5 Cardinality

Definition 5.1. Two sets S and T are called equinumerous, denoted by $S \sim T$, if there exists a bijection from S onto T.

Equinumerosity defines an equivalence relation on families of sets. The equivalence classes separate sets into classes of equal size, and we associate cardinal numbers to each class for the size of the sets in that particular class.

Definition 5.2. A set S is said to be finite if there exists $n \in \mathbb{N}$ and a bijection $f : I_n = \{1, 2, \ldots, n\} \to S$. If a set is not finite, it is called infinite

A set S is called denumerable, if there exists a bijection $f : \mathbb{N} \to S$. If a set is finite or denumerable, it is called countable.

Countable sets are those whose elements can be indexed by the natural numbers, i.e. $S = \{s_1, s_2, \ldots\}$. The indexing is given by the bijection $f : I_n \to S$ or $f : \mathbb{N} \to S$, in which case $s_1 = f(1)$, $s_2 = f(2), \ldots, s_n = f(n), \ldots$. One can show that any subset of a countable set is also countable.

The cardinal number of $\varnothing$ is taken to be zero, the cardinal number of $I_n = \{1, 2, \ldots, n\}$ is n, and the cardinal number of $\mathbb{N}$ is denoted by $\aleph_0$ (aleph-null). In general, if a cardinal number is not finite, it is called transfinite.

Theorem 5.3. *Let S be a nonempty set, then the following three statements are equivalent:*
(a) S is countable
(b) There exists an injection $f : S \to \mathbb{N}$
(c) There exists a surjection $g : \mathbb{N} \to S$.

Using the last theorem one can show that if S and T are countable, then so is $S \cup T$ and $S \times T$. as well as $\bigcup_{n=1}^{\infty} S_n$ for a family of countable sets $\{S_n\}$.

The sets $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$ are all countable. However, the set $\mathbb{R}$ is uncountable, and we denote its cardinality by c.

Definition 5.4. The cardinal number of a set S is denoted by $|S|$. Two sets have the same cardinality, $|S| = |T|$, iff S and T are equinumerous, i.e. there exists a bijection $f : S \to T$. We say that $|S| \leq |T|$, if there exists an injection $f : S \to T$. And we say $|S| < |T|$, If $|S| \leq |T|$ and $|S| \neq |T|$.

Cardinal numbers satisfy the following ordering properties (S, T, U are sets):

$$(a)\ S \subseteq T \Rightarrow |S| \leq |T|, \quad \text{in particular },|S| \leq |S|$$
$$(b)\ |S| \leq |T| \text{ and } |T| \leq |U| \Rightarrow |S| \leq |U|$$
$$(c) \text{ if } S \text{ is finite, then } |S| < \aleph_0$$
$$(d)\ \aleph_0 < c$$

Definition 5.5. Let S be a set. The collection of all of its subsets is called the power set of S and is denoted by $\mathscr{P}(S)$.

Theorem 5.6. *For any set S, $|S| < |\mathscr{P}(S)|$.*

By the last theorem there exists an increasing sequence of cardinal numbers

$$\aleph_0 = |\mathbb{N}| < |\mathscr{P}(\mathbb{N})| < |\mathscr{P}(\mathscr{P}(\mathbb{N}))| < \ldots$$

One can also show that $|\mathscr{P}(\mathbb{N})| = |\mathbb{R}| = c$. Sometimes the notation $|\mathscr{P}(S)| = 2^{|S|}$ is used as an extension of the identity for finite sets. Using this notation, we would have $c = 2^{\aleph_0}$.