

### 13 Waves on the half-line

Similar to the last lecture on the heat equation on the half-line, we will use the reflection method to solve the boundary value problems associated with the wave equation on the half-line  $0 < x < \infty$ . Let us start with the Dirichlet boundary condition first, and consider the initial boundary value problem

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0, & 0 < x < \infty, 0 < t < \infty, \\ v(x, 0) = \phi(x), & v_t(x, 0) = \psi(x), & x > 0, \\ v(0, t) = 0, & & t > 0. \end{cases} \quad (1)$$

For the vibrating string, the boundary condition of (1) means that the end of the string at  $x = 0$  is held fixed. We reduce the Dirichlet problem (1) to the whole line by the reflection method. The idea is again to extend the initial data, in this case  $\phi, \psi$ , to the whole line, so that the boundary condition is automatically satisfied for the solutions of the IVP on the whole line with the extended initial data. Since the boundary condition is in the Dirichlet form, one must take the odd extensions

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -\phi(-x) & \text{for } x < 0. \end{cases} \quad \psi_{\text{odd}}(x) = \begin{cases} \psi(x) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -\psi(-x) & \text{for } x < 0. \end{cases} \quad (2)$$

Consider the IVP on the whole line with the extended initial data

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & -\infty < x < \infty, 0 < t < \infty, \\ u(x, 0) = \phi_{\text{odd}}(x), & u_t(x, 0) = \psi_{\text{odd}}(x). \end{cases} \quad (3)$$

Since the initial data of the above IVP are odd, we know from a homework problem that the solution of the IVP,  $u(x, t)$ , will also be odd in the  $x$  variable, and hence  $u(0, t) = 0$  for all  $t > 0$ . Then defining the restriction of  $u(x, t)$  to the positive half-line  $x \geq 0$ ,

$$v(x, t) = u(x, t)|_{x \geq 0}, \quad (4)$$

we automatically have that  $v(0, t) = u(0, t) = 0$ . So the boundary condition of the Dirichlet problem (1) is satisfied for  $v$ . Obviously the initial conditions are satisfied as well, since the restrictions of  $\phi_{\text{odd}}(x)$  and  $\psi_{\text{odd}}(x)$  to the positive half-line are  $\phi(x)$  and  $\psi(x)$  respectively. Finally,  $v(x, t)$  solves the wave equation for  $x > 0$ , since  $u(x, t)$  satisfies the wave equation for all  $x \in \mathbb{R}$ , and in particular for  $x > 0$ . Thus,  $v(x, t)$  defined by (4) is a solution of the Dirichlet problem (1). It is clear that the solution must be unique, since the odd extension of the solution will solve IVP (3), and therefore must be unique.

Using d’Alembert’s formula for the solution of (3), and taking the restriction (4), we have that for  $x \geq 0$ ,

$$v(x, t) = \frac{1}{2}[\phi_{\text{odd}}(x + ct) + \phi_{\text{odd}}(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{odd}}(s) ds. \quad (5)$$

Notice that if  $x \geq 0$  and  $t > 0$ , then  $x + ct > 0$ , and  $\phi_{\text{odd}}(x + ct) = \phi(x + ct)$ . If in addition  $x - ct > 0$ , then  $\phi_{\text{odd}}(x - ct) = \phi(x - ct)$ , and over the interval  $s \in [x - ct, x + ct]$ ,  $\psi_{\text{odd}}(s) = \psi(s)$ . Thus, for  $x > ct$ , we have

$$v(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds, \quad (6)$$

which is exactly d’Alembert’s formula.

For  $0 < x < ct$ , the argument  $x - ct < 0$ , and using (2) we can rewrite the solution (5) as

$$v(x, t) = \frac{1}{2}[\phi(x + ct) - \phi(ct - x)] + \frac{1}{2c} \left[ \int_{x-ct}^0 -\psi(-s) ds + \int_0^{x+ct} \psi(s) ds \right].$$

Making the change of variables  $s \mapsto -s$  in the first integral on the right, we get

$$\begin{aligned} v(x, t) &= \frac{1}{2}[\phi(x + ct) - \phi(ct - x)] + \frac{1}{2c} \left[ \int_{ct-x}^0 \psi(s) ds + \int_0^{x+ct} \psi(s) ds \right] \\ &= \frac{1}{2}[\phi(x + ct) - \phi(ct - x)] + \frac{1}{2c} \int_{ct-x}^{x+ct} \psi(s) ds. \end{aligned}$$

One could also use the fact that the integral of the odd function  $\psi_{\text{odd}}(s)$  over the symmetric interval  $[x - ct, ct - x]$  is zero, thus  $\int_{x-ct}^{x+ct} \psi_{\text{odd}}(s) ds = \int_{ct-x}^{x+ct} \psi(s) ds$ .

The two different cases giving different expressions are illustrated in Figures 1 and 2 below. Notice how one of the characteristics from a point with  $x_0 < ct_0$  gets reflected from the “wall” at  $x = 0$  in Figure 2.

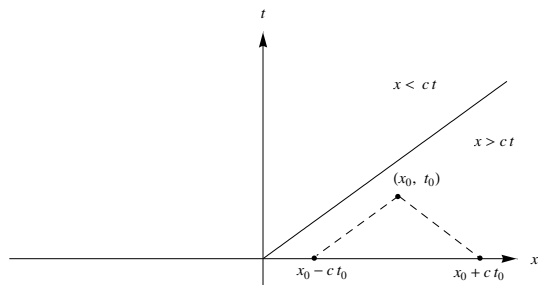


Figure 1: The case with  $x_0 > ct_0$ .

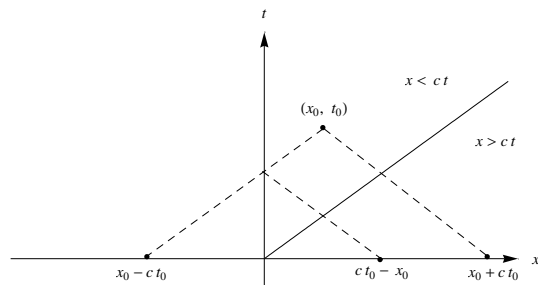


Figure 2: The case with  $x_0 < ct_0$ .

Combining the two expressions for  $v(t, x)$  over the two regions, we arrive at the solution

$$v(x, t) = \begin{cases} \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds, & \text{for } x > ct \\ \frac{1}{2}[\phi(x + ct) - \phi(ct - x)] + \frac{1}{2c} \int_{ct-x}^{x+ct} \psi(s) ds, & \text{for } 0 < x < ct. \end{cases} \quad (7)$$

The minus sign in front of  $\phi(ct - x)$  in the second expression above, as well as the reduction of the integral of  $\psi$  to the smaller interval are due to the cancellation stemming from the *reflected wave*. The next example illustrates this phenomenon.

**Example 13.1.** Solve the Dirichlet problem (1) with the following initial data

$$\phi(x) = \begin{cases} h & \text{for } a < x < 2a, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \psi(x) \equiv 0.$$

The initial data defines a box-like displacement over the interval  $(a, 2a)$  and zero initial velocity for the string. It is clear that for some small time the wave will propagate just like in the case of the IVP on the whole line, i.e. without “seeing” the boundary. This is due to the finite speed of propagation property of the wave equation, according to which it takes some time, specifically  $a/c$  time for the initial displacement to reach the boundary. Thus, we expect that the box-like wave will break into two box-waves, each with half the height of the initial box-like displacement, which travel with speed  $c$  in opposite directions. The box-wave traveling in the right direction will never hit the boundary at  $x = 0$ , and will continue traveling unaltered for all time. However, the left box-wave hits the wall (the fixed end of the vibrating string), and gets reflected in an odd fashion, that is the displacement gets the minus sign in the second expression of (7).

To find the values of the solution at any point  $(x_0, t_0)$ , we draw the backward characteristics from that point and trace the point back to the  $x$  axis, where the initial data is defined. Since the initial data is nonzero only over the interval  $(a, 2a)$ , only those characteristics that hit the  $x$  axis between the points

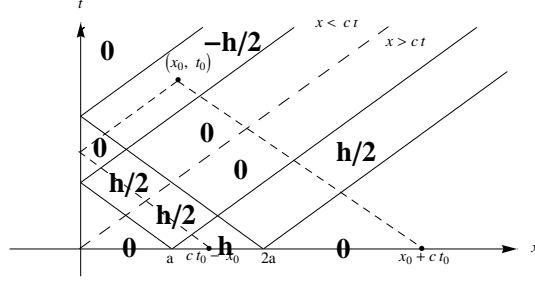


Figure 3: The values of  $v(x, t)$  carried forward in time by the characteristics.

$a$  and  $2a$  will carry nonzero values forward in time. Notice also that if a characteristic hits the interval  $(a, 2a)$  after being reflected from the wall  $x = 0$ , then the value it gets must be taken with a minus sign.

The different values determined by this method are illustrated in Figure 3 above. Notice how the left box-wave with height  $h/2$  flips after hitting the boundary, and travels in the opposite direction with negative height  $-h/2$ .

One can then use the values from Figure 3 to draw the profile of the string at any time  $t$ .  $\square$

### 13.1 Neumann boundary condition

For the Neumann problem on the half-line,

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, & 0 < x < \infty, 0 < t < \infty, \\ w(x, 0) = \phi(x), & w_t(x, 0) = \psi(x), & x > 0, \\ w_x(0, t) = 0, & t > 0, \end{cases} \quad (8)$$

we use the reflection method with even extensions to reduce the problem to an IVP on the whole line. Define the even extensions of the initial data

$$\phi_{\text{even}} = \begin{cases} \phi(x) & \text{for } x \geq 0, \\ \phi(-x) & \text{for } x \leq 0, \end{cases} \quad \psi_{\text{even}} = \begin{cases} \psi(x) & \text{for } x \geq 0, \\ \psi(-x) & \text{for } x \leq 0. \end{cases} \quad (9)$$

and consider the following IVP on the whole line

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & -\infty < x < \infty, 0 < t < \infty, \\ u(x, 0) = \phi_{\text{even}}(x), & u_t(x, 0) = \psi_{\text{even}}(x). \end{cases} \quad (10)$$

Clearly, the solution  $u(x, t)$  to the IVP (10) will be even in  $x$ , and since the derivative of an even function is odd,  $u_x(x, t)$  will be odd in  $x$ , and hence  $u_x(0, t) = 0$  for all  $t > 0$ . Similar to the case of the Dirichlet problem, the restriction

$$w(x, t) = u(x, t)|_{x \geq 0}$$

will be the unique solution of the Neumann problem (8).

Using d'Alembert's formula for the solution  $u(x, t)$  of (10), and taking the restriction to  $x \geq 0$ , we get

$$w(x, t) = \frac{1}{2}[\phi_{\text{even}}(x + ct) + \phi_{\text{even}}(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{even}}(s) ds. \quad (11)$$

One again needs to consider the two cases  $x > ct$  and  $0 < x < ct$  separately. Notice that with the even extensions we will get additions, rather than cancellations. Using the definitions (9), the solution (11) can be written as

$$w(x, t) = \begin{cases} \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds, & \text{for } x > ct \\ \frac{1}{2}[\phi(x + ct) + \phi(ct - x)] \\ \quad + \frac{1}{2c} \left[ \int_0^{ct-x} \psi(s) ds + \int_0^{x+ct} \psi(s) ds \right], & \text{for } 0 < x < ct. \end{cases} \quad (12)$$

The Neumann boundary condition corresponds to a vibrating string with a free end at  $x = 0$ , since the string tension, which is proportional to the derivative  $v_x(x, t)$ , vanishes at  $x = 0$ . In this case the reflected wave adds to the original wave, rather than canceling it.

**Example 13.2.** Solve the Neumann problem (8) with the following initial data

$$\phi(x) = \begin{cases} h & \text{for } a < x < 2a, \\ 0 & \text{otherwise,} \end{cases} \quad \psi(x) \equiv 0.$$

The initial data is exactly the same as in the previous example for the Dirichlet problem. The only difference from the Dirichlet case is that the free end reflects the wave with a plus sign. The values carried forward in time by the characteristics are determined in the same way as before. Figure 4 illustrates this method. Notice that the reflected wave has the same (positive) height  $h/2$  as the wave right before the reflection.

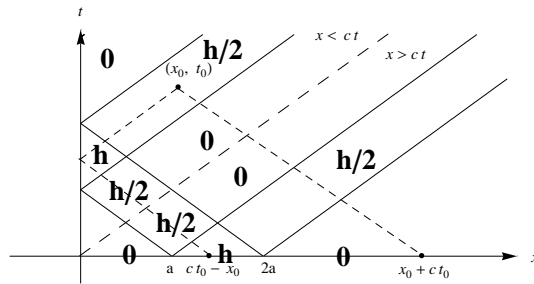


Figure 4: The values of  $v(x, t)$  carried forward in time by the characteristics.

One can again use the values from Figure 4 to draw the profile of the string at any time  $t$ . □

### 13.2 Conclusion

We derived the solution to the wave equation on the half-line in much the same way as was done for the heat equation. That is, we reduced the initial/boundary value problem to the initial value problem over the whole line through appropriate extension of the initial data. In this case the characteristics nicely illustrate the reflection phenomenon. We saw that the characteristics that hit the initial data after reflection from the boundary wall  $x = 0$  carry the values of the initial data with a minus sign in the case of the Dirichlet boundary conditions, and with a plus sign in the case of the Neumann boundary conditions. This corresponds to our intuition of reflected waves from a fixed end, and free end respectively.