STABLE RANK OF CORNER RINGS

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Abstract. B. Blackadar recently proved that any full corner pAp in a unital C*-algebra A has K-theoretic stable rank greater than or equal to the stable rank of A. (Here p is a projection in A, and fullness means that $ApA = A$.) This result is extended to arbitrary (unital) rings A in the present paper: If p is a full idempotent in A, then $\mathrm{sr}(pAp) \geq \mathrm{sr}(A)$. The proofs rely partly on algebraic analogs of Blackadar's methods, and partly on a new technique for reducing problems of higher stable rank to a concept of stable rank one for skew (rectangular) corners pAq. The main result yields estimates relating stable ranks of Morita equivalent rings. In particular, if $B \cong \mathrm{End}_A(P)$ where P_A is a finitely generated projective generator, and P can be generated by n elements, then $\mathrm{sr}(A) \leq n\cdot \mathrm{sr}(B) - n + 1$.

Introduction

The theory of stable range of rings was developed by H. Bass [**2**] and L. N. Vaserstein [**10**]. As is now common, we define the *stable rank* of a ring A, denoted $\mathrm{sr}(A)$, to be the least positive integer n such that A satisfies Bass's n-th stable range condition, or ∞ if no such n exists. It is well known that stable rank is not Morita invariant. In fact, Vaserstein [**10**] computed the stable rank of a matrix ring $M_n(A)$, obtaining the following amazing formula

$$\mathrm{sr}(M_n(A)) = \left\lceil \frac{\mathrm{sr}(A) - 1}{n} \right\rceil + 1,$$

where $\lceil r \rceil$ denotes the least integer greater than or equal to a real number r. If B is a ring Morita equivalent to A, then $B \cong pM_n(A)p$ for some full idempotent $p \in M_n(A)$. Thus, to understand the behavior of stable rank under Morita equivalence, it remains to see what happens to stable rank under the passage from a ring to a full corner. Vaserstein's formula already contains some information in this direction, namely that $\mathrm{sr}(A) \geq \mathrm{sr}(M_n(A))$ for all $n \in \mathbb{N}$. Since A is isomorphic to a corner ring in $M_n(A)$, corresponding to the full idempotent e_{11}, this suggests the inequality $\mathrm{sr}(pAp) \geq \mathrm{sr}(A)$ for any full corner pAp of

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A. Such a formula was conjectured by Blackadar [**3**, Remark A7] to hold for the *topological stable rank* introduced by Rieffel in [**9**]. It was subsequently proved by Herman and Vaserstein [**5**] that the Rieffel topological stable rank and the Bass stable rank agree for any C*-algebra. Blackadar has recently verified the corner conjecture in [**4**, Theorem 4.5]. His methods are focussed on the topological stable rank, and rely on norm estimates for differences of row vectors.

Previous work on stable rank of corners gave weaker inequalities of the following form. If p is a full projection in a unital C*-algebra A, then, for some n, there exist n pairs $(a_i, b_i) \in A^2$ such that $\sum_{i=1}^n a_i p b_i = 1$. Blackadar showed in [**3**, Lemma A6] that in this situation, $\mathrm{sr}(A) \leq \mathrm{sr}(pAp) + n - 1$. That this result extends to full idempotents in arbitrary rings was noted by the present authors in [**1**, Remark 1.4]. In particular, this inequality suffices to show that finiteness of the stable rank is Morita invariant.

Here we prove that the inequality $\mathrm{sr}(A) \leq \mathrm{sr}(pAp)$ holds for any full corner pAp in any unital ring A (Theorem 7). The structure of the proof has been modelled after Blackadar's paper [**4**], but we have had to replace his topological methods with purely algebraic ones. Of crucial importance is the notion of stable rank one for skew, or rectangular, corners pAq, where p and q are distinct idempotents of A. This allows us to work only with stable rank one conditions, thus avoiding higher rank conditions. By combining our main result with Vaserstein's formula, we obtain estimates comparing the stable ranks of Morita equivalent rings (Theorem 9).

We note that Lam and Dugas [**8**] have recently shown that the reverse inequality $\mathrm{sr}(A) \geq \mathrm{sr}(eAe)$ holds for any quasi-duo ring A and any idempotent e in A. By definition, a *quasi-duo ring* is a ring in which every maximal one-sided ideal is an ideal. It is clear that the only full idempotent in a quasi-duo ring is 1, so our result does not give any further insight into Lam and Dugas's, nor vice versa.

Stable rank and skew corners

Throughout, let A be a unital ring. We start by recalling the definition of the (Bass) stable rank:

Definition. An n-row $(a_1, \dots, a_n) \in A^n$ is said to be *right unimodular* if $\sum_{i=1}^n a_i A = A$. An $(n+1)$-row $(a_1, \dots, a_n, b) \in A^{n+1}$ is *reducible* in case there is an n-row $(c_1, \dots, c_n) \in A^n$ such that the n-row $(a_1 + bc_1, \dots, a_n + bc_n)$ is right unimodular. The *stable rank* of A, denoted $\mathrm{sr}(A)$, is the least positive integer n such that every right unimodular $(n+1)$-row in A^{n+1} is reducible, or ∞ if no such n exists.

We next recall some useful terminology. Two idempotents p and q in A are said to be *orthogonal*, written $p \perp q$, in case $pq = qp = 0$. The set of all idempotents of A is partially ordered by declaring $p \leq q$ if and only if $p = pq = qp$. The idempotents p and q are *equivalent*, written $p \sim q$, in case there are elements $a \in pAq$ and $b \in qAp$ such that $p = ab$ and $q = ba$. (Note that p and q are equivalent if and only if the right (respectively, left) ideals generated by p and q are isomorphic as a right (respectively, left) R-modules [**6**, Proposition 21.20].) We write $p \lesssim q$ in case there is an idempotent p' such that $p' \leq q$ and $p \sim p'$; this occurs if and only if there exist elements $a \in pAq$ and $b \in qAp$ such that $ab = p$. For any idempotents $p, q \in A$, we write $p \oplus q$ for the idempotent $\operatorname{diag}(p, q)$ in $M_2(A)$. Accordingly, the notation $n{\cdot}p$ is used for the idempotent $\operatorname{diag}(p, p, \dots, p)$ in $M_n(A)$.

For all $m, n \in \mathbb{N}$, identify $M_n(A)$ with the $n \times n$ upper left corner subring of $M_{n+m}(A)$. In particular, $A = M_1(A)$ is then identified with a subring of each $M_n(A)$. With this identification, 1_A equals the matrix unit e_{11}, and $n{\cdot}1_A$ equals the identity matrix in $M_n(A)$. These identifications allow us to work in as large a matrix ring as is convenient. When the size of the matrices is not relevant, we write $M_\bullet(A)$ to stand for $M_n(A)$ with n unspecified.

We say that an idempotent p in A is *full* in case p generates A as a two-sided ideal, that is, $ApA = A$. It is standard (and an easy exercise) that p is a full idempotent if and only if $1_A \lesssim t{\cdot}p$ for some $t \in \mathbb{N}$. A *corner* of A is any subring of the form pAp, where p is an idempotent, and we say that pAp is a *full corner* in case p is a full idempotent. A *skew* (*rectangular*) *corner* in A is any subset of the form pAq, for idempotents $p, q \in A$. Note that pAq is a (pAp, qAq)-bimodule, and that the ring multiplication in A induces bimodule homomorphisms $pAq \otimes_{qAq} qAp \to pAp$ and $qAp \otimes_{pAp} pAq \to qAq$.

Definition. Let $p, q \in A$ be idempotents. We say that the skew corner pAq has (right) *stable rank* 1, abbreviated $\operatorname{sr}(pAq) = 1$, provided the following condition holds: Whenever $a \in pAq$, $x \in qAp$, and $b \in pAp$ such that $ax + b = p$, there exist $y \in pAq$ and $z \in qAp$ such that $(a + by)z = p$. Note that in case $p = q$, there is no conflict between this definition and the statement that the stable rank of the ring pAp is 1.

The key to our methods is the following lemma, which reduces stable rank calculations to questions of stable rank 1 for skew corners. Note that $1_A M_n(A)$ is a skew corner, since it equals $1_A M_n(A)(n{\cdot}1_A)$.

Lemma 1. *Let $n \in \mathbb{N}$. Then $\operatorname{sr}(A) \le n$ if and only if $\operatorname{sr}\big(1_A M_n(A)\big) = 1$.*

Proof. ($\Longrightarrow$): Let $\alpha \in 1_A M_n(A)$, $\chi \in M_n(A)1_A$, and $\beta \in 1_A M_n(A) 1_A$ such that $\alpha\chi + \beta = 1_A$. Then

$$(\dagger)\quad \alpha = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad \chi = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ x_n & 0 & \cdots & 0 \end{bmatrix}, \qquad \beta = \begin{bmatrix} b & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

for some $a_i, x_i, b \in A$ such that $a_1x_1 + \cdots + a_nx_n + b = 1$. Since $\operatorname{sr}(A) \le n$, there exist $y_1, \dots, y_n \in A$ such that the row $(a_1 + by_1, \dots, a_n + by_n)$ is right unimodular, that is, $(a_1 + by_1)z_1 + \cdots + (a_n + by_n)z_n = 1$ for some $z_i \in A$. Setting

$$(\ddagger)\quad \zeta = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in 1_A M_n(A), \qquad \xi = \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ z_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ z_n & 0 & \cdots & 0 \end{bmatrix} \in M_n(A)1_A,$$

we have $(\alpha + \beta\zeta)\xi = 1_A$.

($\Longleftarrow$): If $(a_1, \dots, a_n, b) \in A^{n+1}$ is a right unimodular row, there exist $x_1, \dots, x_n, x \in A$ such that $a_1x_1 + \cdots + a_nx_n + bx = 1$. After replacing b by bx, we may assume that $x = 1$. Define matrices $\alpha \in 1_AM_n(A)$, $\chi \in M_n(A)1_A$, and $\beta \in 1_AM_n(A)1_A$ as in (†) above, and observe that $\alpha\chi + \beta = 1_A$. Since $\mathrm{sr}\big(1_AM_n(A)\big) = 1$, there exist matrices ζ and ξ as in (‡) such that $(\alpha + \beta\zeta)\xi = 1_A$. It follows that the row $(a_1 + by_1, \dots, a_n + by_n)$ is right unimodular. $\square$

We next show how stable rank 1 can be transferred from certain skew corners to others.

Lemma 2. *Let $p, q \in A$ be idempotents.*

(a) *If $\mathrm{sr}(pAq) = 1$, then $p \lesssim q$.*

(b) *If $p', q' \in A$ are idempotents such that $p' \sim p$ and $q' \sim q$, then $\mathrm{sr}(pAq) = 1$ if and only if $\mathrm{sr}(p'Aq') = 1$.*

Proof. (a) Consider the equation $0{\cdot}0 + p = p$, where we view the first $0 \in pAq$, the second $0 \in qAp$, and $p \in pAp$. The hypothesis $\mathrm{sr}(pAq) = 1$ then gives us $y \in pAq$ and $z \in qAp$ such that $(0 + py)z = p$. Hence, $pyqzp = p$, and it follows that $p \lesssim q$.

(b) There are elements $u \in pAp'$ and $u' \in p'Ap$ such that $uu' = p$ and $u'u = p'$, and elements $v \in qAq'$ and $v' \in q'Aq$ such that $vv' = q$ and $v'v = q'$.

Assume that $\mathrm{sr}(pAq) = 1$, and consider elements $a \in p'Aq'$, $x \in q'Ap'$, and $b \in p'Ap'$ such that $ax + b = p'$. Then we have $uav' \in pAq$, $vxu' \in qAp$, and $ubu' \in pAp$ satisfying $(uav')(vxu') + (ubu') = p$. Since $\mathrm{sr}(pAq) = 1$, there exist $y \in pAq$ and $z \in qAp$ such that $(uav' + ubu'y)z = p$. Then $u'yv \in p'Aq'$ and $v'zu \in q'Ap'$ are elements satisfying the equation $(a + bu'yv)v'zu = p'$, which proves that $\mathrm{sr}(p'Aq') = 1$. The converse follows by symmetry. $\square$

Lemma 3. *Let $p, q, s \in A$ be idempotents such that $s \perp q$. If $\mathrm{sr}(pAq) = 1$, then also $\mathrm{sr}\big(pA(q + s)\big) = 1$.*

Proof. Let $a \in pA(q + s)$, $x \in (q + s)Ap$, and $b \in pAp$ such that $ax + b = p$. Rewrite this equation as $(aq)(qx) + (b + asx) = p$, where $aq \in pAq$, $qx \in qAp$, and $b + asx \in pAp$. Since $\mathrm{sr}(pAq) = 1$, there exist $y \in pAq$ and $z \in qAp$ such that

$$\big[(aq) + (b + asx)y\big]z = p.$$

Note that $sz = 0$ because $s \perp q$, which allows us to rewrite the equation above as

$$\big[a(q + s + sxy) + by\big]z = p.$$

Since $ys = 0$, the element sxy is nilpotent, and so the element $u := q + s + sxy$ is a unit in the ring $(q + s)A(q + s)$. Now

$$(a + byu^{-1})(uz) = (au + by)z = p$$

with $yu^{-1} \in pA(q + s)$ and $uz \in (q + s)Ap$, which completes the proof. $\square$

Lemma 4. *Let $p, q, r \in A$ be idempotents such that $p, q \perp r$. If* $\mathrm{sr}\big((p+r)A(q+r)\big) = 1$, *then* $\mathrm{sr}(pAq) = 1$.

Proof. Let $a \in pAq$, $x \in qAp$, and $b \in pAp$ such that $ax + b = p$. Then the elements $a+r \in (p+r)A(q+r)$ and $x+r \in (q+r)A(p+r)$ satisfy $(a+r)(x+r)+b = p+r$. Since $\mathrm{sr}\big((p+r)A(q+r)\big) = 1$, there exist $y \in (p+r)A(q+r)$ and $z \in (q+r)A(p+r)$ such that $(a+r+by)z = p+r$. Observe that

$$rz = r(a+r+by)z = r \qquad\qquad (a+by)z = p(a+r+by)z = p.$$

Then $z = (q+r)z = qz + r$, and so $zp = qzp \in qAp$. Since

$$(a+bpyq)(zp) = (a+by)zp = p,$$

we have shown that $\mathrm{sr}(pAq) = 1$. □

The main results

The final ingredient needed to prove our main theorem is a (partial) converse to Lemma 4, which holds when $r \in ApA$. The following observations will be helpful.

Observation 5. If we are trying to establish $\mathrm{sr}(pAq) = 1$ for some idempotents $p, q \in A$, then we are given $ax+b = p$ for some $a \in pAq$, $x \in qAp$, and $b \in pAp$, and we seek $y \in pAq$ and $z \in qAp$ such that $(a+by)z = p$. Several reduction steps are possible, in which we may do any of the following:

(1) Replace a by $a + bc$ for any $c \in pAq$;
(2) Replace a by au for any unit u of qAq;
(3) Replace a by va for any unit v of pAp;
(4) Replace A by $M_n(A)$ for any $n \in \mathbb{N}$.

In cases (1)–(3), the replacement of a by another element of pAq must be accompanied by corresponding replacements for x and b.

To see why (1) is allowed, for instance, observe that

$$(a+bc)x + b(p-cx) = p$$

with $a+bc \in pAq$ and $b(p-cx) \in pAp$; if there exist $y' \in pAq$ and $z' \in qAp$ such that

$$\big[(a+bc) + b(p-cx)y'\big]z' = p,$$

then $\big[a+b(c+y'-cxy')\big]z' = p$, with $c+y'-cxy' \in pAq$. For (2), we have $(au)(u^{-1}x)+b = p$, and if $(au+by')z' = p$, then $(a+by'u^{-1})(uz') = p$. In the case of (3), we have $(va)(xv^{-1}) + (vbv^{-1}) = p$, and if $(va+vbv^{-1}y')z' = p$, then $(a+bv^{-1}y')(z'v) = p$.

Finally, we address (4). Because of our identification of A with the corner $e_{11}M_n(A)e_{11}$, we have $pAq = pM_n(A)q$, and similarly for qAp and pAp. Thus, if there exist $y \in pM_n(A)q$ and $z \in qM_n(A)p$ satisfying $(a+by)z = p$, then $y \in pAq$ and $z \in qAp$, and the equation holds in A.

Proposition 6. *Let $p, q, r \in A$ be idempotents such that $p, q \perp r$. If* $\mathrm{sr}(pAq) = 1$ *and* $r \lesssim n{\cdot}p$ *for some* $n \in \mathbb{N}$, *then* $\mathrm{sr}\big((p+r)A(q+r)\big) = 1$.

Proof. By Lemma 2, $p \sim p'$ for some idempotent $p' \leq q$, and we may replace p by p'. Thus, we may assume that $p \leq q$.

We claim that it suffices to establish the proposition under the additional hypothesis $r \sim p$. Given that case, it follows by induction that $\mathrm{sr}\big((2^m{\cdot}p)M_\bullet(A)(q \oplus (2^m-1){\cdot}p)\big) = 1$ for all $m \in \mathbb{N}$. In the general case, we choose m large enough that $2^m - 1 \geq n$, so that $(2^m-1){\cdot}p = r' \oplus r''$ for some orthogonal idempotents r' and r'' with $r' \sim r$. In a lower right corner of a suitably large matrix ring, we can find an idempotent s such that $s \perp 1_A$ and $s \sim r''$. Thus, $p, q, r \perp s$ and $(2^m-1){\cdot}p \sim r+s$. Consequently, $\mathrm{sr}\big((p{+}r{+}s)M_\bullet(A)(q{+}r{+}s)\big) = 1$, and the desired result follows from Lemma 4. Therefore the claim holds, and we may assume that $p \sim r$. In particular, $\mathrm{sr}(rAq) = 1$.

To prove that $\mathrm{sr}\big((p+r)A(q+r)\big) = 1$, we may work within the ring $(q+r)A(q+r)$. Hence, we may assume, for convenience, that $q{+}r = 1$. Now elements $a \in A$ can be viewed as formal matrices of the form
$$\begin{bmatrix} qaq & qar \\ raq & rar \end{bmatrix}.$$
We mimic the proof that stable rank 1 passes from a ring to its 2×2 matrix ring, which would be exactly our present situation in case $p = q$. In order to allow for the possibility that $p < q$, we must be careful to modify our matrices starting at the lower right corner (where the entries come from rAr), rather than starting at the upper left corner, where we can control pAq but not qAq.

Let $a \in (p+r)A$, $x \in A(p+r)$, and $b \in (p+r)A(p+r)$ such that $ax+b = p+r$. Note that $qaq = paq$ and $qar = par$. Now
$$(raq)(qxr) + (rarxr + rbr) = r(ax+b)r = r.$$
Since $\mathrm{sr}(rAq) = 1$, there exist $y_1 \in rAq$ and $z_1 \in qAr$ such that
$$\big[(raq) + (rarxr + rbr)y_1\big]z_1 = r.$$
The factor q in raq can be dropped from the last equation because $qz_1 = z_1$. Hence,
$$r\big[a(1+rxy_1) + by_1\big]z_1 = r.$$
Since $y_1 r = 0$, the element $1 + rxy_1$ is a unit in A. In view of Observation 5, we may replace a by $a(1{+}rxy_1) + by_1$. Thus, we may now assume that there exists $z_1 \in qAr$ such that $raz_1 = r$.

Next, we replace a by the element
$$\begin{bmatrix} paq & par \\ raq & rar \end{bmatrix}\begin{bmatrix} q & z_1(r-rar) \\ 0 & r \end{bmatrix} = \begin{bmatrix} paq & * \\ raq & r \end{bmatrix},$$

which is allowed since $\begin{bmatrix} q & z_1(r - rar) \\ 0 & r \end{bmatrix}$ is a unit in A. At this stage, we have $rar = r$. After replacing a by

$$\begin{bmatrix} p & -par \\ 0 & r \end{bmatrix} \begin{bmatrix} paq & par \\ raq & r \end{bmatrix} = \begin{bmatrix} * & 0 \\ raq & r \end{bmatrix}$$

(note that $\begin{bmatrix} p & -par \\ 0 & r \end{bmatrix}$ is a unit in $(p + r)A(p + r)$), we may assume in addition that $par = 0$.

Now return to the equation $ax + b = p + r$, and observe that

$$(paq)(qxp) + (pbp) = p(ax + b)p = p,$$

because $pa = paq$. Since $\mathrm{sr}(pAq) = 1$, there exist $y_2 \in pAq$ and $z_2 \in qAp$ such that $(paq + pbpy_2)z_2 = p$. Consequently, $p(a + by_2)z_2 = p$. Note that $(a + by_2)r = ar$, and so neither of the conditions $rar = r$ and $par = 0$ is lost on replacing a by $a + by_2$. Thus, we may assume that there exists $z_2 \in qAp$ with $paz_2 = p$, whence

$$\begin{bmatrix} paq & 0 \\ raq & r \end{bmatrix} \begin{bmatrix} z_2 & 0 \\ -raz_2 & r \end{bmatrix} = \begin{bmatrix} p & 0 \\ 0 & r \end{bmatrix}.$$

In other words, we have found an element $z = \begin{bmatrix} z_2 & 0 \\ -raz_2 & r \end{bmatrix}$ in $A(p+r)$ such that $az = p+r$, and therefore we have proved that $(p + r)A$ has stable rank 1. $\square$

Theorem 7. *If p is a full idempotent in A, then* $\mathrm{sr}(A) \leq \mathrm{sr}(pAp)$.

Proof. Assume that $\mathrm{sr}(pAp) = n < \infty$. By Lemma 1, $pM_n(pAp) = pM_n(A)(n{\cdot}p)$ has stable rank 1. Since p is full, $1_A \lesssim t{\cdot}p$ for some $t \in \mathbb{N}$. Working in a suitably large matrix ring $R = M_\bullet(A)$, we have $\mathrm{sr}(pR(n{\cdot}p)) = 1$ and we have room for an idempotent r which is equivalent to $(t - 1){\cdot}p$ and orthogonal to both p and $n{\cdot}p$. Proposition 6 now implies that $\mathrm{sr}\big((p+r)R(n{\cdot}p+r)\big) = 1$, that is, $\mathrm{sr}\big((t{\cdot}p)R((n+t-1){\cdot}p)\big) = 1$. We also have $t{\cdot}p = e + f$ and $(n + t - 1){\cdot}p = e + f + g$ for some orthogonal idempotents e, f, g with $e \sim 1_A$ and $g \sim (n - 1){\cdot}p$. Since $\mathrm{sr}\big((e + f)R(e + f + g)\big) = 1$, we can use Lemma 4 to see that $\mathrm{sr}\big(eR(e + g)\big) = 1$. Finally, using Lemma 3 to increase $e + g$ by an orthogonal idempotent equivalent to $(n - 1){\cdot}(1_A - p)$, we conclude that $\mathrm{sr}\big(eR(e \oplus (n - 1){\cdot}1_A)\big) = 1$. Since $e \sim 1_A$, we thus have $\mathrm{sr}\big(1_A R(n{\cdot}1_A)\big) = 1$, and so $\mathrm{sr}\big(1_A M_n(A)\big) = 1$. Therefore $\mathrm{sr}(A) \leq n$, by Lemma 1. $\square$

An upper bound for $\mathrm{sr}(pAp)$ in terms of $\mathrm{sr}(A)$ can be obtained from Theorem 7 and Vaserstein's formula, as follows.

Theorem 8. *If p is a full idempotent in A and $\sum_{i=1}^n a_i p b_i = 1$ for some $a_i, b_i \in A$, then* $\mathrm{sr}(pAp) \leq n{\cdot}\,\mathrm{sr}(A) - n + 1$.

Proof. We may clearly assume that each $a_i \in Ap$ and each $b_i \in pA$. Set

$$\alpha = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in 1_A M_n(A)(n{\cdot}p)$$

$$\beta = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ b_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ b_n & 0 & \cdots & 0 \end{bmatrix} \in (n{\cdot}p) M_n(A) 1_A,$$

and observe that $\alpha\beta = 1_A$. Then the matrix $q := \beta\alpha$ is an idempotent in the ring $(n{\cdot}p)M_n(A)(n{\cdot}p)$, which we identify with $M_n(pAp)$. Since $(n{\cdot}p)\alpha q \beta (n{\cdot}p) = p$, we see that q is full in $M_n(pAp)$. Moreover, $1_A \sim q$ in $M_n(A)$, and so

$$A \cong 1_A M_n(A) 1_A \cong q M_n(A) q = q M_n(pAp) q.$$

Hence, $\mathrm{sr}(M_n(pAp)) \leq \mathrm{sr}(A)$ by Theorem 7. According to Vaserstein's formula [**10**, Theorem 3],

$$\mathrm{sr}(M_n(pAp)) \geq \frac{\mathrm{sr}(pAp) - 1}{n} + 1,$$

and the theorem follows. $\square$

To conclude, we derive the following estimates for the stable ranks of Morita equivalent rings.

Theorem 9. *Let A and B be Morita equivalent rings; then $B \cong \mathrm{End}_A(P)$ for some finitely generated projective generator P_A. If P can be generated by n elements as a right A-module, then*

$$\mathrm{sr}(A) \leq n{\cdot}\,\mathrm{sr}(B) - n + 1.$$

If there are t homomorphisms $f_i \in \mathrm{Hom}_A(P, A)$ such that $\sum_{i=1}^t f_i(P) = A$, then

$$\mathrm{sr}(B) \leq t{\cdot}\,\mathrm{sr}(A) - t + 1.$$

Proof. There exists a split epimorphism $A^n \twoheadrightarrow P$, so that $P \cong pA^n$ for some idempotent $p \in M_n(A)$, and $B \cong pM_n(A)p$. Since P is a generator, p is full. Thus, by Vaserstein's formula and Theorem 7,

$$\frac{\mathrm{sr}(A) - 1}{n} + 1 \leq \mathrm{sr}(M_n(A)) \leq \mathrm{sr}(B).$$

We now identify B with $\mathrm{End}_A(P)$, and view P as a left B-module. Then ${}_BP$ is a finitely generated projective generator, and $\mathrm{End}_B(P) \cong A$ (e.g., [**7**, Propositions 18.17, 18.22]). There exist elements $x_i \in P$ such that $\sum_{i=1}^t f_i(x_i) = 1$ in A. Given any element $x \in P$, there are endomorphisms $xf_i \in B$ (sending any $y \mapsto xf_i(y)$) such that $\sum_{i=1}^t (xf_i)x_i = x\sum_{i=1}^t f_i(x_i) = x$. This shows that P is generated as a left B-module by $x_1, \dots, x_t$. Therefore the inequality $\mathrm{sr}(B) \leq t{\cdot}\,\mathrm{sr}(A) - t + 1$ follows from the first part of the theorem, on replacing A and B by B^{op} and A^{op}, respectively. $\square$

References

1. P. Ara and K. R. Goodearl, *The almost isomorphism relation for simple regular rings*, Publ. Mat. (Barcelona) **36** (1992), 369–388.

2. H. Bass, *Algebraic K-theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

3. B. Blackadar, *A stable cancellation theorem for simple C*-algebras*, Proc. London Math. Soc. **47** (1983), 303–305.

4. ______, *The stable rank of full corners in C*-algebras*, Proc. Amer. Math. Soc. (to appear).

5. R. H. Herman, L. N. Vaserstein, *The stable range of C*-algebras*, Invent. Math. **77** (1984), 553–555.

6. T. Y. Lam, *A First Course in Noncommutative Rings*, Grad. Texts in Math. 131, Springer-Verlag, New York-Berlin, 1991.

7. ______, *Lectures on Modules and Rings*, Grad. Texts in Math. 189, Springer-Verlag, New York-Berlin, 1999.

8. T. Y. Lam and A. Dugas, *Quasi-duo rings and stable range descent* (to appear).

9. M. A. Rieffel, *Dimension and stable rank in the K-theory of C*-algebras*, Proc. London Math. Soc. **46** (1983), 301–333.

10. L. N. Vaserstein, *The stable range of rings and the dimension of topological spaces*, Funkcional Anal. i Prilozen. **5(2)** (1971), 17–27.

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