11 Comparison of wave and heat equations

In the last several lectures we solved the initial value problems associated with the wave and heat equations on the whole line $x \in \mathbb{R}$. We would like to summarize the properties of the obtained solutions, and compare the propagation of waves to conduction of heat.

Recall that the solution to the wave IVP on the whole line

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \\ u(x,0) = \phi(x), \\ u_t(x,0) = \psi(x), \end{cases} \tag{1}$$

is given by d'Alambert's formula

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds. \tag{2}$$

Most of the properties of this solution can be deduced from the solution formula, which can be understood fairly well, if one thinks in terms of the characteristic coordinates. This is how we arrived at the properties of finite speed of propagation, propagation of discontinuities of the data along the characteristics, and others.

On the other hand, the solution to the heat IVP on the whole line

$$\begin{cases} u_t - k u_{xx} = 0, \\ u(x,0) = \phi(x), \end{cases} \tag{3}$$

is given by the formula

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, dy. \tag{4}$$

We saw some of the properties of the solutions to the heat IVP, for example the smoothing property, in the case of the fundamental solution or the heat kernel

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}, \tag{5}$$

which had the Dirac delta function as its initial data. The solution u given by (4) can be written in terms of the heat kernel, and we use this to prove the properties for solutions to the general IVP (3). In terms of the heat kernel the solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\, dy = \int_{-\infty}^{\infty} S(z,t)\phi(x-z)\, dz,$$

where we made the change of variables $z = x - y$ to arrive at the last integral. Making a further change of variables $p = z/\sqrt{kt}$, the above can be written as

$$u(x,t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} \phi(x - p\sqrt{kt})\, dp. \tag{6}$$

This last form of the solution will be handy when proving the smoothing property of the heat equation, the precise statement of which is contained in the following.

Theorem 11.1. *Let $\phi(x)$ be a bounded continuous function for $-\infty < x < \infty$. Then (4) defines an infinitely differentiable function $u(x,t)$ for all $x \in \mathbb{R}$ and $t > 0$, which satisfies the heat equation, and* $\lim_{t\to 0+} u(x,t) = \phi(x)$, $\forall x \in \mathbb{R}$.

The proof is rather straightforward, and amounts to pushing the derivatives of $u(x,t)$ onto the heat kernel inside the integral. All one needs to guarantee for this procedure to go through is the uniform convergence of the resulting improper integrals. Let us first take a look at the solution itself given by (4). Notice that using the form in (6), we have

$$|u(x,t)| \leq \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \left| e^{-p^2/4} \phi(x - p\sqrt{kt}) \right| dp \leq \frac{1}{\sqrt{4\pi}} (\max|\phi|) \int_{-\infty}^{\infty} e^{-p^2/4}\, dp = \max|\phi|,$$

which shows that u, given by the improper integral, is well-defined, since ϕ is bounded. One can also see the maximum principle in the above inequality. We will use similar logic to show that the improper integrals appearing in the derivatives of u converge uniformly in x and t.

Notice that formally

$$\frac{\partial u}{\partial x} = \int_{-\infty}^{\infty} \frac{\partial S}{\partial x}(x-y,t)\phi(y)\, dy. \tag{7}$$

To make this rigorous, one must prove the uniform convergence of the integral. For this, we use expression (5) for the heat kernel to write

$$\int_{-\infty}^{\infty} \frac{\partial S}{\partial x}(x-y,t)\phi(y)\, dy = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left[e^{-(x-y)^2/4kt}\right]\phi(y)\, dy = -\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \frac{x-y}{2kt} e^{-(x-y)^2/4kt}\phi(y)\, dy.$$

Making the change of variables $p = (x-y)/\sqrt{kt}$ in the above integral, we get

$$\int_{-\infty}^{\infty} \frac{\partial S}{\partial x}(x-y,t)\phi(y)\, dy = \frac{1}{4\sqrt{\pi kt}} \int_{-\infty}^{\infty} pe^{-p^2/4}\phi(x - p\sqrt{kt})\, dp \leq \frac{c}{\sqrt{t}}(\max|\phi|) \int_{-\infty}^{\infty} |p|e^{-p^2/4}\, dp,$$

where $c = 1/(4\sqrt{\pi k})$ is a constant. The last integral is finite, so the integral in the formal derivative (7) converges uniformly and absolutely for all $x \in \mathbb{R}$ and $t > \epsilon > 0$, where ϵ can be taken arbitrarily small. So the derivative $u_x = \partial u/\partial x$ exists and is given by (7).

The above argument works for the t derivative, and all the higher order derivatives as well, since for the n^{th} order derivatives one will end up with the integral $\int_{-\infty}^{\infty} |p|^n e^{-p^2/4}\, dp$, which is finite for all $n \in \mathbb{N}$. This proves the infinite differentiability of the solution, even though the initial data is only continuous.

We have already seen that u given by (4) solves the heat equation, due to the invariance properties. It then only remains to prove that $\lim_{t\to 0+} u(x,t) = \phi(x)$, $\forall x$. Recall that our previous proofs of this used the derivative of $\phi(x)$, or the language of distributions to employ the Dirac δ, where we assumed that ϕ is a test function, i.e. infinitely differentiable with compact support. To prove that $u(x,t)$ satisfies the initial condition in (3) in the case of continuous initial data ϕ as well, one can either use a *density argument*, in which $\phi(x)$ is uniformly approximated by smooth functions, and make a use of our earlier proofs, or provide a direct proof. The basic idea behind the direct proof is given next. We need to show that the difference $u(x,t) - \phi(x)$ becomes arbitrarily small when $t \to 0+$. First notice that

$$u(x,t) - \phi(x) = \int_{-\infty}^{\infty} S(x-y,t)[\phi(y) - \phi(x)]\, dy = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4}[\phi(x - p\sqrt{kt}) - \phi(x)]\, dp, \tag{8}$$

where we used the same change of variables as before, $p = (x-y)/\sqrt{kt}$. To see that the last integral becomes arbitrarily small as t goes to zero, notice that if $p\sqrt{kt}$ is small, then $|\phi(x - p\sqrt{kt}) - \phi(x)|$ is small due to the continuity of ϕ, and the rest of the integral is finite. Otherwise, when $p\sqrt{kt}$ is large, then p is large, and the exponential in the integral becomes arbitrarily small, while the ϕ term is bounded. Thus, one estimates the above integral by breaking it into the following two integrals

$$\begin{aligned}&\frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4}[\phi(x - p\sqrt{kt}) - \phi(x)]\, dp \\ &\quad = \frac{1}{\sqrt{4\pi}} \int_{|p|<\delta/\sqrt{kt}} e^{-p^2/4}[\phi(x - p\sqrt{kt}) - \phi(x)]\, dp + \frac{1}{\sqrt{4\pi}} \int_{|p|\geq\delta/\sqrt{kt}} e^{-p^2/4}[\phi(x - p\sqrt{kt}) - \phi(x)]\, dp.\end{aligned}$$

For some small δ, the first integral is small due to the continuity of ϕ, while for arbitrarily small t the second integral is the tail of a converging improper integral, and is hence small. You should try to fill in the rigorous details. This completes the proof of Theorem 11.1.

It turns out, that the result in the above theorem can be proved even if the assumption of continuity of ϕ is relaxed to piecewise continuity. One then has the following.

Theorem 11.2. *Let $\phi(x)$ be a bounded piecewise-continuous function for $-\infty < x < \infty$. Then* (4) *defines an infinitely differentiable function $u(x,t)$ for all $x \in \mathbb{R}$ and $t > 0$, which satisfies the heat equation, and*

$$\lim_{t\to 0+} u(x,t) = \frac{1}{2}[\phi(x+) + \phi(x-)], \qquad \text{for all } x \in \mathbb{R}, \tag{9}$$

where $\phi(x+)$ and $\phi(x-)$ stand for the right hand side and left hand side limits of ϕ at x.

Of course the fact that ϕ has jump discontinuities will not effect the convergence of the improper integrals encountered in the proof of Theorem 11.1. To see why (9) holds, notice that the integral in the right hand side of (6) can be broken into integrals over positive and negative half-lines

$$u(x,t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{0} e^{-p^2/4} \phi(x - p\sqrt{kt})\, dp + \frac{1}{\sqrt{4\pi}} \int_{0}^{\infty} e^{-p^2/4} \phi(x - p\sqrt{kt})\, dp.$$

Then, since $p < 0$ in the first integral, $\phi(x - p\sqrt{kt})$ goes to $\phi(x+)$ as $t \to 0+$, while it goes to $\phi(x-)$ in the second integral, due to p being positive. So one can make the obvious changes in the proof of the previous theorem to show (9). This curious fact is one of the reasons why some people prefer to define the value of the Heaviside step function at $x = 0$ to be $H(0) = \frac{1}{2}$. Then one has

$$\lim_{t\to 0+} Q(x,t) = H(x) \qquad \text{for all } x \in \mathbb{R} \quad (\text{including } x = 0!),$$

where $Q(x,t)$ was the solution arising from the initial data given by $H(x)$.

11.1 Comparison of wave to heat

We now summarize and compare the fundamental properties of the wave and heat equations in the table below. Brief discussion of each of the properties will follow.

	Property	**Wave** ($u_{tt} - c^2 u_{xx} = 0$)	**Heat** ($u_t - k u_{xx} = 0$)
(i)	Speed of propagation	Finite (speed $\le c$)	Infinite
(ii)	Singularities for $t > 0$	Transported along characteristics (speed $= c$)	Lost immediately
(iii)	Well-posed for $t > 0$	Yes	Yes (for bounded solutions)
(iv)	Well-posed for $t < 0$	Yes	No
(v)	Maximum principle	No	Yes
(vi)	Behaviour as $t \to \infty$	Does not decay	Decays to zero (if ϕ is integrable)
(vii)	Information	Transported	Lost gradually

Let us now recall why each of the properties listed in the table holds or does not for each equation.

(i) Finite speed of propagation for the wave equation is immediately seen from d'Alambert's formula (2).

The infinite speed of propagation for the heat equation was seen in the example of the heat kernel, which is strictly positive for all $x \in \mathbb{R}$ for $t > 0$, but has Dirac δ function as its initial data, and hence is zero for all $x \neq 0$ initially.

(ii) We saw in the box-wave (initial displacement in the form of a box, no initial velocity) and the "hammer blow" (no initial displacement, initial box-shaped velocity) that singularities are preserved and are transported along the characteristics. The same is seen from (2).

For the heat equation we saw in the last section that the solution (4) is infinitely differentiable even for piecewise continuous initial data (this is true for even weaker conditions on ϕ).

(iii) Well-posedness for the wave IVP is seen immediately from d'Alambert's formula.

In the case of the heat equation, we proved uniqueness and stability using either the maximum principle, or alternatively, the energy method. Existence follows from our construction of the explicit solution (4).

(iv) For the wave equation, this follows from the invariance under time reversion. Indeed, if $u(x,t)$ is a solution, then so is $u(x,-t)$, which has data $(\phi(x), -\psi(x))$.

If we reverse the time in the heat equation, we get $u_t + ku_{xx} = 0, t > 0$. One can solve this equation in much the same way as the heat equation, and due to the symmetry in t, will get the solution

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{(x-y)^2/4kt} \phi(y)\, dy,$$

which diverges for all $x \in \mathbb{R}$ (unless ϕ decays to zero faster than e^{-p^2}). So the heat equation is not well-posed *backward in time*. This makes physical sense as well, since the processes described by the heat equation, namely diffusion, heat flow and random motion are irreversible processes.

(v) The fact that there is no maximum principle for the wave equation is apparent from the "hammer blow" example, where the solution was everywhere zero initially, but due to the nonzero initial velocity, had nonzero displacement for any time $t > 0$. For the heat equation, the maximum principle was proved rigorously in a previous lecture.

(vi) We saw that the energy is conserved for the wave equation, so the solutions do not decay. We also saw this in the box-wave example, in which the initial box-shaped data split into two box-shaped waves of half the height that traveled in opposite directions without changing the shape.

For the heat equation, the decay is seen from formula (4), since $S(x-y,t) \to 0$ as $t \to \infty$, and the integral will be bounded if ϕ is integrable. Notice that in the example we considered in the last lecture, with $\phi(x) = e^x$, the solution did not decay, but rather "traveled" from right to left. This was due to ϕ being non-integrable.

(vii) The fact that information is transported by the solutions of the wave equation is seen from the fact that the initial data is propagated along the characteristics. So the information will travel along the characteristics as well.

In the case of the heat equation, the information is gradually lost, which can be seen from the graph of a typical solution (think of the heat kernel). The heat from the higher temperatures gets dissipated and after a while it is not clear what the original temperatures were.

11.2 Conclusion

Although the wave and heat equations are both second order linear constant coefficient PDEs, their respective solutions posses very different properties. By now we have learned how to solve the initial value problems on the whole line for both of these equations, and understood these solutions in terms of the physics behind the corresponding problems. We also saw that the properties of the solutions of the respective equations correspond to our intuition for each of the physical phenomena described by the equations.