $q \in P$ such that $p \subset q$, and $\langle x_0, x_1 \rangle \in \ker(f_{pq})$ by construction, whence $s_0 = (\eta_q \circ f_{pq})(x_0) = (\eta_q \circ f_{pq})(x_1) = s_1$. Therefore f is an isomorphism. □

Example. In the language consisting of a binary operation symbol $+$ and a constant symbol 0, one can consider the quasivariety of *commutative monoids*. Finitely generated free commutative monoids are exactly the positive cones of simplicial groups. The Triangle Lemma in this context is a reformulation of the corresponding Triangle Lemma for partially ordered abelian groups (Shen's condition), see for example [14, Proposition 3.16]. It is to be noted that Lemma 3.4 cannot be directly applied to partially ordered abelian groups, because of the binary *relation* symbol $\leq$. However, this is easily finessed here by considering the positive cones instead of the full ordered groups.

We will see another application of Lemma 3.4 in Section 6, in the case of the variety of *semilattices*.

4. Temperate powers of $\mathbb{Q}$

The purpose of this section is to demonstrate that Problem 1 (see Section 10) has a positive solution for distributive 0-*lattices*. Moreover, the construction developed here will allow us, in the following section, to demonstrate a positive solution to Problem 1 for countable distributive semilattices.

Throughout this section, we shall fix a set X and a sublattice D of the powerset lattice $\mathcal{P}(X)$, such that $\varnothing \in D$. Define $\mathbf{B}(D)$ to be the generalized Boolean subalgebra of $\mathcal{P}(X)$ generated by D. Equivalently, the elements of $\mathbf{B}(D)$ are finite unions of the form

$$\bigcup_{i<2n} (a_{2i} \setminus a_{2i+1}),$$

where $\langle a_0, a_1, \dots, a_{2n}\rangle$ is a finite decreasing sequence of elements of D (see [17, II.4]).

Further, let $\mathbb{Q}\langle D\rangle$ be the set of all functions $f\colon X \to \mathbb{Q}$ with finite range such that f is *measurable* with respect to the generalized Boolean algebra $\mathbf{B}(D)$, that is, $f^{-1}\{r\}$ belongs to $\mathbf{B}(D)$ for every nonzero $r \in \mathbb{Q}$.

Lemma 4.1. *The set $\mathbb{Q}\langle D\rangle$ is a subalgebra of the $\mathbb{Q}$-algebra $\mathbb{Q}^X$. Furthermore, for all f, $g \in \mathbb{Q}\langle D\rangle$, the map $(f : g)$, defined componentwise by*

$$(f : g)(x) = \begin{cases} f(x)/g(x) & \text{if } g(x) \neq 0, \\ 0 & \text{if } g(x) = 0 \end{cases}$$

belongs to $\mathbb{Q}\langle D\rangle$.

Proof. It is obvious that $\mathbb{Q}\langle D\rangle$ is closed under multiplication by rational scalars.

For all f, $g \in \mathbb{Q}\langle D\rangle$, both f and g have finite range, thus so does $f + g$. Furthermore, for all $r \in \mathbb{Q}$, we have

$$(f + g)^{-1}\{r\} = \bigcup\{f^{-1}\{s\} \cap g^{-1}\{t\}\colon s \in \operatorname{rng}(f),\ t \in \operatorname{rng}(g) \text{ and } s + t = r\},$$

where $\operatorname{rng}(f)$ denotes the range of f, and thus $(f + g)^{-1}\{r\} \in \mathbf{B}(D)$. Hence, $f+g \in \mathbb{Q}\langle D\rangle$. Similarly, the product fg and the element $(f : g)$ belong to $\mathbb{Q}\langle D\rangle$. □

For every element f of $\mathbb{Q}\langle D\rangle$, note that the *support* of f,

$$\operatorname{supp}(f) = \{x \in X\colon f(x) \neq 0\},$$

belongs to $\mathbf{B}(D)$.

Notation. Let $\mathbb{Q}^+\langle D\rangle$ be the set of all functions $f \in \mathbb{Q}\langle D\rangle$ such that $f(x) \geq 0$ for all $x \in X$, and $\operatorname{supp}(f) \in D$.

Proposition 4.2. *$\mathbb{Q}^+\langle D\rangle$ is the positive cone of a structure of dimension vector space on $\mathbb{Q}\langle D\rangle$.*

Proof. It is easy to verify that $\mathbb{Q}^+\langle D\rangle$ is the positive cone of a structure of partially ordered vector space on $\mathbb{Q}\langle D\rangle$: one has to verify that $\mathbb{Q}^+\langle D\rangle$ is an additive submonoid of $\mathbb{Q}^+\langle D\rangle$, closed under multiplication by positive rational numbers, and that $\mathbb{Q}^+\langle D\rangle \cap (-\mathbb{Q}^+\langle D\rangle) = \{0\}$; this is straightforward.

Denote by $\leq$ the pointwise ordering of $\mathbb{Q}\langle D\rangle$, and by $\leq^+$ the ordering of $\mathbb{Q}\langle D\rangle$ with positive cone $\mathbb{Q}^+\langle D\rangle$.

Every element $f \in \mathbb{Q}\langle D\rangle$ is majorized (for $\leq$) by some $n \cdot \chi_A$, where $n \in \mathbb{N}$ and $A \in D$ (here χ_Y denotes the characteristic function of a subset Y of X). Therefore, the support of $(n+1)\cdot\chi_A - f$ is equal to A, so that $f \leq^+ (n+1)\cdot\chi_A$. Hence, the partial ordering $\leq^+$ is directed.

It remains to verify interpolation. It is convenient to use Lemma 2.5, that is, to verify that $\mathbb{Q}^+\langle D\rangle$ satisfies the refinement property.

Thus let f_0, f_1, g_0 and g_1 be elements of $\mathbb{Q}^+\langle D\rangle$ such that $f_0 + f_1 = g_0 + g_1$. Put $h = f_0 + f_1$. For all i, $j < 2$, put (with the notation of Lemma 4.1)

$$h_{ij} = (f_i g_j : h).$$

By Lemma 4.1, h_{ij} belongs to $\mathbb{Q}\langle D\rangle$. It is obvious that $0 \leq h_{ij}$. To prove that $0 \leq^+ h_{ij}$, note that $\operatorname{supp}(h_{ij}) = \operatorname{supp}(f_i) \cap \operatorname{supp}(g_j)$. But D is closed under finite intersection, whence $\operatorname{supp}(h_{ij}) \in D$. Thus $h_{ij} \in \mathbb{Q}^+\langle D\rangle$. Finally, it is obvious that $f_i = h_{i0} + h_{i1}$ and $g_j = h_{0j} + h_{1j}$ for i, $j < 2$. □

In the sequel, we shall identify in notation $\mathbb{Q}\langle D\rangle$ with the dimension vector space $\langle \mathbb{Q}\langle D\rangle, +, 0, \mathbb{Q}^+\langle D\rangle\rangle$, and we will call it the *temperate power* of $\mathbb{Q}$ by D.

Lemma 4.3. *Let f and g be two elements of $\mathbb{Q}^+\langle D\rangle$. Then the following are equivalent:*

(i) *There exists $n \in \mathbb{N}$ such that $f \leq^+ ng$.*
(ii) *There exists $n \in \mathbb{N}$ such that $f \leq ng$.*
(iii) $\operatorname{supp}(f) \subseteq \operatorname{supp}(g)$.

In particular, for $f,\ g \in \mathbb{Q}^+\langle D\rangle$, there is no ambiguity on the notation $f \propto g$, whether $\leq$ or $\leq^+$ is used to order the vector space $\mathbb{Q}\langle D\rangle$.

Proof. (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) are trivial.

Assume (iii). Since $(f : g)$ has finite range, it is majorized by some positive integer n. Let $x \in X$; we prove that $f(x) \leq ng(x)$. This is trivial when $f(x) = 0$. If $x \in \operatorname{supp}(f)$, that is, $f(x) > 0$, then, by assumption, $g(x) > 0$, thus $(f : g)(x) = f(x)/g(x)$; but $n \geq (f : g)(x)$, so that $f(x) \leq ng(x)$. It follows easily that the support of $(n+1)g - f$ is equal to the support of g; whence $f \leq^+ (n+1)g$. □

By putting together Proposition 4.2 and Lemma 4.3, one obtains the following result:

Theorem 4.4. *For every distributive 0-lattice D, there exists a dimension vector space E such that $\nabla(E^+)$ is isomorphic to D as a semilattice.*

Proof. By Stone's Theorem (see [17, Corollary II.1.21]), there exists a set X such that D embeds into $\mathcal{P}(X)$. Since D has a zero, the embedding can be arranged in such a way that its range includes $\varnothing$. Thus, without loss of generality, we may assume that D is a sublattice of $\mathcal{P}(X)$ containing $\varnothing$. Put $E = \mathbb{Q}\langle D\rangle$. By Proposition 4.2, E is a dimension vector space. By Lemma 4.3, the maximal semilattice quotient of E^+ is isomorphic to D (*via* the support map). □

5. Lifting countable distributive semilattices to dimension groups

In this section, we shall see how an easy application of the results of Section 4 yields a solution of Problem 1 in the case of *countable* semilattices.

For every partially ordered set P, denote by $\mathrm{H}(P)$ the distributive lattice of all lower subsets of P (that is, the subsets X of P such that if $p \leq x$ and $x \in X$, then $p \in X$). Put $\mathbb{Q}(P) = \mathbb{Q}\langle \mathrm{H}(P)\rangle$, and $\mathbb{Q}^+(P) = \mathbb{Q}^+\langle \mathrm{H}(P)\rangle$. We will call $\mathbb{Q}(P)$ (with positive cone $\mathbb{Q}^+(P)$) the *temperate power* of $\mathbb{Q}$ by P.

In case P is finite, one can give a direct description of the dimension vector space $\mathbb{Q}(P)$, since the generalized Boolean algebra $\mathbf{B}(\mathrm{H}(P))$ just equals $\mathcal{P}(P)$ in this case. The underlying space of $\mathbb{Q}(P)$ is $\mathbb{Q}^P$, and $\mathbb{Q}^+(P)$ consists of those functions $u \colon P \to \mathbb{Q}^+$ whose support belongs to $\mathrm{H}(P)$.

By Lemma 4.3, one can define an *isomorphism* $\iota_P \colon \nabla(\mathbb{Q}^+(P)) \to \mathrm{H}(P)$, by the formula

$$\iota_P([x]) = \operatorname{supp}(x), \qquad \text{for all } x \in \mathbb{Q}^+(P).$$

Lemma 5.1. *Let P and Q be two finite partially ordered sets, and let $f \colon \mathrm{H}(P) \to \mathrm{H}(Q)$ be a 0-semilattice homomorphism. Then there exists a positive homomorphism $g \colon \mathbb{Q}(P) \to \mathbb{Q}(Q)$ such that $\iota_Q \circ \nabla(g^+) = f \circ \iota_P$.*

The last condition of the statement above means that the following diagram commutes:

$$\begin{array}{ccc}
\nabla(\mathbb{Q}^+(P)) & \xrightarrow{\iota_P} & \mathrm{H}(P) \\
{\scriptstyle \nabla(g^+)}\downarrow & & \downarrow{\scriptstyle f} \\
\nabla(\mathbb{Q}^+(Q)) & \xrightarrow{\iota_Q} & \mathrm{H}(Q)
\end{array}$$

Proof. Denote by $\langle \dot{p} \colon p \in P \rangle$ the canonical basis of $\mathbb{Q}^P$, where $\dot{p} = \chi_{\{p\}}$, and by $\langle \dot{q} \colon q \in Q \rangle$ the canonical basis of $\mathbb{Q}^Q$. Let g be the unique linear map from $\mathbb{Q}^P$ to $\mathbb{Q}^Q$ defined by the formula

$$g(\dot{p}) = \sum_{q \in f(\downarrow p)} \dot{q}$$

(recall the notation $\downarrow p = \{x \in P \colon x \leq p\}$). Let $x \in \mathbb{Q}^+(P)$, written as $x = \sum_{p \in P} x_p \dot{p}$, where all x_p are elements of $\mathbb{Q}^+$. Then we have

$$g(x) = \sum_{p \in P} \left(x_p \sum_{q \in f(\downarrow p)} \dot{q} \right) = \sum_{q \in Q} y_q \dot{q},$$

where we put $D_q = \{p \in P \colon q \in f(\downarrow p)\}$ and $y_q = \sum_{p \in D_q} x_p$ for all $q \in Q$. It is obvious that all y_q belong to $\mathbb{Q}^+$.

Put $U = \operatorname{supp}(x)$; by assumption, U belongs to $\mathrm{H}(P)$. If $q \in f(U)$, then, since $U = \bigcup_{p \in U} \downarrow p$ and since f is a 0-semilattice homomorphism, there exists $p \in U$ such that $q \in f(\downarrow p)$, that is, $p \in D_q$. Since $p \in U$, we have $x_p > 0$, whence $y_q > 0$. Conversely, if $q \notin f(U)$, then, for all $p \in D_q$, we have $p \notin U$ and thus $x_p = 0$; hence, $y_q = 0$. This shows that $\operatorname{supp}(g(x)) = f(U) \in \mathrm{H}(Q)$.

It follows that g is a positive homomorphism, and that, for all $x \in \mathbb{Q}^+(P)$, we have

$$\operatorname{supp}(g(x)) = f(\operatorname{supp}(x)).$$

Hence, g satisfies the required condition. □

By using Pudlák's Lemma (see the Introduction), we can now conclude:

Theorem 5.2. *Every countable distributive* 0*-semilattice S is isomorphic to the maximal semilattice quotient of the positive cone of some countable dimension vector space E. If, in addition, S is bounded, then E has an order-unit.*

Proof. Let S be a countable distributive 0-semilattice. By Pudlák's result, one can write S as a countable, increasing union $S = \bigcup_{n \in \omega} S_n$, where all the S_n are finite distributive subsemilattices of S, containing 0. Then each S_n is a distributive lattice. Denote by P_n the set of all (nonzero) join-irreducible elements of S_n, ordered by the restriction of the ordering of S_n, and by τ_n the natural isomorphism from $\mathrm{H}(P_n)$ onto S_n. Put $f_n = \tau_{n+1}^{-1}|_{S_n} \circ \tau_n$. By Lemma 5.1, there exists a positive homomorphism $g_n \colon \mathbb{Q}(P_n) \to \mathbb{Q}(P_{n+1})$ such that $\iota_{P_{n+1}} \circ \nabla(g_n^+) = f_n \circ \iota_{P_n}$. The information can be partly visualized in the following commutative diagram:

$$\begin{array}{ccccc}
\nabla(\mathbb{Q}^+(P_n)) & \xrightarrow{\iota_{P_n}} & \mathrm{H}(P_n) & \xrightarrow{\tau_n} & S_n \\
{\scriptstyle \nabla(g_n^+)}\downarrow & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle \subseteq} \\
\nabla(\mathbb{Q}^+(P_{n+1})) & \xrightarrow{\iota_{P_{n+1}}} & \mathrm{H}(P_{n+1}) & \xrightarrow{\tau_{n+1}} & S_{n+1}
\end{array}$$

Consider the direct system $\mathcal{S}$ of partially ordered $\mathbb{Q}$-vector spaces whose objects are the $\mathbb{Q}(P_n)$, for $n \in \omega$, and whose morphisms are the maps $g_{n-1} \circ \cdots \circ g_m$, for $m \leq n$. By Lemma 2.1, if E denotes the direct limit of $\mathcal{S}$, then $\nabla(E^+)$ is isomorphic to the direct limit of the S_n with the inclusion maps, that is, to S. Since all the $\mathbb{Q}(P_n)$ are dimension vector spaces (by Proposition 4.2), so is E. It is clear that E is countable.

Finally, suppose that S is bounded. Then $\nabla(E^+)$ has a largest element, call it 1. Let $u \in E^+$ be an element whose $\asymp$-class is 1. Hence, all elements $x \in E^+$ satisfy $x \propto u$, and so u is an order-unit of the monoid E^+. Since E is directed, u must also be an order-unit for E. □

6. Boolean direct limit representation of distributive semilattices

The Triangle Lemma for distributive semilattices can be proved in a very similar fashion as the corresponding result for dimension groups (*i.e.*, [14, Proposition 3.16]). However, we present here a different proof, that shows at the same time a stronger property of distributive semilattices (Proposition 6.3). Furthermore, this proof is specific to semilattices, *e.g.*, the analogue of Proposition 6.3 for dimension groups and simplicial groups does *not* hold.

Lemma 6.1. *Let S be a distributive semilattice. Let $n \in \omega$ and let a, b, c_i $(i < n)$ be elements of S such that $a \leq b \vee c_i$ for all $i < n$. Then there exists $x \in S$ such that $a \leq b \vee x$ and $x \leq c_i$ for all $i < n$.*

Proof. It suffices to prove the lemma for $n = 2$. Since S is distributive, there are $b_i \leq b$ and $d_i \leq c_i$ such that $a = b_i \vee d_i$ (for all $i < 2$). Since $d_1 \leq a \leq b_0 \vee d_0$, there are, further, $b_2 \leq b_0$ and $x \leq d_0$ such that $d_1 = b_2 \vee x$. Therefore $x \leq d_i \leq c_i$ for all $i < 2$, and

$$a = b_1 \vee b_2 \vee x \leq b \vee x.$$ □

Lemma 6.2. *Let S be a distributive semilattice. Let m, $n \in \omega$, and let a_i, b_i $(i < m)$ and c_j $(j < n)$ be elements of S such that $a_i \leq b_i \vee c_j$ for all $i < m$ and $j < n$. Then there exists $x \in S$ such that*

$$(\forall i < m)(a_i \leq b_i \vee x) \quad \textit{and} \quad (\forall j < n)(x \leq c_j).$$

Proof. This is an immediate consequence of [32, Lemma 1.5]. However, we present here a self-contained proof.

By Lemma 6.1, for all $i < m$, there exists $x_i \in S$ such that $a_i \leq b_i \vee x_i$ and $x_i \leq c_j$ for all $j < n$. Then $x = \bigvee_{i<m} x_i$ satisfies the required conditions. □

Proposition 6.3 (Finite injectivity for distributive semilattices)**.** *Let S be a distributive semilattice, and let A be a subsemilattice of a finite semilattice B. Then every semilattice homomorphism from A to S extends to a semilattice homomorphism from B to S.*

Proof. Let f be a homomorphism from A to S.

We consider first the case where there exists $b \in B \setminus A$ such that B is generated by $A \cup \{b\}$. Therefore,

$$B = A \cup \{b\} \cup \{x \vee b \colon x \in A\}. \tag{1}$$

Let $\{\langle x_i, y_i\rangle \colon i < m\}$ list all the pairs $\langle x, y\rangle$ of elements of A such that $x \leq y \vee b$, and let $\{z_j \colon j < n\}$ list all elements z of A such that $b \leq z$. For all $i < m$ and all $j < n$, we have $x_i \leq y_i \vee z_j$, and thus $f(x_i) \leq f(y_i) \vee f(z_j)$. Therefore, by Lemma 6.2, there exists $\alpha \in S$ such that

$$\begin{aligned} f(x_i) &\leq f(y_i) \vee \alpha && \text{for all } i < m,\\ \alpha &\leq f(z_j) && \text{for all } j < n. \end{aligned}$$

Let $\{w_k \colon k < r\}$ list all elements w of A such that $w \leq b$. Then $f(w_k) \leq f(z_j)$ for all $k < r$ and $j < n$. Set $\beta = \alpha \vee \bigvee_{k<r} f(w_k)$ (β is defined as being equal to α if $r = 0$), and observe that

$$\begin{aligned} f(x_i) &\leq f(y_i) \vee \beta && \text{for all } i < m,\\ f(w_k) &\leq \beta && \text{for all } k < r,\\ \beta &\leq f(z_j) && \text{for all } j < n. \end{aligned} \tag{2}$$

It follows from (2) that

$$\begin{aligned} f(x) \vee \beta &= f(y) \vee \beta && \text{for all } x, y \in A \text{ such that } x \vee b = y \vee b\\ f(x) \vee \beta &= f(z) && \text{for all } x, z \in A \text{ such that } x \vee b = z\\ f(x) \vee \beta &= \beta && \text{for all } x \in A \text{ such that } x \vee b = b. \end{aligned} \tag{3}$$

By (1) and (3), f extends to a well-defined map $g \colon B \to S$ such that $g(b) = \beta$ and $g(x \vee b) = f(x) \vee \beta$ for all $x \in A$. Since f is a homomorphism, it follows easily that g is a homomorphism.

In the general case, there exists a finite chain of subsemilattices

$$A = B_0 \subset B_1 \subset \cdots \subset B_k = B$$

such that each B_i ($i > 0$) is generated by $B_{i-1} \cup \{b_i\}$ for some $b_i \in B_i \setminus B_{i-1}$. Thus, we conclude by an easy induction argument. $\square$

It is to be noted that Proposition 6.3 is also an immediate consequence of Pudlák's Lemma (see the Introduction) and the *injectivity* of every finite distributive semilattice in the class of semilattices. The latter result follows immediately from [35, Theorem 3.11], but it can also be proved directly. Moreover, our proof here is self-contained.

A finite semilattice is *Boolean* if it is isomorphic to $\mathbf{2}^n$ for some $n \in \omega$, where $\mathbf{2}$ is the two element semilattice.

Lemma 6.4 (folklore). *Every finite semilattice (0-semilattice) has a (zero-preserving) embedding into a finite Boolean lattice.*

Proof. If S is a finite semilattice, let $B = \mathcal{P}(S)$ be the powerset semilattice of S, and embed S into B *via* the map $j \colon S \to B$ defined by the rule

$$j(s) = \{x \in S \colon s \nleq x\}. \qquad \square$$

A better embedding (from the computational viewpoint) can be obtained by replacing $\mathcal{P}(S)$ by $\mathcal{P}(P)$, where P denotes the set of meet-irreducible elements of S (here meet-irreducibility means with respect to whatever meets might exist); the map j is defined similarly. We can now prove the Triangle Lemma for distributive semilattices:

Corollary 6.5. *Let S be a distributive semilattice. Let A be a finite semilattice, and let f be a homomorphism from A to S. Then there exist a finite Boolean semilattice B and homomorphisms $\varphi \colon A \to B$ and $g \colon B \to S$ such that $f = g \circ \varphi$ and* $\ker(f) = \ker(\varphi)$.

Proof. Put $A' = A/\ker(f)$, and denote by $\pi\colon A \twoheadrightarrow A'$ the quotient map. There exists a unique homomorphism $f'\colon A' \to S$ such that $f = f' \circ \pi$. By Lemma 6.4, there exists an embedding j from A' into some finite Boolean semilattice B. By Proposition 6.3, there exists a homomorphism $g\colon B \to S$ such that $f' = g \circ j$. Put $\varphi = j \circ \pi$. The situation can be described by the following commutative diagram:

$$\begin{tikzcd}
 & S & \\
A \arrow[ur, "f"] \arrow[r, "\pi"'] & A' \arrow[u, "f'"'] \arrow[r, "j"'] & B \arrow[ul, "g"']
\end{tikzcd}$$

We obtain the following:

$$f = f' \circ \pi = g \circ \varphi.$$

Furthermore, j is one-to-one, and thus $\ker(\varphi) = \ker(\pi) = \ker(f)$. □

We can now deduce a general representation result for distributive semilattices:

Theorem 6.6. *Every distributive semilattice is a direct limit of finite Boolean semilattices and semilattice homomorphisms.*

Proof. We consider the first-order language consisting of one binary operation symbol $\vee$, the variety $\mathbf{V}$ of semilattices, and the subclass $\mathbf{F}$ of finite Boolean semilattices. Since the class of finite (not necessarily Boolean) semilattices is closed under finite coproducts (because every finitely generated semilattice is finite), the assumption (ii) of Lemma 3.4 is, by Corollary 6.5, satisfied. Since the assumption (i) of Lemma 3.4 is trivially satisfied, the theorem follows. □

Say that a partially ordered set is *bounded* if it has a least and a greatest element, which we denote by 0 and 1.

Corollary 6.7.

(i) *Every distributive* 0*-semilattice is a direct limit of finite Boolean semilattices and* 0*-preserving semilattice homomorphisms.*
(ii) *Every bounded distributive semilattice is a direct limit of finite Boolean semilattices and* $0, 1$*-preserving semilattice homomorphisms.*

Proof. We prove, for example, (i). The proof for (ii) is similar. Let S be a distributive 0-semilattice. By Theorem 6.6, S is a direct limit of a direct system

$$\mathcal{S} = \langle \langle S_i, f_{ij} \rangle : i \leq j \text{ in } I \rangle$$

where I is a directed set, the S_i are finite Boolean semilattices and the f_{ij} are semilattice homomorphisms, with respect to limiting homomorphisms $f_i\colon S_i \to S$.

Without loss of generality, I has a least element, denoted by 0, and $0_S = f_0(0_{S_0})$. For all $i \in I$, put $0_i = f_{0i}(0_{S_0})$ and $T_i = \{x \in S_i\colon 0_i \leq x\}$. Then T_i is a finite Boolean semilattice, and f_{ij} maps T_i to T_j for $i \leq j$. Furthermore, the least element of T_i is 0_i, and $f_{ij}(0_i) = 0_j$ for $i \leq j$. Thus each f_{ij} restricts to a 0-preserving semilattice homomorphism $g_{ij}: T_i \to T_j$. Finally, S is the direct limit of the system

$$\mathcal{T} = \langle \langle T_i, g_{ij} \rangle : i \leq j \text{ in } I \rangle.$$

□

Example 6.8. Consider the three element chain $S = \{0, 1, 2\}$, viewed as a bounded join-semilattice. Although Corollary 6.7 allows us to express S as a direct limit of finite Boolean semilattices, the result is puzzling, because S itself, although finite, is not Boolean.

Here is an explicit description of S as a direct limit of finite Boolean semilattices. Consider the 0-semilattice homomorphism $r\colon \mathbf{2}^2 \to \mathbf{2}^2$ defined by $r(a) = a$ and $r(b) = a \vee b$, where a and b are the two atoms of $\mathbf{2}^2$. It is not difficult to verify that S is the direct limit of the sequence

$$\mathbf{2}^2 \xrightarrow{\quad r \quad} \mathbf{2}^2 \xrightarrow{\quad r \quad} \mathbf{2}^2 \xrightarrow{\quad r \quad} \cdots,$$

with the limiting 0-semilattice homomorphism $f\colon \mathbf{2}^2 \to S$ defined by $f(a) = 1$ and $f(b) = 2$.

7. Regular rings and the functors V, $\widetilde{V}$

We recall the definition and some basic facts about regular rings, their idempotents, and their ideal lattices.

For every ring R, denote by $\mathcal{L}(R)$ the semilattice of all finitely generated right ideals of R, ordered by inclusion. A ring R is (von Neumann) *regular* if for all $x \in R$, there exists $y \in R$ such that $xyx = x$.

A 0-lattice L is *sectionally complemented* if for all elements $a \leq b$ of L, there exists a sectional complement of a in b, that is, an element x of L such that $a \wedge x = 0$ and $a \vee x = b$.

Proposition 7.1. *If R is a regular ring, then $\mathcal{L}(R)$ is a sectionally complemented modular lattice.*

Proof. This was first proved by von Neumann in the unital case [33, Theorem 2]; his argument easily extends to the non-unital situation, as noted in [11, 3.2]. $\square$

Let R be a ring. For all $n \in \mathbb{N}$, embed the ring $M_n(R)$ of all $n \times n$ square matrices over R into $M_{n+1}(R)$, *via* the map

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}.$$

Furthermore, denote by $M_\infty(R)$ the direct limit of the system

$$R \to M_2(R) \to M_3(R) \to M_4(R) \to \cdots.$$

Define an equivalence relation $\sim$ on the set of all idempotent elements of $M_\infty(R)$ by

$$e \sim f \iff (\exists x, y \in M_\infty(R))(xy = e \text{ and } yx = f).$$

Equivalently, $e \sim f$ if and only if $e \cdot M_\infty(R) \cong f \cdot M_\infty(R)$ as right $M_\infty(R)$-modules. For every idempotent e of $M_\infty(R)$, denote by $[e]$ the $\sim$-equivalence class of a, and put

$$V(R) = \{\, [e] \colon e \in R,\ e^2 = e \,\}.$$

There is a well-defined addition on $V(R)$ given by

$$[e] + [f] = [e \oplus f]$$

where

$$e \oplus f = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix},$$

and $\langle V(R), +, [0] \rangle$ is a commutative monoid. Now, V extends to a *functor* from the category of rings to the category of commutative monoids. It is well known (and also easy to see) that this functor preserves finite direct products and direct limits.

Since we shall often work with the maximal semilattice quotients of the monoids $V(R)$, let us introduce the notation $\widetilde{V}$ for the composition of V with the functor ∇ (see Section 2). Thus $\widetilde{V}$ is a functor from the category of rings to the category of semilattices, and it preserves direct limits and finite direct products. Given a ring R, write $|e|$ for the $\asymp$-class of $[e] \in V(R)$, where e is any idempotent in $M_\infty(R)$.

Proposition 7.2. *Let R be a regular ring. Then $V(R)$ is a conical refinement monoid, and $\widetilde{V}(R)$ is a distributive semilattice.*

Proof. That $V(R)$ is a conical refinement monoid follows from Theorem 2.8 in [12] (the fact that R does not necessarily have a unit does not affect the proof). By Lemma 2.4, $\widetilde{V}(R)$ is a distributive semilattice. □

It is well known that for any ring R, the lattice $\operatorname{Id} R$ of (two-sided) ideals of R is algebraic. The semilattice $\operatorname{Id_c} R$ of compact elements of $\operatorname{Id} R$ consists of the finitely generated ideals of R, that is, all two-sided ideals of R of the form $\sum_{i<n} Rx_iR$, where $n \in \mathbb{N}$ and all x_i belong to R. Note that by Proposition 1.1, $\operatorname{Id} R \cong \operatorname{Id}(\operatorname{Id_c} R)$.

Proposition 7.3. *Let R be a regular ring. Then all three semilattices $\operatorname{Con_c} \mathcal{L}(R)$, $\operatorname{Id_c} R$ and $\widetilde{V}(R)$ are pairwise isomorphic. Furthermore, they are distributive 0-semilattices. Moreover, the lattices $\operatorname{Id} R$, $\operatorname{Id} V(R)$, and $\operatorname{Id} \widetilde{V}(R)$ are pairwise isomorphic, and these are distributive algebraic lattices.*

Proof. The semilattice isomorphisms follow from [37, Corollary 4.4 and Proposition 4.6]. Again, the fact that R does not necessarily have a unit does not affect the proofs. Then by Proposition 7.2, these semilattices are distributive. (Recall that more generally, the semilattice of all compact congruences of any lattice is distributive, see Section 1).

Now $\operatorname{Id} R \cong \operatorname{Id}(\operatorname{Id_c} R) \cong \operatorname{Id} \widetilde{V}(R) \cong \operatorname{Id} V(R)$. By Proposition 2.6, $\operatorname{Id} \widetilde{V}(R)$ is distributive, and the proposition is proved. (One can also prove directly that $\operatorname{Id} R$ is distributive; this is well known and easy.) □

As a byproduct of the proof of Proposition 7.3, we have the following:

Proposition 7.4. *Let R be a regular ring.*

(i) *Let J be an ideal of $V(R)$. Then there exists a two-sided ideal I of R such that $J \cong V(I)$. Namely, I is the ideal of R generated by all idempotents $e \in R$ for which $[e] \in J$.*

(ii) *Let J be an ideal of $\operatorname{Id_c} R$. Then there exists a two-sided ideal I of R such that $J \cong \operatorname{Id_c} I$. Namely, I is the sum of all those ideals of R which are members of J.* □

Note that every two-sided ideal of a regular ring is itself regular (see [12, Lemma 1.3]).

Another consequence of Proposition 7.3 is the following:

Corollary 7.5. *Let D be a distributive algebraic lattice. If there exists a regular ring R such that $D \cong \operatorname{Id} R$, then there exists a sectionally complemented modular lattice L such that $D \cong \operatorname{Con} L$.*

Translated into the language of semilattices, this gives the following: *Let S be a distributive 0-semilattice. If there exists a regular ring R such that $S \cong \operatorname{Id}_{\mathrm{c}} R$, then there exists a sectionally complemented modular lattice L such that $S \cong \operatorname{Con}_{\mathrm{c}} L$.*

Proof. Propositions 1.1, 7.1, and 7.3. □

8. Bergman's Theorem

We are now ready to develop our two new proofs of Bergman's Theorem. Let us first recall some basic definitions. Let K be a field. A *matricial algebra* over K is a finite direct product of the form

$$\prod_{i<k} M_{n_i}(K),$$

where k and the n_i are natural numbers. A *locally matricial* algebra over K is a direct limit of matricial algebras over K and K-algebra homomorphisms. Note that we do not require the ring homomorphisms to preserve the ring units. Note also that locally matricial algebras are very special cases of regular rings. Countable dimensional locally matricial algebras are sometimes called *ultramatricial*, see [12].

Observe that if R is a matricial algebra, then $V(R) \cong (\mathbb{Z}^+)^n$ for some positive integer n (see [12, Lemma 15.22] for an analogous result with the same proof). In particular, $V(R)$ is then cancellative ($x + a = x + b$ implies $a = b$). Since the functor V preserves direct limits, $V(R)$ is also cancellative for any locally matricial algebra R. Thus $V(R) \cong K_0(R)^+$ for any such R, since $K_0(R)$ is constructed as the universal enveloping group of $V(R)$. We shall use this observation to translate results from the literature, stated in the language of K_0, into $V(R)$-form.

Elliott's Lemma (see the Introduction) together with the countable case of the Effros-Handelman-Shen Theorem (Theorem 3.3), which implies that every countable dimension group is the direct limit of a countable sequence of simplicial groups, yields the following result:

Theorem 8.1 (see [12, 2nd. Ed., p. 376])**.** *Let G be a countable dimension group, and let K be a field. Then there exists a locally matricial K-algebra R of countable dimension such that $V(R) \cong G^+$. If, in addition, G has an order-unit, then one can choose R unital.* □

In this section, we will illustrate the interdependency of various parts of this paper, by giving two proofs of Bergman's Theorem (stated in the Introduction).

First Proof of Bergman's Theorem. By Proposition 1.1, it suffices to solve the following problem. We fix a countable distributive 0-semilattice S and a field K; we must find a locally matricial K-algebra R of countable dimension such that $\operatorname{Id}_{\mathrm{c}} R \cong S$. In view of Proposition 7.3, this is the same as to arrange for $\widetilde{V}(R) \cong S$. Further, if S is bounded, we must find a unital such R.

By Theorem 5.2, there exists a countable dimension vector space E such that $\nabla(E^+) \cong S$. By Theorem 8.1, there exists a locally matricial K-algebra R of countable dimension such that $V(R) \cong E^+$; therefore $\widetilde{V}(R) \cong \nabla(E^+) \cong S$.

In addition, if S is bounded, then E has an order-unit, and thus, by Theorem 8.1, one can choose R unital. □

Second Proof of Bergman's Theorem. This proof does not use the results of Elliott, or Grillet, Effros, Handelman and Shen. In fact, it uses nothing more than the countable case of Corollary 6.7.

As in the first proof, we fix a countable distributive 0-semilattice S and a field K, and we find a locally matricial K-algebra R of countable dimension such that $\widetilde{V}(R) \cong S$.

According to Corollary 6.7, we may assume that S is the direct limit of a sequence

$$\mathbf{2}^{n_1} \xrightarrow{f_1} \mathbf{2}^{n_2} \xrightarrow{f_2} \mathbf{2}^{n_3} \xrightarrow{f_3} \cdots \tag{4}$$

in the category of 0-semilattices.

Set $R_1 = K^{n_1}$ (the direct product of n_1 copies of K), and observe that $\widetilde{V}(R_1) \cong \mathbf{2}^{n_1}$. More precisely, if $p_1, \dots, p_{n_1}$ are the primitive central idempotents in R_1 (that is, the atoms of the finite Boolean algebra of central idempotents of R_1), then $|p_1|, \dots, |p_{n_1}|$ are distinct atoms which generate $\widetilde{V}(R_1)$. Hence, if $a_1, \dots, a_{n_1}$ are the distinct atoms in $\mathbf{2}^{n_1}$, there exists an isomorphism $g_1 \colon \widetilde{V}(R_1) \to \mathbf{2}^{n_1}$ such that $g_1(|p_i|) = a_i$ for $i = 1, \dots, n_1$.

Let $b_1, \dots, b_{n_2}$ be the distinct atoms in $\mathbf{2}^{n_2}$. There are integers $s_{ij} \in \{0, 1\}$ such that $f_1(a_i) = s_{i1}b_1 + \cdots + s_{i,n_2}b_{n_2}$ for all i. Choose a positive integer $t(j) \geq s_{1j} + \cdots + s_{n_1,j}$ for each j, and set

$$R_2 = M_{t(1)}(K) \times \cdots \times M_{t(n_2)}(K).$$

Let $\phi_1 : R_1 \to R_2$ be the block diagonal K-algebra homomorphism with multiplicities s_{ij}, that is, each component map $R_1 \to M_{t(j)}(K)$ is given by

$$(\alpha_1, \dots, \alpha_{n_1}) \mapsto \operatorname{diag}(\overbrace{\alpha_1}^{s_{1j}}, \overbrace{\alpha_2}^{s_{2j}}, \dots, \overbrace{\alpha_n}^{s_{n_1,j}}, 0, \dots, 0),$$

where the notation $\overbrace{\alpha_i}^{s_{ij}}$ means that α_i appears if $s_{ij} = 1$ but not if $s_{ij} = 0$. Let

$$q_1 = (I_{t(1)}, 0, \dots, 0), \quad q_2 = (0, I_{t(2)}, 0, \dots, 0), \quad \dots, \quad q_{n_2} = (0, \dots, 0, I_{t(n_2)})$$

be the primitive central idempotents in R_2. Then there exists an isomorphism $g_2 \colon \widetilde{V}(R_2) \to \mathbf{2}^{n_2}$ such that $g_2(|q_j|) = b_j$ for all j, and we observe that $g_2\widetilde{V}(\phi_1) = f_1 g_1$.

Continuing in the same manner, we obtain a sequence

$$R_1 \xrightarrow{\phi_1} R_2 \xrightarrow{\phi_2} R_3 \xrightarrow{\phi_3} \cdots \tag{5}$$

of matricial K-algebras and K-algebra homomorphisms together with 0-semilattice isomorphisms $g_i \colon \widetilde{V}(R_i) \to \mathbf{2}^{n_i}$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
\widetilde{V}(R_1) & \xrightarrow{\widetilde{V}(\phi_1)} & \widetilde{V}(R_2) & \xrightarrow{\widetilde{V}(\phi_2)} & \widetilde{V}(R_3) & \xrightarrow{\widetilde{V}(\phi_3)} & \cdots \\
{\scriptstyle g_1}\big\downarrow & & {\scriptstyle g_2}\big\downarrow & & {\scriptstyle g_3}\big\downarrow & & \\
\mathbf{2}^{n_1} & \xrightarrow{f_1} & \mathbf{2}^{n_2} & \xrightarrow{f_2} & \mathbf{2}^{n_3} & \xrightarrow{f_3} & \cdots
\end{array}$$

Therefore, if R is the direct limit of the sequence (5), then we have $\widetilde{V}(R) \cong S$ as desired.

It remains to modify the proof for the case that S has a greatest element, say 1. As before, we express S as the direct limit of the sequence (4); in view of Corollary 6.7, we may now assume that the maps f_i preserve greatest elements. Thus

$f_i(1_i) = 1_{i+1}$ for all i, where 1_i denotes the greatest element of $\mathbf{2}^{n_i}$ (the sum of all the atoms).

Define R_1 as before, and note that g_1 maps $|1_{R_1}|$ to 1_1.

Let $b_1, \dots, b_{n_2}$ and the s_{ij} be as before. Since

$$1_2 = f_1(1_1) = \sum_{i=1}^{n_1} f_1(a_i) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} s_{ij} b_j,$$

we must have $\sum_{i=1}^{n_1} s_{ij} \neq 0$ for all j, and so we can choose $t(j) = s_{1j} + \cdots + s_{n_1,j}$. Now if R_2 and ϕ_1 are defined as before, ϕ_1 is a unital homomorphism.

Continuing as before, we can obtain a sequence (5) in which all the homomorphisms ϕ_i are unital, and therefore R is a unital algebra. □

9. Ideal lattices in C*-algebras

In this section, we use the methods of the previous section to derive an analogue of Bergman's Theorem for C*-algebras. This result, in turn, has an interesting application to a class of C*-algebras A which have been classified by H. Lin [23] in terms of the invariants $V(A)$, which for this particular class are actually distributive semilattices.

Throughout, we deal only with *complex* C*-algebras. Recall that the natural morphisms in the category of C*-algebras are **-homomorphisms* ($\mathbb{C}$-algebra homomorphisms which preserve the involution *), since such maps are automatically contractions with respect to C*-algebra norms (see, *e.g.*, [25, Theorem 2.1.7]). Every finite-dimensional C*-algebra has the form

$$\prod_{i<k} M_{n_i}(\mathbb{C}),$$

where the matrix algebras $M_{n_i}(\mathbb{C})$ are equipped with the conjugate transpose involution and the operator norm (see, *e.g.*, [7, Theorem III.1.1] or [25, Theorem 6.3.8]). A C*-algebra is said to be *AF* (for "approximately finite-dimensional") if it is isomorphic to a direct limit (in the category of C*-algebras) of a countable sequence of finite-dimensional C*-algebras and *-homomorphisms.

We shall need the fact that the functor V commutes with C*-algebra direct limits (*i.e.*, norm-completions of *-algebra direct limits), see [4, 5.2.4]. Since V of any finite-dimensional C*-algebra is obviously cancellative, it follows that $V(A)$ is also cancellative for all AF C*-algebras A. Hence, $V(A) \cong K_0(A)^+$ when A is AF. It also follows that K_0 of any AF C*-algebra is a countable dimension group, see [7, Theorem IV.3.3].

In the category of C*-algebras, kernels correspond to *closed* ideals (ideals closed in the norm topology). Thus, the natural ideal lattice to study is the lattice $\overline{\operatorname{Id}}\, A$ of closed ideals of a C*-algebra A. Such a lattice is algebraic: infima are given by intersections, suprema are given by closures of sums, and the compact elements are the finitely generated closed ideals. (For an ideal to be finitely generated in the context of closed ideals means that it is the closure of some ideal which is finitely generated in the usual sense.) It is known that the lattice of closed ideals of an AF C*-algebra is distributive. The C*-algebra analogue of Bergman's Theorem can be stated as follows:

Theorem 9.1. *Let L be a distributive algebraic lattice with only countably many compact elements. Then L is isomorphic to the lattice of closed ideals in some AF*

C-algebra A. If, in addition, the greatest element of L is compact, then A can be chosen to be unital.*

First proof of Theorem 9.1. By Proposition 1.1, there is a countable distributive 0-semilattice S such that $\operatorname{Id} S \cong L$. If A is an AF C*-algebra, let $\overline{\operatorname{Id}}_{\mathrm{c}} A$ denote the semilattice of finitely generated closed ideals of A. This semilattice consists precisely of the compact elements of $\overline{\operatorname{Id}} A$, and hence $\operatorname{Id}(\overline{\operatorname{Id}}_{\mathrm{c}} A) \cong \overline{\operatorname{Id}} A$. Thus, it suffices to find an AF C*-algebra A such that $\overline{\operatorname{Id}}_{\mathrm{c}} A \cong S$.

For any AF C*-algebra A, the lattice $\overline{\operatorname{Id}} A$ is isomorphic to $\operatorname{Id} K_0(A)$, see [7, Proposition IV.5.1], and, consequently, $\overline{\operatorname{Id}}_{\mathrm{c}} A$ is isomorphic to $\operatorname{Id}_{\mathrm{c}} K_0(A)$. By Proposition 2.6, $\operatorname{Id}_{\mathrm{c}} K_0(A) \cong \nabla(K_0(A)^+) \cong \widetilde{V}(A)$, and hence $\overline{\operatorname{Id}}_{\mathrm{c}} A \cong \widetilde{V}(A)$. Thus, to find an AF C*-algebra A with $\overline{\operatorname{Id}} A \cong L$ is the same as to find an A with $\widetilde{V}(A) \cong S$.

By Theorem 5.2, there exists a countable dimension vector space E such that $\nabla(E^+) \cong S$. By the Effros-Handelman-Shen Theorem and the C*-algebra analogue of Elliott's Lemma, see [7, Theorem IV.7.3], there exists an AF C*-algebra A such that $V(A) \cong E^+$. Therefore $\widetilde{V}(A) \cong S$, as desired. In addition, if S is bounded, then E has an order-unit, and then A can be chosen to be unital. ☐

Second proof of Theorem 9.1. As above, we just need to find an AF C*-algebra A such that $\widetilde{V}(A)$ is isomorphic to a given countable distributive 0-semilattice S.

The construction in our second proof of Bergman's Theorem yields a sequence (5) of matricial $\mathbb{C}$-algebras and $\mathbb{C}$-algebra homomorphisms such that the direct limit of $\widetilde{V}$ of (5) is isomorphic to S. Each R_i can be viewed as a finite-dimensional C*-algebra. Observe that the block diagonal maps ϕ_i are *-homomorphisms. Hence, the C*-algebra direct limit of the sequence (5) is an AF C*-algebra, say A. Since the functor V commutes with C*-algebra direct limits, we therefore have $\widetilde{V}(A) \cong S$, as desired. ☐

The result of Theorem 9.1 can be extended to other classes of C*-algebras by a simple tensor product argument. For the basic theory of C*-tensor products and the fundamental concept of *nuclearity*, we refer the reader to [25, Chapter 6]. We shall need the fact that all AF C*-algebras are nuclear, see [25, Theorem 6.3.11]. Since all the C*-tensor products we consider will have at least one nuclear factor, the C*-tensor products will be unique, and we will just denote them by $\otimes$.

The following lemma is well known among the cognoscenti, but we have been unable to locate a reference in the literature, and so we outline a proof here. We thank Bruce Blackadar for this argument.

A C*-algebra B is said to be *simple* provided B is nonzero and the only closed ideals of B are 0 and B.

Lemma 9.2. *Let A and B be C*-algebras, at least one of which is nuclear. If B is simple and unital, then $\overline{\operatorname{Id}} A \cong \overline{\operatorname{Id}}(A \otimes B)$, via the map $I \mapsto I \otimes B$.*

Proof. The rule $I \mapsto I \otimes B$ defines an order-preserving map θ from $\overline{\operatorname{Id}} A$ to $\overline{\operatorname{Id}}(A \otimes B)$. Since B is unital, there is a *-homomorphism $\phi\colon A \to A \otimes B$ given by the rule $\phi(a) = a \otimes 1$, and the set map ϕ^{-1} induces an order-preserving map φ from $\overline{\operatorname{Id}}(A \otimes B)$ to $\overline{\operatorname{Id}} A$. Clearly $\varphi\theta$ is the identity on $\overline{\operatorname{Id}} A$. Thus, to prove that θ is a lattice isomorphism, it suffices to show that θ is surjective.

Let $J \in \overline{\operatorname{Id}}(A \otimes B)$, set $I = \phi^{-1}(J)$, and consider the algebraic (*i.e.*, uncompleted) tensor products $R = A \otimes_{\mathrm{alg}} B$ and $S = (A/I) \otimes_{\mathrm{alg}} B$. Note that $K = J \cap R$ is an ideal of R such that $K \cap (A \otimes 1) = I \otimes 1$. Since B is simple and unital, its center is a field as well as a C*-algebra, so the center of B is $\mathbb{C}\cdot 1$. Consequently, $K = I \otimes_{\mathrm{alg}} B$ (see, *e.g.*, [21, Theorem V.6.1]).

Thus the composition of the inclusion map $R \to A \otimes B$ with the quotient map $A \otimes B \to (A \otimes B)/J$ induces a *-algebra *embedding* $\psi\colon S \to (A \otimes B)/J$. The composition of ψ with the quotient norm on $(A \otimes B)/J$ then defines a C*-norm, call it $\|\cdot\|_\psi$, on S. (It is a norm, rather than just a seminorm, because ψ is injective.) By, *e.g.*, [34, Theorem T.6.21], $\|\cdot\|_\psi$ is a cross norm on S. Because of our nuclearity assumption, $\|\cdot\|_\psi$ is the unique C*-cross norm on S, and so the completion of S with respect to $\|\cdot\|_\psi$ yields the C*-tensor product $(A/I) \otimes B$. On the other hand, ψ is an isometry and the image of ψ is dense in $(A \otimes B)/J$. Hence, ψ induces a *-isomorphism of $(A/I) \otimes B$ onto $(A \otimes B)/J$. It follows that the kernel of the induced map $A \otimes B \to (A/I) \otimes B$ is precisely J, and therefore $J = I \otimes B$, as desired. □

Corollary 9.3. *Let B be a simple, unital C*-algebra, and let L be a distributive algebraic lattice with only countably many compact elements. Then there exists an AF C*-algebra A such that $L \cong \overline{\mathrm{Id}}(A \otimes B)$. If, in addition, the greatest element of L is compact, then A can be chosen to be unital.*

Proof. Theorem 9.1 and Lemma 9.2. □

We will apply the above corollary with a special choice of B which will ensure that $V(A \otimes B)$ is a distributive semilattice. This is the *Cuntz algebra* $\mathcal{O}_2$, defined as the unital C*-algebra generated by elements s_1 and s_2 satisfying the relations

$$s_1^* s_1 = s_2^* s_2 = s_1 s_1^* + s_2 s_2^* = 1.$$

It is known that $\mathcal{O}_2$ is simple (see, *e.g.*, [7, Corollary V.4.7]), that all nonzero projections in $\mathcal{O}_2$ are equivalent, see [6, Corollary 3.12], and that $M_n(\mathcal{O}_2) \cong \mathcal{O}_2$ for all $n \in \omega$, see [26]. In particular, it follows that $V(\mathcal{O}_2) \cong \mathbf{2}$.

In [23], Lin classified a class $\mathcal{A}$ of C*-algebras which have trivial K-theory, that is, the groups K_0 and K_1 are both trivial for the algebras in $\mathcal{A}$. We shall not give the precise definition of $\mathcal{A}$ here, but just recall that $\mathcal{A}$ contains $\mathcal{O}_2$ and is closed under the following operations: hereditary C*-subalgebras, quotients, tensor products with AF C*-algebras, countable direct limits, finite tensor products, and extensions, see [23, Theorem 3.14]. For any $A \in \mathcal{A}$, the monoid $V(A)$ is a countable distributive semilattice (*cf.* [23, Proposition 3.4] and [41, Theorem 1.1], or [1, Theorem 7.2 and Corollary 1.3]). Further, $\overline{\mathrm{Id}}\, A \cong \mathrm{Id}\, V(A)$ for any $A \in \mathcal{A}$ (use [23, Proposition 3.4 and Corollary 3.11] to see that A has real rank zero, then use the argument of [41, Theorem 2.3]). Lin proved that the algebras in $\mathcal{A}$ are classified up to isomorphism by the semilattices $V(\cdot)$ together with elements corresponding to approximate identities, see [23, Theorem 3.13]. In particular, unital algebras A, $B \in \mathcal{A}$ are isomorphic if and only if $V(A) \cong V(B)$ [ibid].

Taking $B = \mathcal{O}_2$ in Corollary 9.3, we see that any distributive algebraic lattice with only countably many compact elements can be represented as the lattice of closed ideals of a C*-algebra $A \otimes \mathcal{O}_2$ where A is AF. Such tensor products are in Lin's class $\mathcal{A}$, and we can use the above information to see that all the unital algebras in $\mathcal{A}$ must have this form.

Theorem 9.4. *Each unital C*-algebra in Lin's class $\mathcal{A}$ has the form $A \otimes \mathcal{O}_2$ for some unital AF C*-algebra A.*

Proof. Let $C \in \mathcal{A}$ be unital; then $V(C)$ is a bounded, countable, distributive 0-semilattice. The lattice $L = \operatorname{Id} V(C)$ is a distributive algebraic lattice whose semilattice of compact elements is isomorphic to $V(C)$ and thus is countable. Further, the greatest element of L is compact. By Corollary 9.3, there exists a unital AF C*-algebra A such that $L \cong \overline{\mathrm{Id}}(A \otimes \mathcal{O}_2)$. Hence,

$$V(C) \cong \overline{\mathrm{Id}}_{\mathrm{c}}(A \otimes \mathcal{O}_2) \cong \operatorname{Id}_{\mathrm{c}} V(A \otimes \mathcal{O}_2) \cong V(A \otimes \mathcal{O}_2)/\asymp = V(A \otimes \mathcal{O}_2).$$

Therefore we conclude from Lin's classification theorem, see [23, Theorem 3.13] that $C \cong A \otimes \mathcal{O}_2$. □

10. Open problems

The first circulated versions of the present paper generated some amount of work, which led to solutions to most of the original open problems. The first one of these open problems was the following.

Problem 1 (Lifting distributive semilattices to dimension groups)**.** Let S be a distributive 0-semilattice. Does there exist a dimension group G such that the maximal semilattice quotient of G^+ (that is, $\nabla(G^+)$) is isomorphic to S?

By Theorems 4.4 and 5.2, Problem 1 has positive solutions in the *lattice* case and in the *countable* case. By results of the first author and D.E. Handelman, see [13, Proposition 1.3] and [20, second Corollary], Problem 1 has a positive solution in case S is totally ordered, or—more generally—if every element of S is a (finite) join of join-irreducible elements of S.

On the other hand, P. Růžička solved Problem 1 negatively in [29], for a semilattice S of size $\aleph_2$. The $\aleph_1$ case is still open:

Problem 1′. Let S be a distributive 0-semilattice of size $\aleph_1$. Does there exist a dimension group G such that $\nabla(G^+)$ is isomorphic to S?

The regular ring version of Problem 1′ was the following:

Problem 2 (Lifting distributive semilattices to regular rings)**.** Let S be a distributive 0-semilattice. If $|S| \leq \aleph_1$, does there exists a regular ring R such that $\operatorname{Id}_{\mathrm{c}} R$ is isomorphic to S?

It is to be noted that the size $\aleph_1$ in the statement of Problem 2 is *optimal*: in [37], the second author proved that there exists a distributive 0-semilattice S of size $\aleph_2$ that cannot be isomorphic to $\operatorname{Id}_{\mathrm{c}} R$ for any regular ring R. One positive case of Problem 2 is that in which S is bounded and every element of S is a finite join of join-irreducible elements (no cardinality restriction on S is needed). This follows from work of G.M. Bergman [2, §§2–4] extending the result of Handelman mentioned above.

Finally, Problem 2 was solved positively by the second author in [38].

In [15, Theorem 1.5], Handelman and the first author showed that for every dimension group G of size at most $\aleph_1$, there exists a locally matricial algebra R such that $V(R) \cong G^+$ (in fact, the result is given there in the case where G has an order-unit. In the general case, G embeds as an ideal into a dimension group H with order-unit such that $|H| \leq \aleph_1$—take, for example, $H = \mathbb{Q} \times_{\mathrm{lex}} G$, the lexicographical product of $\mathbb{Q}$ by G—and then we can use Proposition 7.4.(i)). Therefore, by Proposition 7.3, the analogue of Problem 2 for locally matricial algebras (*i.e.*, the question whether the $\aleph_1$ version of Bergman's Theorem holds) is equivalent to Problem $1'$.

Problem 3. Let S be a distributive 0-*lattice*. Does there exist a regular ring R such that $S \cong \mathrm{Id_c}\, R$?

By Theorem 4.4, every distributive 0-lattice is isomorphic to the maximal semilattice quotient of G^+ for some dimension group G. However, this does not help because there are dimension groups of size $\aleph_2$ that are not isomorphic to $K_0(R)$ for any regular ring R (see [36]).

Finally, P. Růžička solved Problem 3 positively in [28].

Natural extensions of the problems above are found when one does not just ask for lifting semilattices, but their *homomorphisms*. The solution of lattice-theoretical analogues of this kind of problem can be found in [40, 39].

Problem 4. Characterize the distributive 0-semilattices S such that for every locally matricial algebra R, every $\{\vee, 0\}$-homomorphism $\varphi\colon \mathrm{Id_c}\, R \to S$ can be lifted, that is, there are a locally matricial algebra R', an algebra homomorphism $f\colon R \to R'$, and an isomorphism $\alpha\colon \mathrm{Id_c}\, R' \to S$ such that $\alpha \circ \mathrm{Id_c}\, f = \varphi$.

Of course, the map $\mathrm{Id_c}\, f$ is defined by the rule

$$(\mathrm{Id_c}\, f)(xR) = f(x)R', \qquad \text{for all } x \in R,$$

thus turning $\mathrm{Id_c}$ into a *functor*.

Pursuing the lattice-theoretical analogy, it is reasonable to ask for the following two-dimensional analogue of Problem 4:

Problem 5. Characterize the distributive 0-semilattices S such that for every diagram $\mathcal{D}$ of locally matricial algebras of the form $f_i\colon R \to R_i$, for $i \in \{1, 2\}$, every homomorphism $\varphi\colon \mathrm{Id_c}\, \mathcal{D} \to S$ can be lifted by some homomorphism $f\colon \mathcal{D} \to R'$, for some locally matricial algebra R'.

The second author's paper [39] studies the 0-semilattices S that satisfy a lattice-theoretical analogue of Problem 5.

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