12 Heat conduction on the half-line

In previous lectures we completely solved the initial value problem for the heat equation on the whole line, i.e. in the absence of boundaries. Next, we turn to problems with physically relevant boundary conditions. Let us first add a boundary consisting of a single endpoint, and consider the heat equation on the half-line $D = (0, \infty)$. The following initial/boundary value problem, or IBVP, contains a Dirichlet boundary condition at the endpoint $x = 0$.

$$\begin{cases} v_t - kv_{xx} = 0, \quad 0 < x < \infty, 0 < t < \infty, \\ v(x, 0) = \phi(x), \quad x > 0, \\ v(0, t) = 0, \quad t > 0. \end{cases} \tag{1}$$

If the solution to the above mixed initial/boundary value problem exists, then we know that it must be unique from an application of the maximum principle. In terms of the heat conduction, one can think of v in (1) as the temperature in an infinite rod, one end of which is kept at a constant zero temperature. The initial temperature of the rod is then given by $\phi(x)$.

Our goal is to solve the IBVP (1), and derive a solution formula, much like what we did for the heat IVP on the whole line. But instead of constructing the solution from scratch, it makes sense to try to reduce this problem to the IVP on the whole line, for which we already have a solution formula. This is achieved by extending the initial data $\phi(x)$ to the whole line. We have a choice of how exactly to extend the data to the negative half-line, and one should try to do this in such a fashion that the boundary condition of (1) is automatically satisfied by the solution to the IVP on the whole line that arises from the extended data. This is the case, if one chooses the *odd extension* of $\phi(x)$, which we describe next.

By definition, a function $\psi(x)$ is odd, if $\psi(-x) = -\psi(x)$. But then plugging in $x = 0$ into this definition, one gets $\psi(0) = 0$ for any odd function. Recall also that the solution $u(x, t)$ to the heat IVP with odd initial data is itself odd in the x variable. This follows from the fact that the sum $[u(x, t)+u(-x, t)]$ solves the heat equation and has zero initial data, hence, it is the identically zero function by the uniqueness of solutions. Then, by our above observation for odd functions, we would have that $u(0, t) = 0$ for any $t > 0$, which is exactly the boundary condition of (1).

This shows that if one extends $\phi(x)$ to an odd function on the whole line, then the solution with the extended initial data automatically satisfies the boundary condition of (1). Let us then define

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{for } x > 0, \\ -\phi(-x) & \text{for } x < 0, \\ 0 & \text{for } x = 0. \end{cases} \tag{2}$$

It is clear that ϕ_{odd} is an odd function, since we defined it for negative x by reflecting the $\phi(x)$ with respect to the vertical axis, and then with respect to the horizontal axis. This procedure produces a function whose graph is symmetric with respect to the origin, and thus it is odd. One can also verify this directly from the definition of odd functions. Now, let $u(x, t)$ be the solution of the following IVP on the whole line

$$\begin{cases} u_t - ku_{xx} = 0, \quad -\infty < x < \infty, 0 < t < \infty, \\ u(x, 0) = \phi_{\text{odd}}(x). \end{cases} \tag{3}$$

From previous lectures we know that the solution to (3) is given by the formula

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t)\phi_{\text{odd}}(y)\, dy, \qquad t > 0. \tag{4}$$

Restricting the x variable to only the positive half-line produces the function

$$v(x, t) = u(x, t)\big|_{x\geq 0}. \tag{5}$$

We claim that this $v(x,t)$ is the unique solution of IBVP (1). Indeed, $v(x,t)$ solves the heat equation on the positive half-line, since so does $u(x,t)$. Furthermore,

$$v(x,0) = u(x,0)\Big|_{x>0} = \phi_{\text{odd}}(x)\Big|_{x>0} = \phi(x),$$

and $v(0,t) = u(0,t) = 0$, since $u(x,t)$ is an odd function of x. So $v(x,t)$ satisfies the initial and boundary conditions of (1).

Returning to formula (4), we substitute the expressions for ϕ_{odd} from (2) and write

$$\begin{aligned} u(x,t) &= \int_0^\infty S(x-y,t)\phi_{\text{odd}}(y)\,dy + \int_{-\infty}^0 S(x-y,t)\phi_{\text{odd}}(y)\,dy \\ &= \int_0^\infty S(x-y,t)\phi(y)\,dy - \int_{-\infty}^0 S(x-y,t)\phi(-y)\,dy. \end{aligned}$$

Making the change of variables $y \mapsto -y$ in the second integral on the right, and flipping the integration limits gives

$$u(x,t) = \int_0^\infty S(x-y,t)\phi(y)\,dy - \int_0^\infty S(x+y,t)\phi(y)\,dy.$$

Using (5) and the above expression for $u(x,t)$, as well as the expression of the heat kernel $S(x,t)$, we can write the solution formula for the IBVP (1) as follows

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt} \right] \phi(y)\,dy. \tag{6}$$

The method used to arrive at this solution formula is called the *method of odd extensions* or the *reflection method*. We can make a physical sense of formula (6) by interpreting the integrand as the contribution from the point y minus the heat loss from this point due to the constant zero temperature at the endpoint.

Example 12.1. Solve the IBVP (1) with the initial data $\phi(x) = e^x$.

Using the solution formula (6), we have

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt} e^y - e^{-(x+y)^2/4kt} e^y \right] dy. \tag{7}$$

Combining the exponential factors of the first product under the integral, we will get an exponential with the following exponent

$$\frac{-[y^2 - 2(x+2kt)y + x^2]}{4kt} = -\left(\frac{y-(x+2kt)}{\sqrt{4kt}} \right)^2 + kt + x = -p^2 + kt + x,$$

where we made the obvious notation

$$p = \frac{y-x-2kt}{\sqrt{4kt}}.$$

Similarly, the exponent of the combined exponential from the second product under integral (7) is

$$\frac{-[y^2 + 2(x-2kt)y + x^2]}{4kt} = -\left(\frac{y+x-2kt}{\sqrt{4kt}} \right)^2 + kt - x = -q^2 + kt - x,$$

with

$$q = \frac{y+x-2kt}{\sqrt{4kt}}.$$

Braking integral (7) into a difference of two integrals, and making the changes of variables $y \mapsto p$, and $y \mapsto q$ in the respective integrals, we will get

$$v(x,t) = e^{kt+x} \frac{1}{\sqrt{\pi}} \int_{\frac{-x-2kt}{\sqrt{4kt}}}^{\infty} e^{-p^2}\, dp - e^{kt-x} \frac{1}{\sqrt{\pi}} \int_{\frac{x-2kt}{\sqrt{4kt}}}^{\infty} e^{-q^2}\, dq. \tag{8}$$

Notice that

$$\frac{1}{\sqrt{\pi}} \int_{\frac{-x-2kt}{\sqrt{4kt}}}^{\infty} e^{-p^2}\, dp = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-p^2}\, dp + \frac{1}{\sqrt{\pi}} \int_{\frac{-x-2kt}{\sqrt{4kt}}}^{0} e^{-p^2}\, dp = \frac{1}{2} + \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x+2kt}{\sqrt{4kt}}\right),$$

and similarly for the second integral. Putting these back into (8), we will arrive at the solution

$$v(x,t) = e^{kt+x}\left[\frac{1}{2} + \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x+2kt}{\sqrt{4kt}}\right)\right] - e^{kt-x}\left[\frac{1}{2} - \frac{1}{2}\mathcal{E}\mathrm{rf}\left(\frac{x-2kt}{\sqrt{4kt}}\right)\right].$$

□

12.1 Neumann boundary conditions

Let us now turn to the Neumann problem on the half-line,

$$\begin{cases} w_t - kw_{xx} = 0, & 0 < x < \infty, 0 < t < \infty, \\ w(x,0) = \phi(x), & x > 0 \\ w_x(0,t) = 0, & t > 0. \end{cases} \tag{9}$$

To find the solution of (9), we employ a similar idea used in the case of the Dirichlet problem. That is, we seek to reduce the IBVP to an IVP on the whole line by extending the initial data $\phi(x)$ to the negative half-axis in such a fashion that the boundary condition is automatically satisfied.

Notice that if $\psi(s)$ is an even function, i.e. $\psi(-x) = \psi(x)$, then its derivative function will be odd. Indeed, differentiating in the definition of the even function, we get $-\psi'(-x) = \psi'(x)$, which is the same as $\psi'(-x) = -\psi'(x)$. Hence, for an arbitrary even function $\psi(x)$, $\psi'(0) = 0$. It is now clear that extending the initial data so that the resulting function is even will produce solutions to the IVP on the whole line that automatically satisfy the Neumann condition of (9).

We define the even extension of $\phi(x)$,

$$\phi_{\text{even}} = \begin{cases} \phi(x) & \text{for } x \geq 0, \\ \phi(-x) & \text{for } x \leq 0, \end{cases} \tag{10}$$

and consider the following IVP on the whole line

$$\begin{cases} u_t - ku_{xx} = 0, & -\infty < x < \infty, 0 < t < \infty, \\ u(x,0) = \phi_{\text{even}}(x). \end{cases} \tag{11}$$

It is clear that the solution $u(x,t)$ of the IVP (11) will be even in x, since the difference $[u(-x,t)-u(x,t)]$ solves the heat equation and has zero initial data. We then use the solution formula for the IVP on the whole line to write

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{even}}(y)\, dy, \qquad t > 0, \tag{12}$$

and take

$$w(x,t) = u(x,t)\big|_{x\geq 0},$$

similar to the case of the Dirichlet problem. One can show that this $w(x,t)$ solves the IBVP (9), and use the expression for the heat kernel, as well as the definition (10), to write the solution formula as follows

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left[e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}\right] \phi(y)\, dy. \tag{13}$$

Notice that the formulas (6) and (13) differ only by the sign between the two exponential terms inside the integral.

In terms of heat conduction, the Neumann condition in (9) means that there is no heat exchange between the rod and the environment (recall that the heat flux is proportional to the spatial derivative of the temperature). The physical interpretation of formula (13) is that the integrand is the contribution of $\phi(y)$ plus an additional contribution, which comes from the lack of heat transfer to the points of the rod with negative coordinates.

Example 12.2. Solve the IBVP (9) with the initial data $\phi(x) \equiv 1$.

Using the formula (13), we can write the solution as

$$\begin{aligned} w(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt} \right] dy \\ &= \frac{1}{\sqrt{\pi}} \int_{\frac{-x}{\sqrt{4kt}}}^\infty e^{-p^2}\, dp + \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-q^2}\, dq, \end{aligned}$$

where we made the changes of variables

$$p = \frac{y-x}{\sqrt{4kt}}, \qquad \text{and} \qquad q = \frac{y+x}{\sqrt{4kt}}.$$

Using the same idea as in the previous example, we can write the solution in terms of the $\mathcal{E}\text{rf}$ function as follows

$$w(x,t) = \left[\frac{1}{2} + \frac{1}{2}\mathcal{E}\text{rf}\left(\frac{x}{\sqrt{4kt}} \right) \right] + \left[\frac{1}{2} - \frac{1}{2}\mathcal{E}\text{rf}\left(\frac{x}{\sqrt{4kt}} \right) \right] \equiv 1.$$

So the solution is identically 1, which is clear if one thinks in terms of heat conduction. Indeed, problem (9) describes the temperature dynamics with identically 1 initial temperature, and no heat loss at the endpoint. Obviously there is no heat transfer between points of equal temperature, so the temperatures remain steady along the entire rod. □.

12.2 Conclusion

We derived the solution to the heat equation on the half-line by reducing the initial/boundary value problem to the initial value problem over the whole line through appropriate extension of the initial data. In the case of zero Dirichlet boundary condition the odd extension of the initial data automatically guarantees that the solution will satisfy the boundary condition. While for the case of zero Neumann boundary condition the appropriate choice is the even extension. This reflection method relies on the fact that the solution to the heat equation on the whole line with odd initial data is odd, while the solution with even initial data is even.