## 14 Subsequences

### Subsequences

**Definition 14.1.** Let $(s_n)_{n=1}^{\infty}$ be a sequence, and let $(n_k)_{k=1}^{\infty}$ be a strictly increasing sequence of natural numbers. Then $(s_{n_k})_{k=1}^{\infty}$ is called a subsequence of $(s_n)$.

For convergent sequences, the subsequences must have the same asymptotic behavior as the entire sequence.

**Theorem 14.2.** *If $(s_n)$ converges to $s$, then every subsequence of $(s_n)$ converges to the same limit $s$.*

However, even if a sequence does not converge, it may have convergent subsequences.

**Theorem 14.3.** *Every bounded sequence has a convergent subsequences.*

On the other hand, if a sequence is not bounded, it must have subsequences diverging to positive or negative infinities.

**Theorem 14.4.** *Let $(s_n)$ be a sequence of real numbers.*
*(a) If $(s_n)$ is unbounded above, then it has an increasing subsequence diverging to $+\infty$.*
*(b) If $(s_n)$ is unbounded below, then it has a decreasing subsequence diverging to $-\infty$.*

We next consider all the limits that subsequences of a given sequence can converge to.

**Definition 14.5.** Let $(s_n)$ be a bounded sequence. A **partial limit** (or *subsequential limit*) of $(s_n)$ is any real number that is the limit of some subsequence of $(s_n)$. Let $S$ be the set of all partial limits of $(s_n)$, then the **limit superior** (or *upper limit*) of $(s_n)$ is defined to be

$$\limsup s_n = \sup S.$$

Similarly, the **limit inferior** (or *lower limit*) of $S$ is defined to be

$$\liminf s_n = \inf S.$$

It is an interesting question whether the $\limsup s_n$ and $\liminf s_n$ must themselves be partial limits of $(s_n)$, which is answered by the following.

**Theorem 14.6.** *Let $(s_n)$ be a bounded sequence and $M = \limsup s_n, m = \liminf s_n$. Then $M$ and $m$ are themselves partial limits of $(s_n)$.*