1 Introduction

Recall that an ordinary differential equation (ODE) contains an independent variable x and a dependent variable u, which is the unknown in the equation. The defining property of an ODE is that derivatives of the unknown function $u' = \frac{du}{dx}$ enter the equation. Thus, an equation that relates the independent variable x, the dependent variable u and derivatives of u is called an *ordinary differential equation*. Some examples of ODEs are:

$$u'(x) = u$$
$$u'' + 2xu = e^x$$
$$u'' + x(u')^2 + \sin u = \ln x$$

In general, and ODE can be written as $F(x, u, u', u'', \dots) = 0$.

In contrast to ODEs, a partial differential equation (PDE) contains partial derivatives of the dependent variable, which is an unknown function in more than one variable $x, y, \dots$. Denoting the partial derivative of $\frac{\partial u}{\partial x} = u_x$, and $\frac{\partial u}{\partial y} = u_y$, we can write the general first order PDE for $u(x, y)$ as

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = 0. \tag{1}$$

Although one can study PDEs with as many independent variables as one wishes, we will be primarily concerned with PDEs in two independent variables. A *solution* to the PDE (1) is a function $u(x, y)$ which satisfies (1) for all values of the variables x and y. Some examples of PDEs (of physical significance) are:

$$u_x + u_y = 0 \quad \text{transport equation} \tag{2}$$
$$u_t + uu_x = 0 \quad \text{inviscid Burger's equation} \tag{3}$$
$$u_{xx} + u_{yy} = 0 \quad \text{Laplace's equation} \tag{4}$$
$$u_{tt} - u_{xx} = 0 \quad \text{wave equation} \tag{5}$$
$$u_t - u_{xx} = 0 \quad \text{heat equation} \tag{6}$$
$$u_t + uu_x + u_{xxx} = 0 \quad \text{KdV equation} \tag{7}$$
$$iu_t - u_{xx} = 0 \quad \text{Shrödinger's equation} \tag{8}$$

It is generally nontrivial to find the solution of a PDE, but once the solution is found, it is easy to verify whether the function is indeed a solution. For example to see that $u(t, x) = e^{t-x}$ solves the wave equation (5), simply substitute this function into the equation:

$$(e^{t-x})_{tt} - (e^{t-x})_{xx} = e^{t-x} - e^{t-x} = 0.$$

1.1 Classification of PDEs

There are a number of properties by which PDEs can be separated into families of similar equations. The two main properties are *order* and *linearity*.

Order. The order of a partial differential equation is the order of the highest derivative entering the equation. In examples above (2), (3) are of first order; (4), (5), (6) and (8) are of second order; (7) is of third order.

Linearity. Linearity means that all instances of the unknown and its derivatives enter the equation linearly. To define this property, rewrite the equation as

$$\mathcal{L}u = 0, \tag{9}$$

where $\mathcal{L}$ is an operator, which assigns u a new function $\mathcal{L}u$. For example $\mathcal{L} = \frac{\partial^2}{\partial x^2} + 1$, then $\mathcal{L}u = u_{xx} + u$. The operator $\mathcal{L}$ is called linear if

$$\mathcal{L}(u + v) = \mathcal{L}u + \mathcal{L}v, \quad \text{and} \quad \mathcal{L}(cu) = c\mathcal{L}u \tag{10}$$

for any functions u, v and constant c. The equation (9) is called linear, if $\mathcal{L}$ is a linear operator. In our examples above (2), (4), (5), (6), (8) are linear, while (3) and (7) are nonlinear (*i.e.* not linear). To see this, let us check, e.g. (6) for linearity:

$$\mathcal{L}(u+v) = (u+v)_t - (u+v)_{xx} = u_t + v_t - u_{xx} - v_{xx} = (u_t - u_{xx}) + (v_t - v_{xx}) = \mathcal{L}u + \mathcal{L}v,$$

and

$$\mathcal{L}(cu) = (cu)_t - (cu)_{xx} = cu_t - cu_{xx} = c(u_t - u_{xx}) = c\mathcal{L}u.$$

So, indeed, (6) is a linear equation, since it is given by a linear operator. To understand how linearity can fail, let us see what goes wrong for equation (3):

$$\mathcal{L}(u+v) = (u+v)_t + (u+v)(u+v)_x = u_t + v_t + (u+v)(u_x + v_x) = (u_t + uu_x) + (v_t + vv_x) + uv_x + vu_x \neq \mathcal{L}u + \mathcal{L}v.$$

You can check that the second condition of linearity fails as well. This happens precisely due to the nonlinearity of the uu_x term, which is quadratic in "u and its derivatives".

Notice that for a linear equation, if u is a solution, then so is cu, and if v is another solution, then $u+v$ is also a solution. In general any linear combination of solutions

$$c_1u_1(x,y) + c_2u_2(x,y) + \cdots + c_nu_n(x,y) = \sum_{i=1}^{n} c_iu_i(x,y)$$

will also solve the equation.

The linear equation (9) is called *homogeneous* linear PDE, while the equation

$$\mathcal{L}u = g(x,y) \tag{11}$$

is called *inhomogeneous* linear equation. Notice that if u^h is a solution to the homogeneous equation (9), and u^p is a particular solution to the inhomogeneous equation (11), then $u^h + u^p$ is also a solution to the inhomogeneous equation (11). Indeed

$$\mathcal{L}(u^h + u^p) = \mathcal{L}u^h + \mathcal{L}u^p = 0 + g = g.$$

Thus, in order to find the general solution of the inhomogeneous equation (11), it is enough to find the general solution of the homogeneous equation (9), and add to this a particular solution of the inhomogeneous equation (check that the difference of any two solutions of the inhomogeneous equation is a solution of the homogeneous equation). In this sense, there is a similarity between ODEs and PDEs, since this principle relies only on the linearity of the operator $\mathcal{L}$.

1.2 Examples

Example 1.1. $u_x = 0$

Remember that we are looking for a function $u(x,y)$, and the equation says that the partial derivative of u with respect to x is 0, so u does not depend on x. Hence $u(x,y) = f(y)$, where $f(y)$ is an arbitrary function of y. Alternatively, we could simply integrate both sides of the equation with respect to x. More on this in the following examples. □

Example 1.2. $u_{xx} + u = 0$

Similar to the previous example, we see that only the partial derivative with respect to one of the variables enters the equation. In such cases we can treat the equation as an ODE in the variable in which partial derivatives enter the equation, keeping in mind that the constants of integration may depend on the other variables. Rewrite the equation as

$$u_{xx} = -u,$$

which, as an ODE, has the general solution

$$u = c_1 \cos x + c_2 \sin x.$$

Since the constants may depend on the other variable y, the general solution of the PDE will be

$$u(x, y) = f(y) \cos x + g(y) \sin x,$$

where f and g are arbitrary functions. To check that this is indeed a solution, simply substitute the expression back into the equation. $\square$

Example 1.3. $u_{xy} = 0$

We can think of this equation as an ODE for u_x in the y variable, since $(u_x)_y = 0$. Then similar to the first example, we can integrate in y to obtain

$$u_x = f(x).$$

This is an ODE for u in the x variable, which one can solve by integrating with respect to x, arriving at at the solution

$$u(x, y) = F(x) + G(y).$$

$\square$

1.3 Conclusion

Notice that where the solution of an ODE contains arbitrary constants, the solution to a PDE contains arbitrary functions. In the same spirit, while an ODE of order m has m linearly independent solutions, a PDE has infinitely many (there are arbitrary functions in the solution!). These are consequences of the fact that a function of two variables contains immensely more (a whole dimension worth) of information than a function of only one variable.