9 Topology of the reals

Definition 9.1. Let $x \in \mathbb{R}, \epsilon > 0$. A neighborhood of x of size ϵ is the set $N(x, \epsilon) = \{y \in \mathbb{R} : |x-y| < \epsilon\}$. A deleted neighborhood of x of size ϵ is the set $N^*(x, \epsilon) = \{y \in \mathbb{R} : 0 < |x-y| < \epsilon\}$.

Clearly $N^*(x, \epsilon) = N(x, \epsilon) \setminus \{x\}$. In our case of the set of real numbers $\mathbb{R}$, $N(x, \epsilon) = (x - \epsilon, x + \epsilon)$, however the above definition of a neighborhood is more general and works for any metric spaces. Using the notion of a neighborhood, points in $\mathbb{R}$ can be classified as interior, boundary or exterior to any particular subset $S \subseteq \mathbb{R}$.

Definition 9.2. Let $S \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called **interior** to S, if there is a neighborhood of x that entirely lies in S, i.e. $x \in N \subseteq S$ $(\exists \epsilon > 0, N(x, \epsilon) \subseteq S)$. If for every neighborhood of x $N \cap S \neq \varnothing$ and $N \cap (\mathbb{R} \setminus S) \neq \varnothing$, then x is called a **boundary** point for S, and x is called **exterior** to S, if there exists a neighborhood of x, such that $N \cap S = \varnothing$. The set of interior points is denoted by int S, and the set of boundary points is denoted by bd S.

Definition 9.3. Let $S \subseteq \mathbb{R}$.
(a) If bd $S \subseteq S$, then S is called a **closed set**
(b) If bd $S \subseteq (\mathbb{R} \setminus S)$, then S is called an **open set**.

An alternative characterization of open and closed sets is given by the following.

Theorem 9.4. *Let $S \subseteq \mathbb{R}$.*
(a) S is open iff $S =$ int S
(b) S is closed iff $\mathbb{R} \setminus S$ is open.

Combining open sets gives an open set, and intersecting closed sets results in a closed set.

Theorem 9.5. *(a) The union of any collection of open sets is open*
(b) The intersection of any finite collection of open sets is open.

And as a direct corollary of the last theorem along with the preceding statement, gives the following.

Theorem 9.6. *(a) The intersection of any collection of closed sets is closed*
(b) The union of any finite collection of closed sets is closed.

Definition 9.7. A point $x \in \mathbb{R}$ is called an **accumulation point** for $S \subseteq \mathbb{R}$, if every deleted neighborhood of x contains a point of S $(\forall \epsilon > 0, N^*(x, \epsilon) \cap S \neq \varnothing)$. The set of all accumulation points for S is denoted by S'. If a point of S is not in S', then it is called an **isolated point** of S.

The union of S with the set of its accumulation points is called the closure of S, $clS = S \cup S'$.
Using the closure of a set, one has an alternative characterization for closed sets (and hence, also for open sets by taking compliments).

Theorem 9.8. *Let $S \subseteq \mathbb{R}$. Then*
(a) S is closed iff $S' \subseteq S$ (S contains all of its accumulation points)
(b) cl S is closed
(c) S is closed iff $S =$ cl S
(d) cl $S = S \cup$ bd S.