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THE DIXMIER-MOEGLIN EQUIVALENCE IN QUANTUM COORDINATE RINGS AND QUANTIZED WEYL ALGEBRAS

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Abstract. We study prime and primitive ideals in a unified setting applicable to quantizations (at nonroots of unity) of $n \times n$ matrices, of Weyl algebras, and of Euclidean and symplectic space. The framework for this analysis is based upon certain iterated skew polynomial algebras A over infinite fields k of arbitrary characteristic. Our main result is the verification, for A, of a characterization of primitivity established by Dixmier and Moeglin for complex enveloping algebras. Namely, we show that a prime ideal P of A is primitive if and only if the center of the Goldie quotient ring of A/P is algebraic over k, if and only if P is a locally closed point – with respect to the Jacobson topology – in the prime spectrum of A.

These equivalences are established with the aid of a suitable group $\mathcal{H}$ acting as automorphisms of A. The prime spectrum of A is then partitioned into finitely many "$\mathcal{H}$-strata" (two prime ideals lie in the same $\mathcal{H}$-stratum if the intersections of their $\mathcal{H}$-orbits coincide), and we show that a prime ideal P of A is primitive exactly when P is maximal within its $\mathcal{H}$-stratum. This approach relies on a theorem of Moeglin-Rentschler (recently extended to positive characteristic by Vonessen), which provides conditions under which $\mathcal{H}$ acts transitively on the set of rational ideals within each $\mathcal{H}$-stratum. In addition, we give detailed descriptions of the strata that can occur in the prime spectrum of A.

For quantum coordinate rings of semisimple Lie groups, results analogous to those obtained in this paper already follow from work of Joseph and Hodges-Levasseur-Toro. For quantum affine spaces, analogous results have been obtained in previous work of the authors.

1. Introduction

For several classes of finitely generated noncommutative algebras, a long-standing common goal has been to classify the primitive ideals. Toward this end it has often been necessary to first identify the primitive ideals within the larger set of prime ideals, and one of the most famous results along these lines was proved by Dixmier [**8**] and Moeglin [**29**]: If U is the enveloping algebra of a finite dimensional complex Lie algebra, and P is a prime ideal of U, then P is primitive if and only if it is rational (i.e., $\mathbb{C}$ is the center of the Goldie quotient ring of U/P), if and only if P is a locally closed point in the prime spectrum of U. Following [**40**], we will say that any noetherian algebra whose primitive ideals can be classified in this fashion satisfies the *Dixmier-Moeglin Equivalence*; over an arbitrary field k, rationality occurs when the center of the appropriate Goldie quotient ring is algebraic over k (cf. [**19**; **20**; **38**]). Our first aim in this paper is to verify the equivalence for a class of iterated skew polynomial rings that includes quantum matrices and quantum Weyl algebras, both considered here in the "non-root-of-unity" case; other

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examples include quantizations of affine, symplectic, and Euclidean spaces. This result can be viewed as an analog of the Dixmier-Moeglin Equivalence for enveloping algebras of solvable Lie algebras, whose proof preceded the general case (cf. [**9**, 4.5.7]), because such enveloping algebras may be constructed as iterated differential operator rings.

Another motivating context for our work arises from quantizations of the function algebra on a complex connected semisimple Lie group G. Recently, detailed descriptions of the prime and primitive spectra have been obtained by Joseph [**23**; **24**; **25**] for $R_q[G]$ and by Hodges-Levasseur-Toro [**17**] for $\mathbb{C}_{q,p}[G]$; both of these studies follow the earlier conjectures in [**15**; **16**] and are concerned with the case when the parameter q is not a root of unity. A key qualitative feature of these algebras is that the prime spectrum of the algebra divides into finitely many "H-strata" (see also [**6**, 2.4]), where H is a maximal torus of G acting as algebra automorphisms, and where two prime ideals are in the same stratum exactly when their H-orbits have the same intersection. It is further proved by these authors that H acts transitively on each stratum of primitive ideals (in the algebraically closed case) and that a prime ideal is primitive if and only if it is maximal within its stratum. For the iterated skew polynomial rings mentioned in the first paragraph, we establish a similar description, which turns out to be an intermediate step toward proving the Dixmier-Moeglin Equivalence in our desired setting. For $R_q[G]$ or $\mathbb{C}_{q,p}[G]$, the equivalence follows directly from the results of Joseph and Hodges-Levasseur-Toro; see (2.4).

Our approach can be divided into two main steps. In the first we consider an affine algebraic group $\mathcal{H}$ acting rationally (cf. (2.5)) by automorphisms on a noetherian algebra A. We then assume that there exist only finitely many $\mathcal{H}$-strata in $\operatorname{Spec} A$, and we prove that the remaining qualitative properties discussed in the preceding paragraph follow for A. More precisely, we establish the equivalence of the following four conditions for prime ideals P of A:

(1) P locally closed in $\operatorname{Spec} A$;
(2) P primitive;
(3) P rational;
(4) P maximal within its $\mathcal{H}$-stratum.

It is well known that (1) $\Longrightarrow$ (2) in Jacobson rings, and that (2) $\Longrightarrow$ (3) in the presence of the Nullstellensatz. Under the assumption that $\operatorname{Spec} A$ contains only finitely many $\mathcal{H}$-strata, it is easy to show that (4) $\Longrightarrow$ (1). Thus our main effort is devoted to establishing (3) $\Longrightarrow$ (4) under suitable additional hypotheses. Our proof of this implication rests upon a result of Moeglin and Rentschler, asserting – given certain extra restrictions – that $\mathcal{H}$ must act transitively on each stratum of rational ideals; included among the required added hypotheses here is the assumption of an algebraically closed base field of characteristic zero. (A positive characteristic version of this theorem has recently been obtained by Vonessen [**46**].) Descent techniques, partly modelled on those of Irving and Small [**20**], now allow us to obtain the equivalence of (1)–(4) for suitable algebras and groups over non-algebraically-closed fields. (Over algebraically closed fields of characteristic zero, the implication (3) $\Longrightarrow$ (1) is already present in unpublished work of Moeglin and Rentschler [**31**]; our paper does not rely on the results therein, and the methods we employ are different.) In the second main step, finally, we use techniques from [**12**] to prove that the skew polynomial algebras in our setting have only finitely many $\mathcal{H}$-ideals, and so only finitely many $\mathcal{H}$-strata, where $\mathcal{H}$ is a suitable affine algebraic group acting rationally by automorphisms.

For rational torus actions we are able to prove, furthermore, that the strata are homeomorphic to the spectra of commutative Laurent polynomial algebras occurring naturally as the centers of suitable localizations; see Section 6.

Examples of finitely generated noetherian algebras that do not satisfy the Dixmier-Moeglin Equivalence may be found in [**19**] and references therein.

2. Finite Stratification

The first goal of this section is to prove, under suitable hypotheses, that an algebra with finitely stratified prime spectrum satisfies the Dixmier-Moeglin Equivalence. This part of our analysis does not explicitly require a rational action of an algebraic group; the major nontrivial assumption of the first theorem is that the group giving the stratification acts transitively on the rational ideals within each stratum. In the second part of the section, we appeal to work of Moeglin-Rentschler and Vonessen to obtain this hypothesis in the algebraically closed case, under certain additional assumptions, where now the acting group is assumed to be affine algebraic. In characteristic zero, results of Irving and Small then show that the Dixmier-Moeglin Equivalence descends to the non-algebraically closed case. We provide an alternate argument, based on algebraic group actions, which allows descent also in positive characteristic. Over algebraically closed fields of characteristic zero, it already follows from an unpublished paper of Moeglin and Rentschler [**31**] that the Dixmier-Moeglin Equivalence holds when the prime spectrum is finitely stratified by a rational action of an affine algebraic group.

2.1 Notation and Preliminaries. Throughout this section, A will denote a noetherian algebra, over a field k, equipped with a group $\mathcal{H}$ acting on A by k-algebra automorphisms. The group of all k-algebra automorphisms of A will be denoted $\operatorname{Aut}_k A$.

(i) Tensor products will be assumed over k, unless indicated otherwise. If k' is a field extension of k, then we will always identify A with its image $A \otimes 1$ in $A \otimes k'$. Observe in this setting that if P is a prime ideal of $A \otimes k'$, then $P \cap A$ is a prime ideal of A. Further information concerning extensions of scalars and prime ideals may be found, for example, in [**43**]. The reader is referred to [**9**] and [**28**] for additional background information.

(ii) The set of (left) primitive ideals of A will be denoted $\operatorname{Prim} A$, and the set of prime ideals of A will be referred to as $\operatorname{Spec} A$. Both $\operatorname{Prim} A$ and $\operatorname{Spec} A$ will be equipped with the Jacobson topology. As before, a prime ideal P of A is *rational* if the center of the Goldie quotient ring of A/P is algebraic over k, and P is said to be *locally closed* if it is a locally closed point of $\operatorname{Spec} A$. (In other words, P is locally closed if and only if it is strictly contained in the intersection of all those prime ideals that properly contain it – following the convention that the intersection of an empty set of ideals is equal to A). The set of rational ideals of A will be denoted $\operatorname{Rat} A$ and will also be provided the Jacobson topology (i.e., $\operatorname{Rat} A$ is a subspace of $\operatorname{Spec} A$). Crucial to the analysis below are the induced actions of $\mathcal{H}$ on $\operatorname{Spec} A$, $\operatorname{Prim} A$, and $\operatorname{Rat} A$.

As previously stated, A satisfies the *Dixmier-Moeglin Equivalence* provided the sets of rational, locally closed, and primitive ideals all coincide. Note that in [**30**], a prime ideal P of A is called "rational" only if $Z(\operatorname{Fract}(A/P))$ coincides with k (rather than allowing this field to be any algebraic extension of k). However, the theorem we shall use from that paper, namely [**30**, 2.12ii] and its extension in [**46**], will only be applied in the case when k is algebraically closed. Hence, the conflict in terminology does not affect our work here.

(iii) Following [**28**, Chapter 9], we will say that A satisfies the *Nullstellensatz (over k)* if A is a Jacobson ring and if all of the endomorphism rings of simple A-modules are algebraic over k. It follows from [**9**, 4.1.6], when A satisfies the Nullstellensatz, that every primitive ideal of A is rational. It is immediate, of course, that a locally closed prime ideal in a Jacobson ring must be primitive.

(iv) An $\mathcal{H}$-stable ideal (or, more briefly, $\mathcal{H}$-ideal) I is said to be *$\mathcal{H}$-prime* when I contains no product of $\mathcal{H}$-ideals that all properly contain it. Next, if J is an ideal of A, set

$$(J : \mathcal{H}) = \bigcap_{h \in \mathcal{H}} h(J) .$$

When P is a prime ideal of A, it is easy to verify that $(P : \mathcal{H})$ is an $\mathcal{H}$-prime ideal of A. Conversely, every $\mathcal{H}$-prime ideal I of A is the intersection of the (finite) $\mathcal{H}$-orbit of prime ideals minimal over I (cf. [**10**, Remarks 4*,5*, p. 338]). The set of $\mathcal{H}$-prime ideals of A is denoted $\mathcal{H}\text{-Spec}\, A$.

The *$\mathcal{H}$-stratum in* $\operatorname{Spec} A$ of a prime ideal P is then the set of prime ideals P' for which $(P' : \mathcal{H}) = (P : \mathcal{H})$. The $\mathcal{H}$-strata in $\operatorname{Rat} A$ and $\operatorname{Prim} A$ are similarly defined. Note that every $\mathcal{H}$-stratum in $\operatorname{Spec} A$ is a union of $\mathcal{H}$-orbits.

2.2. (i) Assume that A is a Jacobson ring. Set $Y = \{\ (P : \mathcal{H}) \mid P \in \operatorname{Prim} A\ \}$. By (2.1iv), Y is a subset of $\mathcal{H}\text{-Spec}\, A$. Now let I be an $\mathcal{H}$-prime ideal of A, and set J equal to the intersection of all of the ideals in Y that properly contain I. Assume further that I is strictly smaller than J; this assumption will be valid, for example, when Y is finite. Since I is a semiprime ideal and A is a Jacobson ring, we may now choose a primitive ideal Q that contains I but does not contain J. Thus $I = (Q : \mathcal{H})$, and so $I \in Y$. It follows, for example, that if Y is finite, then $Y = \mathcal{H}\text{-Spec}\, A$; hence, Y is finite if and only if $\mathcal{H}\text{-Spec}\, A$ is finite. In particular, $\mathcal{H}\text{-Spec}\, A$ is finite if $\operatorname{Prim} A$ consists of only finitely many $\mathcal{H}$-orbits; a partial converse to this assertion will be discussed in (2.7).

(ii) Assume now that $\mathcal{H}\text{-Spec}\, A$ is finite (but that A is not necessarily a Jacobson ring). Let P be a prime ideal that is maximal within its $\mathcal{H}$-stratum, and set $I = (P : \mathcal{H})$. Letting J denote the intersection of all the $\mathcal{H}$-prime ideals of A that properly contain I, it follows that I is strictly contained in J. Observe that every prime ideal of A that properly contains P also contains J, and so P is locally closed.

2.3 Proposition. *Assume that A satisfies the Nullstellensatz, that $\mathcal{H}$ acts transitively on each $\mathcal{H}$-stratum in* $\operatorname{Rat} A$, *and that $\mathcal{H}\text{-Spec}\, A$ is finite.*

(i) *A satisfies the Dixmier-Moeglin Equivalence.*

(ii) *In A, an arbitrary prime ideal is primitive if and only if it is maximal within its $\mathcal{H}$-stratum.*

Proof. Let P be a prime ideal of A. It follows from (2.1iii) and (2.2ii) that P is locally closed if it is maximal within its $\mathcal{H}$-stratum, that P is primitive if it is locally closed, and that P is rational if it is primitive. It remains to show that if P is rational, then P is maximal within its $\mathcal{H}$-stratum.

Choose a prime ideal $P' \supseteq P$ maximal within the $\mathcal{H}$-stratum of P. It follows from (2.2ii) that P' is locally closed, and from (2.1iii) that it is rational. Because P and P' are rational ideals in the same $\mathcal{H}$-stratum, they are contained within the same $\mathcal{H}$-orbit. Hence, $P = P'$, and therefore P is maximal within its $\mathcal{H}$-stratum, as desired. (The last equality follows, for example, from the fact that the classical Krull dimensions of the isomorphic algebras A/P and A/P' must be the same.) $\square$

2.4 Remarks. Let G be a connected, simply connected, semisimple, complex Lie group with maximal torus H.

(i) Define the quantum function algebra $R = R_q[G]$ over $\overline{k(q)}$, as in [**25**, Chapter 9]. In particular, R is a noetherian $\overline{k(q)}$-algebra satisfying the Nullstellensatz [**25**, 9.2.2], where k is assumed to have characteristic zero. An explicit structure theory for $\operatorname{Prim} R$ has been established by Joseph [**23**; **24**; **25**, Chapters 9, 10], following conjectures (verified for $\mathbb{C}_q[SL_n]$) by Hodges-Levasseur [**15**; **16**]. Key to these results is the action on R, by its character group $R^\wedge$, via "winding" automorphisms [**25**, 1.3.4]. Furthermore, $R^\wedge$ may be identified with H [**25**, 10.3.12]. From [**25**, 10.3.2, 10.3.11] it follows that H acts transitively on each H-stratum in $\operatorname{Rat} R$ and that there are only finitely many H-orbits in $\operatorname{Rat} R$. Consequently, $R_q[G]$ satisfies the hypotheses of (2.3), recalling (2.1iii) and (2.2i). However, the second conclusion of the proposition, that the primitive ideals are exactly those prime ideals maximal within their H-strata, is already a fundamental part of the theory [**25**, 10.3.2, 10.3.7]. Moreover, from [**25**, 10.3.4, 10.3.7] it follows that every rational ideal of R is primitive, and by arguing as in [**15**, proof of 4.3.1], one can deduce from [**25**, 10.3.2] (cf. [**4**, 5.6]) that every primitive ideal of R is locally closed.

(ii) Considerations similar to (i) apply to the multiparameter quantum function algebras $\mathbb{C}_{q,p}[G]$, studied by Hodges-Levasseur-Toro [**17**]. In particular, H acts transitively on each H-stratum of rational ideals [**17**, 4.14], and there are only finitely many H-strata of rational ideals [**17**, 4.16]. The Dixmier-Moeglin Equivalence follows from [**17**, §4], arguing as in [**15**, 4.4.1; **16**, 4.2].

(iii) Quantum function algebras at roots of unity (cf. [**7**]) are affine PI algebras, and so the primitive ideals are precisely the maximal ideals. These coincide with the rational ideals (see, e.g., [**45**, 2.6]), and the Dixmier-Moeglin Equivalence immediately follows.

2.5. We now recall the definition of a rational action of an algebraic group. In order to include the case of the group of k-rational points of an algebraic group defined over the algebraic closure $\overline{k}$, assume only that $\mathcal{H}$ is a Zariski-closed subgroup of $GL_n(k)$ for some n. The action of $\mathcal{H}$ on A is then said to be *rational* provided A is a union of finite dimensional $\mathcal{H}$-invariant linear subspaces W_ℓ for which the induced group homomorphisms $\mathcal{H} \to GL(W_\ell)$ are morphisms of algebraic varieties. In particular, rational actions are locally finite, but the converse does not hold. For example, given any group homomorphism $\phi : k^\times \to k^\times$, there is a locally finite action of $k^\times$ on the polynomial ring $k[X]$ by k-algebra automorphisms such that $\alpha.X = \phi(\alpha)X$ for $\alpha \in k^\times$. This action is rational only if ϕ is given by $\phi(\alpha) = \alpha^n$ for some $n \in \mathbb{Z}$.

2.6 Theorem. (Moeglin-Rentschler, Vonessen) *Assume that the field k is algebraically closed, and that the k-algebra A is noetherian. Suppose further that $\mathcal{H}$ is a k-affine algebraic group and that its action on A is rational. Then $\mathcal{H}$ acts transitively on each $\mathcal{H}$-stratum of* $\operatorname{Rat} A$.

Proof. The original version of this result is proved in [**30**, 2.12ii] (cf. [**40**, §2]) under several additional assumptions: that k has characteristic zero, that A is finitely generated as a k-algebra, that $A \otimes k'$ is noetherian for all extension fields k' of k, and that $\mathcal{H}$ is connected. The final assumption is easily removed, given that the identity component of $\mathcal{H}$ is a closed, connected, normal subgroup of finite index (e.g., [**3**, I.1.2]). Vonessen [**46**] has recently shown that the theorem as stated can be proved by a combination of Moeglin and Rentschler's methods together with techniques he developed in [**45**]. □

Combining (2.2), (2.3), and (2.6), we obtain the following corollary; when the base field has characteristic zero, part (ii) may be deduced from [**31**].

2.7 Corollary. *Retain the hypotheses of* (2.6)*, and assume that A satisfies the Nullstellensatz over $k = \overline{k}$.*

(i) *The set of $\mathcal{H}$-orbits in* $\operatorname{Prim} A$ *is finite if and only if* $\mathcal{H}$-$\operatorname{Spec} A$ *is finite. When* $\mathcal{H}$-$\operatorname{Spec} A$ *is finite, the assignment $P \mapsto (P : \mathcal{H})$ induces a bijection from the set of $\mathcal{H}$-orbits in* $\operatorname{Prim} A$ *onto* $\mathcal{H}$-$\operatorname{Spec} A$.

(ii) *Suppose that* $\mathcal{H}$-$\operatorname{Spec} A$ *is finite. Then A satisfies the Dixmier-Moeglin Equivalence, and the primitive ideals of A are precisely the prime ideals maximal within their $\mathcal{H}$-strata.* □

The remainder of this section is devoted to showing, in certain situations, that (2.7ii) descends to non-algebraically-closed fields. Descent of the Dixmier-Moeglin Equivalence in characteristic zero already follows from the work of Irving-Small [**20**]; see [**43**, §8.4] for a more general exposition. In our situation, these results will not be needed, because the Dixmier-Moeglin Equivalence will follow from the maximality of rational ideals within their $\mathcal{H}$-strata.

2.8. Let A' be an over-ring of A, and let $\mathcal{H}'$ be a group acting by automorphisms on A'. If $\mathcal{H}$ is a subgroup of $\mathcal{H}'$, and if the action of each element $h \in \mathcal{H}$ on A' restricts to the given action of h on A, then we say that the action of $\mathcal{H}'$ on A' *extends* the action of $\mathcal{H}$ on A. For instance, the canonical action of a torus $(k^\times)^n$ on a polynomial ring $k[x_1, \dots, x_n]$ extends to the canonical action of $(K^\times)^n$ on $K[x_1, \dots, x_n]$ whenever K is a field extension of k. We shall further say that the action of $\mathcal{H}'$ *densely extends* that of $\mathcal{H}$ if $(P : \mathcal{H}') = (P : \mathcal{H})$ for all $P \in \operatorname{Spec} A'$. In the example above, the action of $(K^\times)^n$ densely extends that of $(k^\times)^n$ as long as k is infinite.

2.9 Lemma. *Assume that k' is an algebraic field extension of k, and that $A' = A \otimes k'$ is noetherian. Suppose that the action of $\mathcal{H}$ on A extends to an action of a group $\mathcal{H}'$ on A' by k'-algebra automorphisms. If* $\mathcal{H}$-$\operatorname{Spec} A$ *is finite, then* $\mathcal{H}'$-$\operatorname{Spec} A'$ *is finite.*

Proof. Let Q be a prime ideal of A, and let $P_1, \dots, P_t$ denote the prime ideals of A' minimal over $Q \otimes k'$. Next, recall that every prime ideal of A' contracts to a prime ideal of A (e.g., [**43**, 2.12.39]). Hence, since Q contains a product of the $P_i \cap A$, it follows that at least one of the P_i contracts to Q. Furthermore, every prime ideal of A' contracting to Q contains $Q \otimes k'$, and so from Incomparability (cf. [**43**, 3.4.13′]) it follows that every prime ideal of A' contracting to Q is in the set $\{P_1, \dots, P_t\}$. We have just shown that intersection with A produces a finite-to-one surjection from $\operatorname{Spec} A'$ onto $\operatorname{Spec} A$. (Cf. [**47**].)

As noted in (2.1iv), the $\mathcal{H}$-prime ideals of A are exactly the intersections of finite $\mathcal{H}$-orbits in $\operatorname{Spec} A$, and a similar statement holds for the $\mathcal{H}'$-prime ideals of A'. It therefore follows from our assumptions that there exist only finitely many finite $\mathcal{H}$-orbits in $\operatorname{Spec} A$. Also, to prove the lemma it suffices to show that there exist only finitely many finite $\mathcal{H}'$-orbits in $\operatorname{Spec} A'$. However, it follows from the preceding paragraph that contraction to A induces a finite-to-one surjection from the set of finite $\mathcal{H}'$-orbits in $\operatorname{Spec} A'$ onto the set of finite $\mathcal{H}$-orbits in $\operatorname{Spec} A$. The lemma follows. $\square$

2.10 Lemma. *Assume that $A' = A \otimes \overline{k}$ is noetherian, and suppose that the action of $\mathcal{H}$ on A extends densely to an action, by $\overline{k}$-algebra automorphisms, of a group $\mathcal{H}'$ on A'. If every $\overline{k}$-rational ideal in A' is maximal within its $\mathcal{H}'$-stratum in* $\operatorname{Spec} A'$, *then every rational ideal of A is maximal within its $\mathcal{H}$-stratum in* $\operatorname{Spec} A$.

Proof. Suppose to the contrary that Q_0 is a rational ideal of A which is properly contained in a prime ideal Q_1 lying within the same $\mathcal{H}$-stratum. Set E_0 equal to the Goldie quotient ring of A/Q_0. Observe that $\overline{E}_0 = E_0 \otimes_k \overline{k}$ is an Ore localization of its noetherian subring $(A/Q_0) \otimes_k \overline{k}$, because $\overline{E}_0$ is obtained therefrom by inverting regular elements of the form $s \otimes 1$. Consequently, $\overline{E}_0$ is noetherian.

By the rationality of Q_0, we may regard $Z = Z(E_0)$ as a subfield of $\overline{k}$, and so we may set $F_0 = E_0 \otimes_Z \overline{k}$. Since $\overline{E}_0$ maps surjectively onto F_0, the latter is noetherian, and it follows from a standard lemma (e.g., [**43**, 1.7.27]) that F_0 is simple. We claim that F_0 is its own Goldie quotient ring, which can be proved as follows. Given a regular element r of F_0, choose a field $k' \subseteq \overline{k}$ such that k' is finite over Z and such that $r \in E_0' = E_0 \otimes_Z k' \subseteq F_0$. Since E_0' is artinian, and since r is regular in E_0', it follows that r is invertible in E_0'. Thus every regular element of F_0 is invertible (in F_0), and the claim is established.

Now let P_0 denote the kernel of the composition of canonical maps

$$A' \to (A/Q_0) \otimes_k \overline{k} \to \overline{E}_0 \to F_0,$$

and observe that $P_0 \cap A = Q_0$. Because the first and third maps in the composition are surjective and the second is an Ore localization, it follows from the preceding paragraph that P_0 is prime and that F_0 is the Goldie quotient ring of A'/P_0. Next, observe (by, e.g., [**43**, 1.7.24]) that $Z(F_0) = \overline{k}$. Hence P_0 is $\overline{k}$-rational.

By [**43**, 2.12.50], there exists a prime ideal P_1 of A' such that $P_1 \cap A = Q_1$ and $P_0 \subset P_1$. From our assumptions on A' it follows that P_0 and P_1 are not in the same $\mathcal{H}'$-stratum.

For $i \in \{0, 1\}$, set J_i equal to the $\mathcal{H}'$-prime ideal $(P_i : \mathcal{H}')$. As in (2.1iv), we may choose a prime ideal M_i minimal over J_i such that $J_i = (M_i : \mathcal{H}')$; the finitely many prime ideals of A' minimal over J_i are precisely the ideals comprising the $\mathcal{H}'$-orbit of M_i. However, any product of all of the prime ideals minimal over J_0 is contained in M_1, and so – without loss of generality – we may assume that $M_0 \subseteq M_1$. Moreover, $M_0 \subsetneq M_1$, because $J_0 \subsetneq J_1$.

Because the action of $\mathcal{H}'$ densely extends that of $\mathcal{H}$, we have

$$J_i \cap A = (P_i : \mathcal{H}') \cap A = (P_i : \mathcal{H}) \cap A = (P_i \cap A : \mathcal{H}) = (Q_i : \mathcal{H}).$$

Similarly, setting $N_i = M_i \cap A$ we have $J_i \cap A = (M_i : \mathcal{H}') \cap A = (N_i : \mathcal{H})$. Since the $\mathcal{H}'$-orbit of M_i is finite, so is the $\mathcal{H}$-orbit of N_i, whence N_i is minimal over $J_i \cap A$. Therefore, since $J_0 \cap A \subseteq N_0 \subseteq N_1$, it follows that $J_0 \cap A = J_1 \cap A$ if and only if $N_0 = N_1$. By Incomparability (cf. [**43**, 3.4.13′]), $M_0 \subsetneq M_1$ implies that $N_0 \subsetneq N_1$, and so $J_0 \cap A \subsetneq J_1 \cap A$. But $J_i \cap A = (Q_i : \mathcal{H})$, and we assumed at the beginning of the proof that $(Q_0 : \mathcal{H}) = (Q_1 : \mathcal{H})$. This contradiction proves the lemma. $\square$

2.11. Set $A' = A \otimes \overline{k}$, and suppose that $\mathcal{H}'$ is a $\overline{k}$-affine algebraic group acting rationally (by $\overline{k}$-algebra automorphisms) on A'. Moeglin and Rentschler proved, for any rational ideal Q of A', that the map $\mathcal{H}' \to \operatorname{Rat}_{\overline{k}} A'$ given by $h' \mapsto h'(Q)$, for $h' \in \mathcal{H}'$, is continuous [**30**, 1.5]. We shall need the corresponding fact for arbitrary prime ideals:

Lemma. *For any $Q \in \operatorname{Spec} A'$, the map $\mathcal{H}' \to \operatorname{Spec} A'$, given by $h' \mapsto h'(Q)$ for $h' \in \mathcal{H}'$, is continuous.*

Proof. It suffices to show, for any ideal I of A', that the set

$$X = \{h \in \mathcal{H}' \mid h(Q) \supseteq I\}$$

is closed in $\mathcal{H}'$. Write $A' = \bigcup_{j \in J} V_j$ for some finite dimensional $\mathcal{H}'$-invariant $\overline{k}$-subspaces V_j. Then X equals the intersection of the sets

$$X_j = \{h \in \mathcal{H}' \mid h(Q \cap V_j) \supseteq I \cap V_j\},$$

and so it is enough to show that the X_j are all closed in $\mathcal{H}'$.

Fix $j \in J$, and choose idempotents $e_j, f_j \in \operatorname{End}_{\overline{k}}(V_j)$ with images $Q \cap V_j$ and $I \cap V_j$ respectively. The set

$$\{g \in \operatorname{GL}(V_j) \mid (1 - e_j)g^{-1}f_j = 0\}$$

is clearly closed in $\operatorname{GL}(V_j)$. Since the restriction map $\mathcal{H}' \to \operatorname{GL}(V_j)$ is continuous by hypothesis, it follows that X_j is closed in $\mathcal{H}'$, as desired. □

We now (partially) generalize (2.7ii) to arbitrary infinite fields.

2.12. Suppose that k is infinite, that $A \otimes \overline{k}$ is noetherian and satisfies the Nullstellensatz over $\overline{k}$, and that $\mathcal{H}$ is the group of k-rational points of a k-affine algebraic group $\mathcal{H}'$. (Thus $\mathcal{H}'$ is a Zariski-closed subgroup of some $GL_n(\overline{k})$, the polynomials defining $\mathcal{H}'$ can be chosen with coefficients from k, and $\mathcal{H} = \mathcal{H}' \cap GL_n(k)$.) We further assume that the action of $\mathcal{H}$ on A is rational; equivalently, this action arises from a right comodule structure $\mu\colon A \to A \otimes \mathcal{O}(\mathcal{H})$, where $\mathcal{O}(\mathcal{H})$ denotes the coordinate ring of $\mathcal{H}$ over k (see, e.g., [**21**, 2.8]). That $\mathcal{H}$ acts as k-algebra automorphisms means that μ must be a k-algebra homomorphism (cf. [**30**, 1.1]), i.e., A is a right $\mathcal{O}(\mathcal{H})$-comodule algebra via μ.

Theorem. *Suppose that either k is perfect or $\mathcal{H}'$ is reductive. If $\mathcal{H}$-$\operatorname{Spec} A$ is finite, then A satisfies the Dixmier-Moeglin Equivalence, and the primitive ideals of A are precisely the prime ideals maximal within their $\mathcal{H}$-strata.*

Proof. By (2.1iii) and (2.2ii), it suffices to prove that the rational ideals of A are maximal within their $\mathcal{H}$-strata in $\operatorname{Spec} A$. Set $A' = A \otimes \overline{k}$. Tensoring the comodule structure map $\mu\colon A \to A \otimes \mathcal{O}(\mathcal{H})$ with $\overline{k}$ and composing with the identity map tensored with the natural inclusion $\mathcal{O}(\mathcal{H}) \to \mathcal{O}(\mathcal{H}')$, we obtain a $\overline{k}$-algebra homomorphism

$$\mu'\colon A' \to A' \otimes_k \mathcal{O}(\mathcal{H}) \to A' \otimes_k \mathcal{O}(\mathcal{H}') \to A' \otimes_{\overline{k}} \mathcal{O}(\mathcal{H}'),$$

which gives A' the structure of a right $\mathcal{O}(\mathcal{H}')$-comodule algebra. This yields a rational action of $\mathcal{H}'$ on A' by $\overline{k}$-algebra automorphisms, which extends the action of $\mathcal{H}$ on A, in the sense of (2.8). In view of (2.9), $\mathcal{H}'$-$\operatorname{Spec} A'$ is finite.

Now let $\mathcal{H}'_0$ denote the identity component of $\mathcal{H}'$, a (closed connected) normal subgroup in $\mathcal{H}'$ of finite index (cf. [**3**, I.1.2b]). Also, $\mathcal{H}'_0$ is defined over k [ibid]. Note, moreover, that the finiteness of $\mathcal{H}'$-$\operatorname{Spec} A'$ ensures that $\mathcal{H}'_0$-$\operatorname{Spec} A'$ is finite. Setting $\mathcal{H}_0 = \mathcal{H}'_0(k) = \mathcal{H}'_0 \cap \mathcal{H}$, it follows that $\mathcal{H}_0$ is a normal subgroup in $\mathcal{H}$ of finite index.

Because either k is perfect or $\mathcal{H}'$ is reductive, it follows from [**3**, V.18.3] that $\mathcal{H}_0$ is dense in $\mathcal{H}'_0$. We claim that the action of $\mathcal{H}'_0$ on A' densely extends that of $\mathcal{H}_0$ on A, in the sense of (2.8). To prove the claim, consider $P \in \operatorname{Spec} A'$, and write $X = \{P_1 \in \operatorname{Spec} A' \mid P_1 \supseteq (P : \mathcal{H}_0)\}$, a closed subset of $\operatorname{Spec} A'$. By the continuity of the map $h' \mapsto h'(P)$ (see (2.11)), the set $Y = \{h' \in \mathcal{H}'_0 \mid h'(P) \in X\}$ is closed in $\mathcal{H}'_0$. Since Y contains the dense subgroup $\mathcal{H}_0$, it follows that Y must equal $\mathcal{H}'_0$, and so $(P : \mathcal{H}_0) = (P : \mathcal{H}'_0)$. The claim is thus proved. It now follows from (2.7ii) and (2.10) that every rational ideal of A is maximal within its $\mathcal{H}_0$-stratum.

Finally, let Q_0 and Q_1 be prime ideals of A, lying in the same $\mathcal{H}$-stratum, such that Q_0 is rational and $Q_0 \subseteq Q_1$. Let $\{h_1, \dots, h_n\}$ be a transversal for $\mathcal{H}_0$ in $\mathcal{H}$, and observe that

$$\bigcap_{i=1}^{n} h_i(Q_1 : \mathcal{H}_0) = (Q_1 : \mathcal{H}) = (Q_0 : \mathcal{H}) \subseteq (Q_0 : \mathcal{H}_0).$$

Further, each $h_i(Q_1 : \mathcal{H}_0) = (h_i(Q_1) : \mathcal{H}_0)$, an $\mathcal{H}_0$-ideal of A (this uses the normality of $\mathcal{H}_0$ in $\mathcal{H}$). Since $(Q_0 : \mathcal{H}_0)$ is an $\mathcal{H}_0$-prime ideal (2.1iv), it follows that $h_i(Q_1 : \mathcal{H}_0) \subseteq (Q_0 : \mathcal{H}_0)$ for some i. But $(Q_0 : \mathcal{H}_0) \subseteq (Q_1 : \mathcal{H}_0)$ because $Q_0 \subseteq Q_1$, and $h_i(Q_1 : \mathcal{H}_0)$ cannot be properly contained in $(Q_1 : \mathcal{H}_0)$ because A is noetherian. Hence, $h_i(Q_1 : \mathcal{H}_0) = (Q_0 : \mathcal{H}_0) = (Q_1 : \mathcal{H}_0)$; in particular, Q_0 and Q_1 belong to the same $\mathcal{H}_0$-stratum. It follows from the preceding paragraph that $Q_0 = Q_1$, and therefore Q_0 is maximal within its $\mathcal{H}$-stratum. □

3. Skew Polynomial Rings in One Variable

In order to apply the results of the previous section to quantum matrices and quantized Weyl algebras, we must exhibit rational actions of algebraic groups $\mathcal{H}$ on these algebras such that the number of $\mathcal{H}$-prime ideals is finite. Our approach to the latter condition relies on the (known) structure of the given algebras as iterated skew polynomial rings. The purpose of the present section is to provide the induction step, i.e., to develop conditions under which finiteness of $\mathcal{H}$-Spec passes from an algebra R to a skew polynomial extension $R[y; \tau, \delta]$.

3.1 Notation and Preliminaries. Throughout this section, R will denote a noetherian algebra over a field k, and $S = R[y; \tau, \delta]$ a skew polynomial extension of R. We will follow the conventions of [**12**, Chapter 2].

(i) We assume that τ is a k-algebra automorphism of R, and that δ is a k-linear left τ-derivation of R. Recall that the multiplication in S is determined by the rule $yr = \tau(r)y + \delta(r)$, and that S is noetherian [**28**, 1.2.9iv].

(ii) Let $\mathcal{H}$ denote a group acting on S by k-algebra automorphisms. Assume that R is $\mathcal{H}$-stable, that y is an $\mathcal{H}$-eigenvector, and that τ coincides with the action on R of some $h_0 \in \mathcal{H}$. Let $\lambda \colon \mathcal{H} \to k^\times$ denote the $\mathcal{H}$-eigenvalue of y, that is, the character of $\mathcal{H}$ such that $h(y) = \lambda(h)y$ for $h \in \mathcal{H}$.

(iii) It is convenient to denote by q the reciprocal of the h_0-eigenvalue of y, so that $h_0(y) = q^{-1}y$. For $r \in R$, observe that

$$\tau\delta(r) = h_0\big(yr - \tau(r)y\big) = q^{-1}yh_0(r) - h_0^2(r)q^{-1}y = q^{-1}\big(y\tau(r) - \tau^2(r)y\big) = q^{-1}\delta\tau(r).$$

Thus $\delta\tau = q\tau\delta$, and so the pair (τ, δ) is a q-*skew* derivation in the sense of [**12**]. As in [**12**, 2.4ii], it follows that we may extend τ to an automorphism of S such that $\tau(y) = q^{-1}y$. In the present setting, this extension is given by the action of h_0.

(iv) For $d \in R$, the inner τ-derivation $r \mapsto dr - \tau(r)d$ on R will be denoted δ_d.

3.2 Lemma. *If q is not a root of unity, then every $\mathcal{H}$-prime ideal of S contracts to a δ-stable $\mathcal{H}$-prime ideal of R.*

Proof. Let I be an $\mathcal{H}$-prime ideal of S. Then $I \cap R$ is an $\mathcal{H}$-stable ideal of R and in particular is τ-stable. It follows that $I \cap R$ is also δ-stable [**12**, 2.1v].

Choose a prime ideal P minimal over I; then the $\mathcal{H}$-orbit of P is finite, and $(P : \mathcal{H}) = I$. Now choose $Q \in \operatorname{Spec} R$ minimal over $P \cap R$. The prime ideals $h(Q)$, for $h \in \mathcal{H}$, are minimal over the ideals $h(P) \cap R$. But there are only finitely many ideals of the form $h(P) \cap R$, and there are only finitely many prime ideals in R minimal over each such ideal. Thus the $\mathcal{H}$-orbit of Q is finite. In particular, the τ-orbit of Q is finite.

By [**12**, 10.3], P contracts to either Q or the ideal $J = (Q : \tau)$. Hence, $I \cap R = (P \cap R : \mathcal{H})$ equals either $(Q : \mathcal{H})$ or $(J : \mathcal{H})$. However, $(Q : \mathcal{H}) = (J : \mathcal{H})$, because $\tau \in \mathcal{H}$. Therefore $I \cap R = (Q : \mathcal{H})$ is $\mathcal{H}$-prime. □

Recall that R is said to be $\mathcal{H}$-*simple* when its only $\mathcal{H}$-stable ideals are 0 and itself.

3.3 Lemma. *Assume that q is not a root of unity. Suppose that R is $\mathcal{H}$-simple but S is not.*

(i) *There is a unique element $d \in R$ such that $\delta = \delta_d$ and $h(d) = \lambda(h)d$ for all $h \in \mathcal{H}$.*

(ii) *There are precisely two $\mathcal{H}$-prime ideals in S, namely 0 and $(y - d)S$.*

Proof. (i) Let I be a proper nonzero $\mathcal{H}$-ideal in S, and let n be the minimum degree for nonzero elements of I. The set consisting of 0 and the leading coefficients of the elements in I of degree n is then a nonzero $\mathcal{H}$-ideal of R and so equals R. Hence, there exists a monic polynomial $s \in I$ with degree n, say $s = y^n + cy^{n-1} +$ [lower terms].

The usual analysis (as, e.g., in [**11**, 3.5]) of $sa - \tau^n(a)s$, for $a \in R$, shows that $\delta = \delta_d$, where $d = -\binom{n}{1}_q^{-1} q^{n-1}c$. For $h \in \mathcal{H}$, we have

$$h(s) = \lambda(h)^n y^n + \lambda(h)^{n-1}h(c)y^{n-1} + \text{[lower terms]}.$$

Then $h(s) - \lambda(h)^n s$ is an element of I of degree $< n$, whence $h(s) = \lambda(h)^n s$, and consequently $h(c) = \lambda(h)c$. Thus $h(d) = \lambda(h)d$ for all $h \in \mathcal{H}$.

Suppose also that $e \in R$, with $\delta = \delta_e$ and $h(e) = \lambda(h)e$, for all $h \in \mathcal{H}$. Set $f = d - e$: Then $h(f) = \lambda(h)f$ for all $h \in \mathcal{H}$, and $fa = \tau(a)f$ for all $a \in R$. Hence, $fR = Rf$ is an $\mathcal{H}$-ideal of R, and so f is either zero or a unit. However, $\tau(f) = q^{-1}f$, and so $f^2 = q^{-1}f^2$, whence $f^2 = 0$. Therefore $f = 0$, and so $e = d$.

(ii) Set $z = y - d$; then $S = R[z; \tau]$, and $h(z) = \lambda(h)z$ for all $h \in \mathcal{H}$. Then zS is an $\mathcal{H}$-ideal of S, and there is an $\mathcal{H}$-equivariant ring isomorphism $S/zS \cong R$. Therefore,

S/zS is an $\mathcal{H}$-simple ring, and so zS is an $\mathcal{H}$-prime ideal of S. As in the proof of part (i), every nonzero $\mathcal{H}$-ideal of S contains a monic polynomial. Since the product of two monic polynomials is always nonzero, S is an $\mathcal{H}$-prime ring; that is, 0 is an $\mathcal{H}$-prime ideal of S.

Now let P be a nonzero $\mathcal{H}$-prime ideal of S, and let n be the minimum degree for nonzero elements of P. As in part (i), there is a monic (in z) polynomial $p \in P$ of degree n, say $p = z^n + p_{n-1}z^{n-1} + \cdots + p_1 z + p_0$. Since p is monic, the Division Algorithm applies, showing that $P = pS = Sp$. Note that $P \cap R$ is a proper $\mathcal{H}$-ideal of R, whence $P \cap R = 0$. Thus $n \geq 1$.

For $a \in R$, observe that $pa - \tau^n(a)p$ is an element of P with degree less than n, whence $pa = \tau^n(a)p$. Thus $p_{n-i}a = \tau^i(a)p_{n-i}$ for all $a \in R$ and all $i = 1, \dots, n$. Similarly, given $h \in \mathcal{H}$, observe that $h(p) - \lambda(h)^n p$ is an element of P with degree less than n, whence $h(p) = \lambda(h)^n p$. Thus $h(p_{n-i}) = \lambda(h)^i p_{n-i}$ for all $h \in \mathcal{H}$ and all $i = 1, \dots, n$. Now each $p_{n-i}R = Rp_{n-i}$ is an $\mathcal{H}$-ideal of R, and therefore p_{n-i} is either zero or a unit.

We next compute that

$$\begin{aligned} zp &= z^{n+1} + \tau(p_{n-1})z^n + \cdots + \tau(p_1)z^2 + \tau(p_0)z \\ &= z^{n+1} + q^{-1}p_{n-1}z^n + \cdots + q^{-n+1}p_1 z^2 + q^{-n}p_0 z, \end{aligned}$$

and so $zp - pz = (q^{-1}-1)p_{n-1}z^n + \cdots + (q^{-n+1}-1)p_1 z^2 + (q^{-n}-1)p_0 z$. Since $zp - pz$ is an element in P of degree n, it must follow that $zp - pz = ap$, where $a = (q^{-1}-1)p_{n-1}$. Thus $(q^{-i}-1)p_{n-i} = ap_{n-i+1}$, for all $i = 1, \dots, n$, and $ap_0 = 0$.

Suppose that p_0 is a unit. Then $a = 0$, and since q is not a root of unity, it follows that $p_{n-i} = 0$ for all i. Thus $p = z^n$. Furthermore, because $zS = Sz$ is an $\mathcal{H}$-ideal and P is $\mathcal{H}$-prime, $z \in P$. Therefore, $n = 1$ in this case, whence $p = z$ and $P = zS$.

Now assume that $p_0 = 0$. Then $p = fz$, where $f = z^{n-1} + p_{n-1}z^{n-2} + \cdots + p_1$. By the minimality of n, $f \notin P$. Since zS is an $\mathcal{H}$-ideal, it follows that $z \in P$, and we conclude – as in the previous case – that $P = zS$. □

3.4 Proposition. *Assume that q is not a root of unity.*

(i) *There are at most twice as many $\mathcal{H}$-prime ideals in S as in R.*

(ii) *If all $\mathcal{H}$-prime ideals of R are prime (completely prime), then the same is true for S.*

Proof. (i) By (3.2), every $\mathcal{H}$-prime ideal of S contracts to a δ-stable $\mathcal{H}$-prime ideal of R. Hence, it suffices to show that at most two $\mathcal{H}$-prime ideals of S can contract to any given δ-stable $\mathcal{H}$-prime ideal Q in R. After passing to R/Q and localizing (see, e.g., [**12**, 2.3]), we may assume that R is an $\mathcal{H}$-simple artinian ring. If S is $\mathcal{H}$-simple, then S has just one $\mathcal{H}$-prime ideal, namely 0. If S is not $\mathcal{H}$-simple, then by (3.3), S has just two $\mathcal{H}$-prime ideals.

(ii) As in part (i), we can reduce to the case that R is $\mathcal{H}$-simple artinian. If R is prime (completely prime), the same is true of S; see, for example, [**28**, 1.2.9]. If there exists a nonzero $\mathcal{H}$-prime ideal P in S, then by (3.3), P is of the form $(y-d)S$, and $S/P \cong R$. In this case, P is prime (completely prime) if and only if R is a prime (completely prime) ring, and we are done. □

4. Iterated Skew Polynomial Rings

Finite stratification is proved for certain iterated skew polynomial extensions in (4.2), and the Dixmier-Moeglin Equivalence is established for them in (4.4) and (4.7).

4.1 Notation and Assumptions. Let $A = k[y_1][y_2; \tau_2, \delta_2] \cdots [y_n; \tau_n, \delta_n]$ be an iterated skew polynomial ring over the field k. Let $\mathcal{H}$ denote a group, acting as k-algebra automorphisms of A, for which $y_1, \dots, y_n$ are all $\mathcal{H}$-eigenvectors. (Note that the image of $\mathcal{H}$ in $\mathrm{Aut}_k A$ must therefore be abelian.) For $1 \le i \le n$, set $A_i = k[y_1][y_2; \tau_2, \delta_2] \cdots [y_i; \tau_i, \delta_i]$.

We make the following three assumptions:

(a) There are infinitely many distinct eigenvalues for the action of $\mathcal{H}$ on y_1. (This of course implies that k must be infinite.)

(b) Each τ_i is a k-algebra automorphism of A_{i-1} and each δ_i is a k-linear τ_i-derivation of A_{i-1}.

(c) For $2 \le i \le n$, there exists $h_i \in \mathcal{H}$ such that the restriction of h_i to A_{i-1} coincides with τ_i and the h_i-eigenvalue of y_i is not a root of unity.

4.2 Proposition. *The $\mathcal{H}$-prime ideals of A are all completely prime, and there are at most 2^n of them.*

Proof. By assumption (a), the only $\mathcal{H}$-prime ideals of A_1 are 0 and y_1A_1. Thus the desired conclusions hold for A_1, and they follow for A by inductively applying (3.4). $\square$

The fact that the $\mathcal{H}$-prime ideals of A are completely prime will allow us, in Section 6, to develop an explicit picture of the $\mathcal{H}$-strata in $\mathrm{Spec}\, A$.

4.3. If k' is an arbitrary field extension of k, then $A \otimes k'$ is a noetherian k'-algebra satisfying the Nullstellensatz; see, for example, [**28**, 1.2.9iv, 9.4.21] for details.

4.4 Theorem. *Assume that $\mathcal{H}$ is the group of k-rational points of a k-affine algebraic group $\mathcal{H}'$, and that the action of $\mathcal{H}$ on A is rational. Suppose further that either k is perfect or $\mathcal{H}'$ is reductive. Then A satisfies the Dixmier-Moeglin Equivalence, and the primitive ideals of A are precisely the prime ideals maximal within their $\mathcal{H}$-strata.*

Proof. By (4.2) and (4.3), the desired conclusions follow from (2.12). $\square$

4.5. In our applications (Section 5), $\mathcal{H}$ will be a torus of the form $(k^\times)^m$, and so we can take $\mathcal{H}' = (\overline{k}^\times)^m$, which is linearly reductive (see, e.g., [**34**, Chapter 5, Theorem 36]). Thus, we shall be able to apply (4.4) in arbitrary characteristic.

Furthermore, we can always reduce the situation of (4.1) to a case in which $\mathcal{H}$ is replaced by a closed subgroup of a torus. While this process (explained in the next paragraph) may change the $\mathcal{H}$-strata, it has no effect on the validity of the Dixmier-Moeglin Equivalence, and so we will obtain the equivalence whenever the hypotheses of (4.1) are satisfied.

4.6. Assuming – as we can – that the action of $\mathcal{H}$ on A is faithful, there is an embedding of $\mathcal{H}$ into the torus $\mathcal{D} = (k^\times)^n$ such that the closure of $\mathcal{H}$ also acts via automorphisms of A. To start, observe that the ordered monomials $y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n}$ form a k-basis for A. Hence, there is a faithful and rational action of $\mathcal{D}$ on A by vector space automorphisms, where each ordered monomial $y_1^{m_1} y_2^{m_2} \cdots y_n^{m_n}$ is an eigenvector whose eigenvalue is the character $(\alpha_1,\dots,\alpha_n)\mapsto \alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_n^{m_n}$. We identify $\mathcal{H}$ and $\mathcal{D}$ with their images in the group of vector space automorphisms of A, and we set $\mathcal{E}=\mathcal{D}\cap \operatorname{Aut}_k A$. Then $\mathcal{H}\subseteq\mathcal{E}$.

Observe, for $a,b\in A$, that the set $\{\phi\in\mathcal{D}\mid \phi(ab)=\phi(a)\phi(b)\}$ is Zariski-closed in $\mathcal{D}$. Therefore $\mathcal{E}$ is a closed subgroup of $\mathcal{D}$, acting rationally on A by k-algebra automorphisms. Further, $\mathcal{E}$ equals the group of k-rational points of an appropriate k-affine closed subgroup $\mathcal{E}'\subseteq(\overline{k}^{\times})^n$, namely $(\overline{k}^{\times})^n\cap \operatorname{Aut}_{\overline{k}}(A\otimes_k\overline{k})$, and $\mathcal{E}'$ is (linearly) reductive [ibid]. Since $\mathcal{H}\subseteq\mathcal{E}$, we may replace $\mathcal{H}$ by $\mathcal{E}$, thus obtaining the extra hypotheses of (4.4). Hence, we have proved the following theorem:

4.7 Theorem. *Let A be an iterated skew polynomial ring over the field k, equipped with a group $\mathcal{H}$ acting via k-algebra automorphisms, as in* (4.1). *Then A satisfies the Dixmier-Moeglin Equivalence.* □

5. Applications

Assume throughout this section that k is an infinite field. We discuss various quantized algebras for which the Dixmier-Moeglin Equivalence can be obtained via (4.4).

5.1 Quantum Affine Space. Let $A=\mathcal{O}_{\boldsymbol{q}}(k^n)$ be the multiparameter coordinate ring of quantum affine n-space over k, where $\boldsymbol{q}=(q_{ij})$ is a multiplicatively antisymmetric $n\times n$ matrix over k; that is, $q_{ii}=1$ and $q_{ji}=q_{ij}^{-1}$ for all i,j. The k-algebra A is generated by elements $y_1,\dots,y_n$ subject only to the relations $y_iy_j=q_{ij}y_jy_i$ for $i,j=1,\dots,n$. There is a natural action of the torus $\mathcal{H}=(k^{\times})^n$ by k-algebra automorphisms on A such that $(\alpha_1,\dots,\alpha_n).y_i=\alpha_iy_i$ for $(\alpha_1,\dots,\alpha_n)\in\mathcal{H}$ and all i. It is already known that the Dixmier-Moeglin Equivalence holds in A [**13**, 2.5], and it follows from [**13**, 2.3, 2.11] that the primitive ideals of A are precisely the prime ideals maximal in their $\mathcal{H}$-strata. In case k contains a non-root of unity, these results can also be obtained from (4.4); we leave the details to the reader.

5.2 Quantum Symplectic Space. (i) Next, let $A=\mathcal{O}_q(\mathfrak{sp}\,k^{2n})$ be the one-parameter coordinate ring of quantum symplectic $2n$-space over k as in [**42**, Definition 14] or [**32**, 1.1], where q is a nonzero element of k. (To our knowledge, precise relations for multiparameter quantum symplectic spaces have not appeared in the literature, although such algebras surely exist.) The algebra A is generated by elements $y_1,\dots,y_{2n}$ satisfying relations originally worked out in [**42**, p. 210]. We give the equivalent, simpler set of relations found by Musson [**32**, 1.1] (cf. [**35**, 1.1]):

$$y_iy_j=qy_jy_i \qquad (i<j;\ j\neq i')$$

$$y_iy_{i'}=q^2y_{i'}y_i+(q^2-1)\sum_{l=1}^{i-1}q^{l-i}y_ly_{l'} \qquad (i\leq n),$$

where $i'=2n+1-i$.

(ii) For each n-tuple $\alpha\in(k^{\times})^n$, there is a k-algebra automorphism $\omega(\alpha)$ of A such that $\omega(\alpha)(y_i)=\alpha_iy_i$ and $\omega(\alpha)(y_{i'})=\alpha_i^{-1}y_{i'}$, for all $i=1,\dots,n$. In particular, this gives a faithful rational action of the torus $\mathcal{H}=(k^{\times})^n$ on A by k-algebra automorphisms.

(iii) Musson has shown that A can be presented as an iterated skew polynomial ring of the form

$$k[y_1][y_{1'};\tau_1'][y_2;\tau_2][y_{2'};\tau_2',\delta_2']\cdots[y_n;\tau_n][y_{n'};\tau_n',\delta_n'],$$

where the k-algebra automorphisms τ_i,τ_i' and the k-linear τ_i'-derivations δ_i' are determined as follows:

$$\tau_i(y_j)=\begin{cases} q^{-1}y_j & (j<i)\\ qy_j & (j>i')\end{cases}$$

$$\tau_i'(y_j)=\begin{cases} q^{-1}y_j & (j<i)\\ qy_j & (j>i')\\ q^{-2}y_j & (j=i)\end{cases}$$

$$\delta_i'(y_j)=\begin{cases} 0 & (j\neq i)\\ (q^{-2}-1)\sum_{l=1}^{i-1}q^{l-i}y_ly_{l'} & (j=i)\end{cases}$$

[**32**, 1.2] (cf. [**35**, 1.10]). Here we have interchanged q and q^{-1} relative to [**32**, 1.2] due to our conventions for skew polynomial rings. Observe that τ_i and τ_i' coincide with the restrictions to the subalgebras $k\langle y_1,y_{1'},\dots,y_{i-1},y_{(i-1)'}\rangle$ and $k\langle y_1,y_{1'},\dots,y_{i-1},y_{(i-1)'},y_i\rangle$ of automorphisms of the form

$$h_i=\omega(q^{-1},q^{-1},\dots,q^{-1})$$
$$h_i'=\omega(q^{-1},q^{-1},\dots,q^{-1},q^{-2},1,1,\dots,1).$$

With these choices, the h_i-eigenvalue of y_i is q^{-1}, and the h_i'-eigenvalue of $y_{i'}$ is q^2.

In view of (5.2), we may now conclude the following from (4.2) and (4.4):

5.3 Theorem. *Let $A=\mathcal{O}_q(\mathfrak{sp}\,k^{2n})$ as in* (5.2i)*, and let the torus $\mathcal{H}=(k^\times)^n$ act on A as in* (5.2ii)*. Assume that q is not a root of unity.*

(i) *The $\mathcal{H}$-prime ideals of A are all completely prime, and there are at most 2^{2n} of them.*

(ii) *The Dixmier-Moeglin Equivalence holds for A, and its primitive ideals are precisely the prime ideals maximal within their $\mathcal{H}$-strata.* □

5.4 Quantum Euclidean Space. Let $A=\mathcal{O}_q(\mathfrak{o}\,k^n)$ be the one-parameter coordinate ring of quantum Euclidean n-space as in [**42**, Definition 12], with generators $y_1,\dots,y_n$. (In case n is odd, we must require that q has a square root in k.) As in the case of quantum symplectic space, a simpler set of relations is given in [**32**, 2.1, 2.2; **36**, 1.1], and an iterated skew polynomial ring presentation is given in [**32**, 2.3; **36**, 1.2]; we do not write out the details here.

Let $m=\lfloor n/2\rfloor$ denote the integer part of $n/2$. For $\alpha\in\mathcal{H}=(k^\times)^m$, there is a k-algebra automorphism $\omega(\alpha)$ of A such that $\omega(\alpha)(y_i)=\alpha_iy_i$ and $\omega(\alpha)(y_{i'})=\alpha_i^{-1}y_{i'}$ for $i=1,\dots,m$, and such that $\omega(y_{m+1})=y_{m+1}$ if n is odd. (Here $i'=n+1-i$.) This gives a rational action of $\mathcal{H}$ on A, and one checks easily that the automorphisms appearing in the iterated skew polynomial presentation of A coincide with restrictions of automorphisms $\omega(\alpha)$ satisfying (4.1c). Hence, we obtain the following analog of (5.3) from (4.2) and (4.4):

5.5 Theorem. *Let $A = \mathcal{O}_q(\mathfrak{o}\, k^n)$ as in* (5.4)*, and let the torus $\mathcal{H} = (k^\times)^{\lfloor n/2 \rfloor}$ act on A as in* (5.4)*. Assume that q is not a root of unity.*

(i) *The $\mathcal{H}$-prime ideals of A are all completely prime, and there are at most 2^n of them.*

(ii) *The Dixmier-Moeglin Equivalence holds for A, and its primitive ideals are precisely the prime ideals maximal within their $\mathcal{H}$-strata.* □

5.6 Quantum Matrices. (i) Let $A = \mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ be the multiparameter coordinate ring of quantum $n \times n$ matrices over k, as studied in [**2**, **27**, **41**, **44**]. Here, $\boldsymbol{p} = (p_{ij})$ is a multiplicatively antisymmetric $n \times n$ matrix over k, and λ is a nonzero element of k not equal to -1. The k-algebra A is generated by variables $y_{11}, y_{12}, \ldots, y_{nn}$ subject only to the following relations:

$$y_{\ell m} y_{ij} = \begin{cases} p_{\ell i} p_{jm} y_{ij} y_{\ell m} + (\lambda - 1) p_{\ell i} y_{im} y_{\ell j} & \text{when } \ell > i \text{ and } m > j \\ \lambda p_{\ell i} p_{jm} y_{ij} y_{\ell m} & \text{when } \ell > i \text{ and } m \leq j \\ p_{jm} y_{ij} y_{\ell m} & \text{when } \ell = i \text{ and } m > j. \end{cases}$$

(ii) Given any n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ in $(k^\times)^n$, there are k-algebra automorphisms $\theta_l(\alpha)$ and $\theta_r(\beta)$ of A such that $\theta_l(\alpha)(y_{ij}) = \alpha_i y_{ij}$ and $\theta_r(\beta)(y_{ij}) = \beta_j y_{ij}$ for all i, j. Set $\mathcal{H} = (k^\times)^n \times (k^\times)^n$. There is an action of $\mathcal{H}$ on A defined by mapping $\mathcal{H} \to \operatorname{Aut}_k A$ according to the rule $(\alpha, \beta) \mapsto \theta_l(\alpha)\theta_r(\beta)$. This action is clearly rational.

(iii) Apply the lexicographic ordering to the double indices $11, 12, \ldots, nn$. If the variables y_{ij} are adjoined in this order, A can be presented as an iterated skew polynomial ring

$$k[y_{11}][y_{12}; \tau_{12}] \cdots [y_{ij}; \tau_{ij}, \delta_{ij}] \cdots [y_{nn}; \tau_{nn}, \delta_{nn}],$$

as shown in [**2**, pp. 890-891]. Choose $\ell m \in \{11, \ldots, nn\}$. Let R_{lm} denote the k-subalgebra of A generated by $\{y_{ij} \mid ij < \ell m\}$, and let $S_{\ell m}$ denote the subalgebra generated by $R_{\ell m} \cup \{y_{\ell m}\}$. Then $S_{\ell m} = R_{\ell m}[y_{\ell m}; \tau_{\ell m}, \delta_{\ell m}]$, where $\tau_{\ell m}$ and $\delta_{\ell m}$ are a k-algebra automorphism and a k-linear $\tau_{\ell m}$-derivation of $R_{\ell m}$ determined by their actions on the generators of $R_{\ell m}$ as follows:

$$\tau_{\ell m}(y_{ij}) = \begin{cases} p_{\ell i} p_{jm} y_{ij} & \text{when } \ell \geq i \text{ and } m > j \\ \lambda p_{\ell i} p_{jm} y_{ij} & \text{when } \ell > i \text{ and } m \leq j \end{cases}$$

$$\delta_{\ell m}(y_{ij}) = \begin{cases} (\lambda - 1) p_{\ell i} y_{im} y_{\ell j} & \text{when } \ell > i \text{ and } m > j \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\tau_{\ell m}$ coincides with the restriction to $R_{\ell m}$ of the automorphism

$$h_{\ell m} = \theta_l(p_{\ell 1}, \ldots, p_{\ell n}) \theta_r(p_{1m}, \ldots, p_{m-1,m}, \lambda p_{mm}, \lambda p_{m+1,m}, \ldots, \lambda p_{nm}),$$

and that the $h_{\ell m}$-eigenvalue of $y_{\ell m}$ is λ.

5.7. Let A and $\mathcal{H}$ be as in (5.6).

(i) It follows from (4.2), when λ is not a root of unity, that the $\mathcal{H}$-prime ideals of A are completely prime, and that there are no more than 2^{n^2} of them. (For instance, when $n = 2$ there are 14.)

(ii) Note that (i) holds in some cases when the prime ideals of A are not all completely prime: If $n = 2$ and $p_{12} = p_{21} = -1$, then $A/\langle y_{21}, y_{22}\rangle \cong k\langle y_{11}, y_{12}\rangle/\langle y_{11}y_{12} + y_{12}y_{11}\rangle$. Hence, $\langle y_{21}, y_{22}, y_{11}^2 - 1\rangle$ is a prime ideal of A that is not completely prime.

(iii) There is a $\mathbb{Z}\times\mathbb{Z}$-grading on A arising from the double indexing of the variables y_{ij}, and it is straightforward to verify that all of the $\mathcal{H}$-prime ideals of A are homogeneous with respect to this grading. Now assume that k is algebraically closed. Then $\mathcal{H}$ is a divisible group, and so the only finite $\mathcal{H}$-orbits in $\operatorname{Spec} A$ are singletons. It therefore follows directly from (2.1iv), in the algebraically closed case, that every $\mathcal{H}$-prime ideal is prime.

Combining (4.4) and (5.7i), we now obtain:

5.8 Theorem. *Let $A = \mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ as in* (5.6i)*, assuming that λ is not a root of unity. Let the torus $\mathcal{H} = (k^\times)^n \times (k^\times)^n$ act on A as in* (5.6ii)*. Then the Dixmier-Moeglin Equivalence is satisfied by A, and its primitive ideals are precisely the prime ideals maximal within their $\mathcal{H}$-strata.* $\square$

5.9 Remarks. (i) The multiparameter quantum group $\mathcal{O}_{\lambda,\boldsymbol{p}}(GL_n(k))$ is obtained by inverting the quantum determinant of $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$, which is a normal (but not necessarily central) element of $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$; see, e.g., [**2**, Theorem 3]. Also, the action of $\mathcal{H}$ extends to $\mathcal{O}_{\lambda,\boldsymbol{p}}(GL_n(k))$, since the quantum determinant is an $\mathcal{H}$-eigenvector. It is now easy to see that the conclusions of (5.8), again assuming that λ is not a root of unity, extend to $\mathcal{O}_{\lambda,\boldsymbol{p}}(GL_n(k))$.

(ii) The Dixmier-Moeglin Equivalence also holds for $A = \mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ when λ is a root of unity – see [**13**, 3.2]. This case cannot be handled with the methods used above, since A can have infinitely many $\mathcal{H}$-prime ideals. For example, let $n = 2$ and let q be a primitive t-th root of unity (in k) for some odd integer $t > 1$; then take $\lambda = q^{-2}$ and $p = \left(\begin{smallmatrix} 1 & q^{-1} \\ q & 1 \end{smallmatrix}\right)$. (Then A reduces to the standard one-parameter case, often denoted $\mathcal{O}_q(M_2(k))$.) In this case, y_{ij}^t is central for all i, j [**37**, 7.2.1]. Now consider the $\mathcal{H}$-ideals $J_\alpha = \langle y_{11}^t y_{22}^t - \alpha y_{12}^t y_{21}^t\rangle$, for $\alpha \in k$. Any prime ideal P containing two different J_α's must contain both $y_{11}^t y_{22}^t$ and $y_{12}^t y_{21}^t$. Since y_{12} and y_{21} are normal in A, and since y_{11} and y_{22} are normal modulo either $\langle y_{12}\rangle$ or $\langle y_{21}\rangle$, it follows that P must contain one of y_{12}, y_{21} and one of y_{11}, y_{22}; hence, P has height at least 2. On the other hand, any prime minimal over a J_α has height 1 by the Principal Ideal Theorem for normal elements [**28**, 4.1.11]. Therefore no prime ideal of A can be minimal over two different J_α's. Consequently, if we choose an $\mathcal{H}$-prime ideal P_α minimal over J_α for each α, we obtain infinitely many distinct $\mathcal{H}$-prime ideals in A.

5.10 Quantum Weyl Algebras. (i) Let $A = A_n^{Q,\Gamma}(k)$ be the multiparameter quantized Weyl algebra over k as in [**26**, **1**, **5**]. Here, $Q = (q_1, \dots, q_n) \in (k^\times)^n$, and $\Gamma = (\gamma_{ij})$ is a multiplicatively antisymmetric $n \times n$ matrix over k. The algebra A is generated by elements $x_1, y_1, \dots, x_n, y_n$ subject only to the following relations:

$$\begin{cases} y_i y_j = \gamma_{ij} y_j y_i & (\text{all } i, j) \\ x_i x_j = q_i \gamma_{ij} x_j x_i & (i < j) \\ x_i y_j = \gamma_{ji} y_j x_i & (i < j) \\ x_i y_j = q_j \gamma_{ji} y_j x_i & (i > j) \\ x_j y_j = 1 + q_j y_j x_j + \sum_{m<j} (q_m - 1) y_m x_m & (\text{all } j) \end{cases}$$

(ii) For each $\alpha \in \mathcal{H} = (k^\times)^n$, there is a k-algebra automorphism $\omega(\alpha)$ of A such that $\omega(\alpha)(x_i) = \alpha_i x_i$ and $\omega(\alpha)(y_i) = \alpha_i^{-1} y_i$, for all i. In particular, this gives a faithful rational action of $\mathcal{H}$ on A by k-algebra automorphisms.

(iii) For $\ell = 1, \dots, n$, let $S_{2\ell-1}$ denote the k-subalgebra of A generated by $y_1, x_1, \dots, y_{\ell-1}, x_{\ell-1}, y_\ell$, and let $S_{2\ell}$ denote the subalgebra generated by $y_1, x_1, \dots, y_\ell, x_\ell$. We can write A as an iterated skew polynomial ring,

$$A = k[y_1][x_1; \tau_2, \delta_2][y_2; \tau_3][x_2; \tau_4, \delta_4] \cdots [y_n; \tau_{2n-1}][x_n; \tau_{2n}, \delta_{2n}],$$

where the τ_ℓ and δ_ℓ are k-algebra automorphisms and k-linear τ_ℓ-derivations determined by their actions on generators as follows:

$$\begin{cases} \tau_{2\ell-1}(y_i) = \gamma_{\ell i} y_i & (i < \ell) \\ \tau_{2\ell-1}(x_i) = \gamma_{i\ell} x_i & (i < \ell) \\ \tau_{2\ell}(y_i) = q_i \gamma_{i\ell} y_i & (i \le \ell) \\ \tau_{2\ell}(x_i) = q_i^{-1} \gamma_{\ell i} x_i & (i < \ell) \\ \delta_{2\ell}(y_i) = 0 & (i < \ell) \\ \delta_{2\ell}(y_\ell) = 1 + \sum_{m<\ell} (q_m - 1) y_m x_m & \\ \delta_{2\ell}(x_i) = 0 & (i < \ell) \end{cases}$$

(cf. [**22**, 2.1, 2.8]). Observe that $\tau_{2\ell}$ coincides with the restriction to $S_{2\ell-1}$ of the automorphism

$$h_{2\ell} = \omega(q_1^{-1}\gamma_{\ell 1}, \dots, q_{\ell-1}^{-1}\gamma_{\ell,\ell-1}, q_\ell^{-1}, 1, \dots, 1),$$

and that $\tau_{2\ell-1}$ coincides with the restriction to $S_{2\ell-2}$ of

$$h_{2\ell-1} = \omega(\gamma_{1\ell}, \dots, \gamma_{\ell-1,\ell}, q_\ell^{-1}, 1, \dots, 1).$$

With these choices, the $h_{2\ell}$-eigenvalue of x_ℓ is q_ℓ^{-1}, and the $h_{2\ell-1}$-eigenvalue of y_ℓ is q_ℓ.

In view of (5.10), we may now conclude the following from (4.2) and (4.4):

5.11 Theorem. *Let $A = A_n^{Q,\Gamma}(k)$ as in (5.10i), and let the torus $\mathcal{H} = (k^\times)^n$ act on A as in (5.10ii). Assume that $q_1, \dots, q_n$ are not roots of unity.*

(i) *The $\mathcal{H}$-prime ideals of A are all completely prime, and there are at most 2^{2n} of them.*

(ii) *The Dixmier-Moeglin Equivalence holds for A, and its primitive ideals are precisely the prime ideals maximal within their $\mathcal{H}$-strata.* $\square$

6. Strata Under Torus Actions: The Completely Prime Case

In the previous section we considered certain algebras, equipped with actions by tori $\mathcal{H}$, for which all of the $\mathcal{H}$-prime ideals are completely prime. In this section we present a detailed description of the strata occurring in such cases.

6.1. Let A be a noetherian algebra over an infinite field k (of arbitrary characteristic), and let $\mathcal{H} = (k^\times)^r$ be a torus, of (finite) rank r over k, acting rationally on A by k-algebra automorphisms. (For example, A and $\mathcal{H}$ can be assumed to be as described in (5.1), (5.2), (5.4), (5.6), or (5.10).) Let $\widehat{\mathcal{H}}$ denote the group of rational characters of $\mathcal{H}$, that is, the set of algebraic group morphisms $\mathcal{H} \to k^\times$ under the operation of pointwise multiplication. Since k is infinite, $\widehat{\mathcal{H}}$ is free abelian of rank r, with a basis consisting of the component projections $(k^\times)^r \to k^\times$.

Because $\mathcal{H}$ acts rationally, A is spanned by its $\mathcal{H}$-eigenvectors [**34**, Ch. 5, Corollary to Theorem 36], and the corresponding eigenvalues lie in $\widehat{\mathcal{H}}$. Thus $A = \bigoplus_{x\in\widehat{\mathcal{H}}} A_x$, where A_x denotes the x-eigenspace of A. Further, $A_xA_y \subseteq A_{xy}$ for $x, y \in \widehat{\mathcal{H}}$, and so A is graded by the group $\widehat{\mathcal{H}}$. (Conversely, any grading of A by $\mathbb{Z}^r$ arises from a rational action of $(k^\times)^r$ on A, as noted in [**39**, p. 784].)

In order to analyze the stratum of $\operatorname{Spec} A$ corresponding to a particular $\mathcal{H}$-prime ideal J (not yet assumed to be completely prime), we first pass to the factor algebra A/J, which is again graded by $\widehat{\mathcal{H}}$. We then try to simplify the situation by passing to an appropriate localization. Unfortunately, a general "graded Goldie Theorem" – an Ore localization with respect to the homogeneous regular elements, resulting in a graded-semisimple ring – is not available (cf. [**33**, Part C, I.1.1]). However, for the case in which J is completely prime, we need only a "graded Ore Theorem", which is quite easy to prove. (As mentioned above, the $\mathcal{H}$-prime ideals of the algebras in (5.1), (5.2), (5.4), (5.6), and (5.10) are all completely prime.)

6.2 Lemma. *Let R be a right Ore domain, graded by a group G, and let $\mathcal{E}$ denote the set of nonzero homogeneous elements of R. Then $\mathcal{E}$ is a right denominator set in R, the localization $R[\mathcal{E}^{-1}]$ is graded by G, and every nonzero homogeneous element of $R[\mathcal{E}^{-1}]$ is invertible.*

Proof. (The reader is referred to [**14**] for definitions of unexplained terms.) To verify that $\mathcal{E}$ is a right denominator set, only the right Ore condition needs to be checked, since the remaining condition (right reversibility) is trivially true in a domain. To prove that $\mathcal{E}$ is right Ore, one easily sees that it suffices to show that $r\mathcal{E} \cap eR \neq \varnothing$ for all $e \in \mathcal{E}$ and all homogeneous elements $r \in R$ (cf. [**33**, Part A, I.6.1]). Since R is a right Ore domain, there exist $s, t \in R$ such that $s \neq 0$ and $rs = et$. Suppose $r \in R_x$ and $e \in R_y$, for some $x, y \in G$, and suppose $s = s_1 + \cdots + s_n$, with $s_i \in R_{z_i}$ for some distinct $z_i \in G$. Then $t = t_1 + \cdots + t_n$, for some $t_i \in R_{y^{-1}xz_i}$, such that $rs_i = et_i$. Since some $s_i \neq 0$ (whence $s_i \in \mathcal{E}$), this verifies the desired common multiple condition.

Having verified that $\mathcal{E}$ is a right denominator set, we obtain the localization $R[\mathcal{E}^{-1}]$, and we observe that there is a well-defined G-grading on $R[\mathcal{E}^{-1}]$ such that re^{-1} has degree xy^{-1} for all $r \in R_x$ and $e \in \mathcal{E} \cap R_y$ (cf. [**33**, Part A, I.6.2]). It is clear that every nonzero homogeneous element of $R[\mathcal{E}^{-1}]$ is invertible. $\square$

6.3. Let R be a ring graded by a (multiplicative) group G, such that every nonzero homogeneous element of R is invertible. One says in this situation that R is a *G-graded division ring* [**33**, Part A, I.4]. Note that the identity component R_1 is then a division ring. Also note, for $x, y \in G$, that $R_xR_y = R_xr_y$ for all nonzero $r_y \in R_y$; in particular, each R_y is either zero or a 1-dimensional vector space over R_1. Lastly, R is said to be *strongly* graded provided $R_xR_y = R_{xy}$, for all $x, y \in G$.

Lemma. *Assume that G is abelian.*

(a) *The center $Z(R)$ is a homogeneous subring of R, strongly graded by the subgroup $G_Z = \{x \in G \mid Z(R) \cap R_x \neq 0\}$ of G.*

(b) *The ring R is a free $Z(R)$-module, in which $Z(R)$ is a direct summand.*

(c) *Suppose that G_Z is free abelian of finite rank, with basis $\{g_1, \dots, g_n\}$. Choose a nonzero element $z_j \in Z(R) \cap R_{g_j}$ for each j. Then $Z(R) = (Z(R) \cap R_1)[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$, a Laurent polynomial ring over the field $Z(R) \cap R_1$.*

Proof. Part (a) is obvious (cf. [**33**, Part A, I.4.5(3)]), and allows us to write $Z(R) = \bigoplus_{x \in G_Z} Z(R)_x$, where $Z(R)_x = Z(R) \cap R_x$.

(b) Note that the set $G' = \{x \in G \mid R_x \neq 0\}$ is a subgroup of G. After replacing G by G', we may assume that $R_x \neq 0$ for all $x \in G$. (In particular, R is now strongly graded by G.)

Observe that $S = \bigoplus_{x \in G_Z} R_x$ is a homogeneous subalgebra of R containing $Z(R)$ and strongly graded by G_Z. Choose a transversal T for G_Z in G, with $1 \in T$. For $t \in T$ choose a nonzero element $r_t \in R_t$, with $r_1 = 1$. Since the multiplication map $G_Z \times T \to G$ is bijective,

$$R = \bigoplus_{x \in G_Z} \bigoplus_{t \in T} R_{xt} = \bigoplus_{x \in G_Z} \bigoplus_{t \in T} R_x r_t = \bigoplus_{t \in T} S r_t.$$

Thus R is a free left S-module with basis $\{r_t \mid t \in T\}$, and since this basis includes $r_1 = 1$, we also see that S is a left S-module direct summand of R. Therefore it suffices to show that S is a free $Z(R)$-module in which $Z(R)$ is a direct summand.

Choose nonzero elements $z_x \in Z(R)_x$ for $x \in G_Z$. Since $Z(R)_1$ is a field, we can choose a basis $\mathcal{B}$ for S_1 as a vector space over $Z(R)_1$, with $1 \in \mathcal{B}$. Hence,

$$S = \bigoplus_{x \in G_Z} S_1 z_x = \bigoplus_{\beta \in \mathcal{B}} \bigoplus_{x \in G_Z} Z(R)_1 \beta z_x = \bigoplus_{\beta \in \mathcal{B}} Z(R)\beta;$$

that is, S is a free $Z(R)$-module with basis $\mathcal{B}$. Since $1 \in \mathcal{B}$, this also shows that $Z(R)$ is a direct summand of S, as desired.

(c) Since the monomials $z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n}$ are homogeneous with distinct degrees $g_1^{i_1} g_2^{i_2} \cdots g_n^{i_n}$, it is clear that $z_1, \dots, z_n$ are algebraically independent over $Z(R)_1$. For each $x \in G_Z$, there exist unique $m_1, \dots, m_n \in \mathbb{Z}$ such that $x = g_1^{m_1} g_2^{m_2} \cdots g_n^{m_n}$, and $z_x = z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$ is a nonzero element of $Z(R)_x$. Therefore $Z(R) = \bigoplus_{x \in G_Z} Z(R)_1 z_x = Z(R)_1[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$. $\square$

6.4 Proposition. *Let G be an abelian group and R a G-graded division ring.*

(a) *If I is an ideal of R, then $I = R(I \cap Z(R))$.*

(b) *If J is an ideal of $Z(R)$, then $J = (RJ) \cap Z(R)$.*

Proof. Part (b) is immediate from (6.3.b).

(a) Set $J = I \cap Z(R)$, and suppose that $I \neq RJ$. Choose an element $r \in I \setminus RJ$ with support $\{x_1, \dots, x_n\}$ of minimal cardinality. Then write $r = r_1 + \cdots + r_n$ for some nonzero elements $r_i \in R_{x_i}$.

First, suppose there exists a nonzero element $s \in I$ whose support is properly contained in $\{x_1, \dots, x_n\}$. After possibly renumbering, we may assume that the support of s is contained in $\{x_1, \dots, x_{n-1}\}$ and includes x_1. Write $s = s_1 + \cdots + s_{n-1}$, with each $s_i \in R_{x_i}$ and $s_1 \neq 0$. Now consider the element $t = r_1^{-1}r - s_1^{-1}s$, and observe that t is an element in I with support contained in $\{x_1^{-1}x_2, \dots, x_1^{-1}x_n\}$. By the minimality of n, both s and t must lie in RJ. But then $r = r_1 t + r_1 s_1^{-1} s \in RJ$, contradicting our assumptions.

Thus there cannot be any nonzero element in I whose support is properly contained in $\{x_1, \dots, x_n\}$.

Now set $r' = r_1^{-1}r$, which is an element of I with support $\{1, x_1^{-1}x_2, \dots, x_1^{-1}x_n\}$ and identity component 1. Given any nonzero homogeneous element $u \in R$, observe that $u^{-1}r'u$ has the same support as r' (because G is abelian), and its identity component is also 1. Hence, the difference $u^{-1}r'u - r'$ has support contained in $\{x_1^{-1}x_2, \dots, x_1^{-1}x_n\}$, and so the element $v = r_1(u^{-1}r'u - r')$ is an element of I with support contained in $\{x_2, \dots, x_n\}$. By the previous paragraph, $v = 0$, whence $u^{-1}r'u = r'$. Since u was an arbitrary nonzero homogeneous element of R, this last equality implies that $r' \in Z(R)$. Hence, $r = r_1 r' \in RJ$, another contradiction.

Therefore $I = RJ$. $\square$

6.5 Corollary. *If G is an abelian group and R a G-graded division ring, then contraction and extension provide:*

(a) *Mutually inverse isomorphisms between the lattices of ideals of R and $Z(R)$;*

(b) *Mutually inverse homeomorphisms between* $\operatorname{Spec} R$ *and* $\operatorname{Spec} Z(R)$. $\square$

We now return to the situation of (6.1) and apply (6.2)–(6.5), obtaining the following description of the strata:

6.6 Theorem. *Let A be a noetherian algebra over an infinite field k, and suppose that a torus $\mathcal{H} = (k^\times)^r$ acts rationally on A by k-algebra automorphisms. Let J be a completely prime $\mathcal{H}$-invariant ideal of A, and let*

$$\mathcal{S}_J = \{P \in \operatorname{Spec} A \mid (P : \mathcal{H}) = J\}$$

be the corresponding $\mathcal{H}$-stratum of $\operatorname{Spec} A$. *Then there exists an Ore set $\mathcal{E}_J$ in the algebra A/J such that*

(a) *The localization map $A \to A/J \to A_J = (A/J)[\mathcal{E}_J^{-1}]$ induces a homeomorphism of $\mathcal{S}_J$ onto* $\operatorname{Spec} A_J$.

(b) *Contraction and extension induce mutually inverse homeomorphisms between* $\operatorname{Spec} A_J$ *and* $\operatorname{Spec} Z(A_J)$.

(c) *$Z(A_J)$ is a commutative Laurent polynomial ring over an extension field of k, in r or fewer indeterminates.*

Proof. We may clearly assume that $J = 0$. As in (6.1), let $\widehat{\mathcal{H}}$ denote the group of rational characters of $\mathcal{H}$, and grade A by $\widehat{\mathcal{H}}$. Let $\mathcal{E}_J$ denote the set of nonzero homogeneous elements of A with respect to this grading. Then by (6.2), $\mathcal{E}_J$ is an Ore set in A and the localization $A_J = A[\mathcal{E}_J^{-1}]$ is an $\widehat{\mathcal{H}}$-graded division ring. Observe that the prime ideals of A disjoint from $\mathcal{E}_J$ are precisely the prime ideals that contain no nonzero $\mathcal{H}$-eigenvectors. The set of such prime ideals is precisely $\mathcal{S}_J$, and therefore part (a) follows from standard localization theory (e.g., [**14**, 9.22], [**28**, 2.1.16vii]). Part (b) is given by (6.5), and part (c) follows from (6.3). □

6.7. The above description of the $\mathcal{H}$-strata has been noted for certain quantized algebras in [**4**, 4.5, 5.1–7]. In view of (5.3), (5.5), (5.7) and (5.11), this description also holds for the $\mathcal{H}$-strata of the algebras $\mathcal{O}_q(\mathfrak{sp}\, k^{2n})$, $\mathcal{O}_q(\mathfrak{o}\, k^n)$, $\mathcal{O}_{\lambda,\boldsymbol{p}}(M_n(k))$ and $A_n^{Q,\Gamma}(k)$ discussed in (5.2), (5.4), (5.6) and (5.10). We emphasize that these results do not require k to be algebraically closed, as one might expect if one analyzed the maximal portion of an $\mathcal{H}$-stratum using the Moeglin-Rentschler-Vonessen transitivity theorem (2.6). Moreover, a transitivity result suitable for our purposes follows easily from (6.2)–(6.5), as we now show.

6.8 Theorem. *Let A, $\mathcal{H}$, J, and $\mathcal{S}_J$ be as described in* (6.6).
(a) *Every rational ideal in $\mathcal{S}_J$ is maximal in $\mathcal{S}_J$.*
(b) *If k is algebraically closed, then $\mathcal{H}$ acts transitively on $\mathcal{S}_J \cap \operatorname{Rat} A$.*

Proof. We may obviously assume that $J = 0$ and that $\mathcal{S}_J \cap \operatorname{Rat} A$ is nonempty. Now A is a noetherian domain, graded by the character group $\widehat{\mathcal{H}}$, which is free abelian of rank r. Let $\mathcal{E}$ denote the set of nonzero homogeneous elements of A. By (6.2), $\mathcal{E}$ is an Ore set in A, and $A[\mathcal{E}^{-1}]$ is an $\widehat{\mathcal{H}}$-graded division ring. Note that the action of $\mathcal{H}$ on A extends naturally to a rational action by k-algebra automorphisms on $A[\mathcal{E}^{-1}]$, and that the $\mathcal{H}$-grading on $A[\mathcal{E}^{-1}]$ corresponding to this action coincides with the one obtained from the localization process. By (6.6a), the localization map $A \to A[\mathcal{E}^{-1}]$ induces a bijection from $\mathcal{S}_J$ onto $\operatorname{Spec} A[\mathcal{E}^{-1}]$.

Consider a prime ideal $P \in \mathcal{S}_J$. Since the Ore set $\mathcal{E}$ is disjoint from P, its image in A/P consists of regular elements (e.g., [**14**, 9.21]). Hence,

$$\operatorname{Fract}(A[\mathcal{E}^{-1}]/PA[\mathcal{E}^{-1}]) \cong \operatorname{Fract}(A/P),$$

and so $PA[\mathcal{E}^{-1}]$ is rational if and only if P is rational. Thus the bijection of $\mathcal{S}_J$ onto $\operatorname{Spec} A[\mathcal{E}^{-1}]$ via localization restricts to a bijection of $\mathcal{S}_J \cap \operatorname{Rat} A$ onto $\operatorname{Rat} A[\mathcal{E}^{-1}]$. Therefore we may replace A by $A[\mathcal{E}^{-1}]$, that is, we may assume that A is an $\widehat{\mathcal{H}}$-graded division ring and that $\mathcal{S}_J = \operatorname{Spec} A$.

Since $\widehat{\mathcal{H}}$ is free abelian of finite rank, so is the subgroup

$$G_Z = \{x \in \widehat{\mathcal{H}} \mid Z(A)_x \neq 0\}.$$

Choose a basis $\{g_1, \ldots, g_n\}$ for G_Z, and choose a nonzero element $z_j \in Z(A)_{g_j}$ for $j = 1, \ldots, n$. By (6.3.c), $Z(A)$ is a Laurent polynomial ring of the form $Z(A)_1[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$.

(a) Let $P \in \operatorname{Rat} A$, and set $Q = P \cap Z(A)$. Then the domain $Z(A)/Q$ embeds in the center of $\operatorname{Fract}(A/P)$, which is algebraic over k by our assumption on P. Since $Z(A)$ is a Laurent polynomial ring over the field $Z(A)_1$, it follows that $Z(A)_1$ is algebraic over k and that $Z(A)/Q$ is a field. In particular, $Q \in \operatorname{Max} Z(A)$, and thus $P \in \operatorname{Max} A$ by (6.5).

(b) Let $P_1, P_2 \in \operatorname{Rat} A$, and set $Q_i = P_i \cap Z(A)$. Since k is now assumed to be algebraically closed, the observations of the preceding paragraph show that $Z(A)_1 = k$ and that each $Q_i = \sum_{j=1}^n Z(A)(z_j - \alpha_{ij})$ for some $\alpha_{ij} \in k^\times$. Thus by (6.4), each $P_i =$

$\sum_{j=1}^{n} A(z_j - \alpha_{ij})$. Recall that each $z_j \in Z(A)_{g_j}$ and that the g_j are linearly independent characters of $\mathcal{H}$. Hence, there exists $h \in \mathcal{H}$ such that $g_j(h) = \alpha_{1j}\alpha_{2j}^{-1}$ for all j [**18**, 16.2C]. Consequently,

$$h(z_j - \alpha_{1j}) = g_j(h)z_j - \alpha_{1j} = \alpha_{1j}\alpha_{2j}^{-1}(z_j - \alpha_{2j})$$

for all j, from which we conclude that $h(P_1) = P_2$. □

We conclude by noting that (6.8) yields a direct proof of the special case of (2.12) corresponding to our present situation.

6.9 Corollary. *Let A and $\mathcal{H}$ be as in* (6.1)*, and assume that A satisfies the Nullstellensatz over k. Assume further that $\mathcal{H}$-*Spec A *is finite, and that all $\mathcal{H}$-prime ideals of A are completely prime. Then A satisfies the Dixmier-Moeglin Equivalence, and the primitive ideals of A are precisely the prime ideals maximal within their $\mathcal{H}$-strata.*

Proof. By (6.8a), the rational ideals of A are maximal within their $\mathcal{H}$-strata. The corollary thus follows from (2.1iii) and (2.2ii). □

References

1. J. Alev and F. Dumas, *Sur les corps de fractions de certaines algèbres de Weyl quantiques*, J. Algebra **170** (1994), 229-265.

2. M. Artin, W. Schelter, and J. Tate, *Quantum deformations of GL_n*, Communic. Pure Appl. Math. **44** (1991), 879-895.

3. A. Borel, *Linear Algebraic Groups*, Second Enlarged Edition, Springer-Verlag, New York, 1991.

4. K. A. Brown and K. R. Goodearl, *Prime spectra of quantum semisimple groups*, Trans. Amer. Math. Soc. **348** (1996), 2465–2502.

5. G. Cauchon, *Quotients premiers de $O_q(\mathfrak{m}_n(k))$*, J. Algebra **180** (1996), 530–545.

6. C. De Concini, V. Kac, and C. Procesi, *Some remarkable degenerations of quantum groups*, Comm. Math. Phys. **157** (1993), 405–427.

7. C. De Concini and V. Lyubashenko, *Quantum function algebras at roots of* 1, Advances in Math. **108** (1994), 205–262.

8. J. Dixmier, *Idéaux primitifs dans les algèbres enveloppantes*, J. Algebra **48** (1977), 96–112.

9. ______, *Enveloping Algebras*, The 1996 printing of the 1977 English translation, Amer. Math. Soc., Providence, 1996.

10. A. W. Goldie and G. O. Michler, *Ore extensions and polycyclic group rings*, J. London Math. Soc. (2) **9** (1974), 337-345.

11. K. R. Goodearl, *Uniform ranks of prime factors of skew polynomial rings*, in Ring Theory, Proc. Biennial Ohio State – Denison Conf. 1992 (S. K. Jain and S. T. Rizvi, eds.), World Scientific, Singapore, 1993, pp. 182-199.

12. K. R. Goodearl and E. S. Letzter, *Prime ideals in skew and q-skew polynomial rings*, Mem. Amer. Math. Soc. **521** (1994).

13. ______, *Prime and primitive spectra of multiparameter quantum affine spaces*, in Trends in Ring Theory. Proc. Miskolc Conf. 1996 (V. Dlab and L. Márki, eds.), Canad. Math. Soc. Conf. Proc. Series **22** (1998), 39-58.

14. K. R. Goodearl and R. B. Warfield, Jr., *An Introduction to Noncommutative Rings*, Cambridge Univ. Press, Cambridge, 1989.

15. T. J. Hodges and T. Levasseur, *Primitive ideals of $\mathbf{C}_q[SL(3)]$*, Commun. Math. Phys. **156** (1993), 581-605.

16. ______, *Primitive ideals of $\mathbf{C}_q[SL(n)]$*, J. Algebra **168** (1994), 455-468.

17. T. J. Hodges, T. Levasseur, and M. Toro, *Algebraic structure of multi-parameter quantum groups*, Advances in Math. **126** (1997), 52-92.

18. J. E. Humphreys, *Linear Algebraic Groups*, Springer-Verlag, New York, 1975/1981.

19. R. S. Irving, *Noetherian algebras and the Nullstellensatz*, in Séminaire d'Algèbre Paul Dubreil 31ème année (Paris 1977-78) (M.-P. Malliavin, ed.), Lecture Notes in Math. 740, Springer-Verlag, Berlin, 1979, pp. 80–87.

20. R. S. Irving and L. W. Small, *On the characterization of primitive ideals in enveloping algebras*, Math. Z. **173** (1980), 217–221.

21. J. C. Jantzen, *Representations of Algebraic Groups*, Pure and Applied Math. Series 131, Academic Press, Boston-Orlando, 1987.

22. D. A. Jordan, *A simple localization of the quantized Weyl algebra*, J. Algebra **174** (1995), 267-281.

23. A. Joseph, *Idéaux premiers et primitifs de l'algèbre des fonctions sur un groupe quantique*, C. R. Acad. Sci. Paris Sér. I Math. **316** (1993), 1139–1142.

24. ———, *On the prime and primitive spectra of the algebra of functions on a quantum group*, J. Algebra **169** (1994), 441–511.

25. ———, *Quantum Groups and Their Primitive Ideals*, Ergebnisse der Math. (3) 29, Springer-Verlag, Berlin, 1995.

26. G. Maltsiniotis, *Calcul différentiel quantique*, Groupe de travail, Université Paris VII (1992).

27. Yu. I. Manin, *Multiparametric quantum deformation of the general linear supergroup*, Comm. Math. Phys. **123** (1989), 163–175.

28. J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Wiley-Interscience, Chichester-New York, 1987.

29. C. Moeglin, *Idéaux primitifs des algèbres enveloppantes*, J. Math. Pures Appl. **59** (1980), 265–336.

30. C. Moeglin and R. Rentschler, *Orbites d'un groupe algébrique dans l'espace des idéaux rationnels d'une algèbre enveloppante*, Bull. Soc. Math. France **109** (1981), 403–426.

31. ———, *Idéaux G-rationnels, Rang de Goldie*, unpublished manuscript (1986).

32. I. M. Musson, *Ring theoretic properties of the coordinate rings of quantum symplectic and Euclidean space*, in Ring Theory, Proc. Biennial Ohio State–Denison Conf., 1992 (S. K. Jain and S. T. Rizvi, eds.), World Scientific, Singapore, 1993, pp. 248-258.

33. C. Năstăsescu and F. Van Oystaeyen, *Graded Ring Theory*, North-Holland, Amsterdam, 1982.

34. D. G. Northcott, *Affine Sets and Affine Groups*, London Math. Soc. Lecture Note Series 39, Cambridge Univ. Press, Cambridge, 1980.

35. S.-Q. Oh, *Primitive ideals of the coordinate ring of quantum symplectic space*, J. Algebra **174** (1995), 531-552.

36. ———, *Primitive ideals of the coordinate ring of quantum Euclidean space* (to appear).

37. B. Parshall and J.-p. Wang, *Quantum linear groups*, Mem. Amer. Math. Soc. **439** (1991).

38. D. Quillen, *On the endomorphism ring of a simple module over an enveloping algebra*, Proc. Amer. Math. Soc. **21** (1969), 171–172.

39. Z. Reichstein and N. Vonessen, *Torus actions on rings*, J. Algebra **170** (1994), 781-804.

40. R. Rentschler, *Primitive ideals in enveloping algebras (general case)*, in Noetherian Rings and their Applications (Oberwolfach, 1983) (L. W. Small, ed.), Math. Surveys Monographs 24, Amer. Math. Soc., Providence, 1987, pp. 37–57.

41. N. Yu. Reshetikhin, *Multiparameter quantum groups and twisted quasitriangular Hopf algebras*, Lett. Math. Phys. **20** (1990), 331–335.

42. N. Yu. Reshetikhin, L. A. Takhtadzhyan, and L. D. Faddeev, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. **1** (1990), 193-225.

43. L. H. Rowen, *Ring Theory*, Volumes I and II, Academic Press, Boston, 1988.

44. A. Sudbury, *Consistent multiparameter quantisation of $GL(n)$*, J. Phys. A **23** (1990), L697-L704.

45. N. Vonessen, *Actions of algebraic groups on the spectrum of rational ideals*, J. Algebra **182** (1996), 383-400.

46. ———, *Actions of algebraic groups on the spectrum of rational ideals, II*, Preprint, August 1997.

47. S. Yammine, *Les théorèmes de Cohen-Seidenberg en algèbre non commutative*, in Séminaire d'Algèbre Paul Dubreil 31ème année (Paris 1977-78) (M.-P. Malliavin, ed.), Lecture Notes in Math. 740, Springer-Verlag, Berlin, 1979, pp. 120–169.

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