15 Heat with a source

So far we considered homogeneous wave and heat equations and the associated initial value problems on the whole line, as well as the boundary value problems on the half-line and the finite line (for wave only). The next step is to extend our study to the inhomogeneous problems, where an external heat source, in the case of heat conduction in a rod, or an external force, in the case of vibrations of a string, are also accounted for. We first consider the inhomogeneous heat equation on the whole line,

$$\begin{cases} u_t - ku_{xx} = f(x,t), & -\infty < x < \infty, t > 0, \\ u(x,0) = \phi(x), \end{cases} \tag{1}$$

where $f(x,t)$ and $\phi(x)$ are arbitrary given functions. The right hand side of the equation, $f(x,t)$, is called the *source* term, and measures the physical effect of an external heat source. It has units of heat flux (left hand side of the equation has the units of u_t, i.e. change in temperature per unit time), thus it gives the instantaneous temperature change due to an external heat source.

From the superposition principle, we know that the solution of the inhomogeneous equation can be written as the sum of the solution of the homogeneous equation, and a particular solution of the inhomogeneous equation. We can thus break problem (1) into the following two problems

$$\begin{cases} u^h_t - ku^h_{xx} = 0, \\ u^h(x,0) = \phi(x), \end{cases} \tag{2}$$

and

$$\begin{cases} u^p_t - ku^p_{xx} = f(x,t), \\ u^p(x,0) = 0. \end{cases} \tag{3}$$

Obviously, $u = u^h + u^p$ will solve the original problem (1).

Notice that we solve for the general solution of the homogeneous equation with arbitrary initial data in (2), while in the second problem (3) we solve for a particular solution of the inhomogeneous equation, namely the solution with zero initial data. This reduction of the original problem to two simpler problems (homogeneous, and inhomogeneous with zero data) using the superposition principle is a standard practice in the theory of linear PDEs.

We have solved problem (2) before, and arrived at the solution

$$u^h(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy, \tag{4}$$

where $S(x,t)$ is the heat kernel. Notice that the physical meaning of expression (4) is that the heat kernel averages out the initial temperature distribution along the entire rod.

Since $f(x,t)$ plays the role of an external heat source, it is clear that this heat contribution must be averaged out, too. But in this case one needs to average not only over the entire rod, but over time as well, since the heat contribution at an earlier time will effect the temperatures at all later times. We claim that the solution to (3) is given by

$$u^p(x,t) = \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,dy\,ds. \tag{5}$$

Notice that the time integration is only over the interval $[0,t]$, since the heat contribution at later times can not effect the temperature at time t. Combining (4) and (5) we obtain the following solution to the IVP (1)

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,dy\,ds, \tag{6}$$

or, substituting the expression of the heat kernel,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, dy + \int_0^t \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4k(t-s)}}{\sqrt{4\pi k(t-s)}} f(y,s)\, dy\, ds.$$

One can draw parallels between formula (6) and the solution to the inhomogeneous ODE analogous to the heat equation. Indeed, consider the IVP for the following ODE.

$$\begin{cases} \frac{d}{dt}u(t) - Au(t) = f(t), \\ u(0) = \phi, \end{cases} \tag{7}$$

where A is a constant (more generally, for vector valued u, the equation will be a system of ODEs for the components of u, and A will be a matrix with constant entries). Using an integrating factor e^{-At}, the ODE in (7) yields

$$\frac{d}{dt}\left(e^{-At}u\right) = e^{-At}\frac{du}{dt} - Ae^{-At}u = e^{-At}(u' - Au) = e^{-At}f(t).$$

But then

$$e^{-At}u = \int_0^t e^{-As} f(s)\, ds + e^{-A\cdot 0}u(0),$$

and multiplying both sides by e^{At} gives

$$u(t) = e^{At}\phi + \int_0^t e^{A(t-s)} f(s)\, ds. \tag{8}$$

The operator $\mathscr{S}(t)$, given by $\mathscr{S}(t)\phi = e^{At}\phi$, which is called the *propagator* operator, maps the initial value ϕ to the solution of the homogeneous equation at later times. In terms of this operator, we can rewrite solution (8) as

$$u(t) = \mathscr{S}(t)\phi + \int_0^t \mathscr{S}(t-s) f(s)\, ds. \tag{9}$$

In the case of the heat equation, the heat propagator operator is

$$\mathscr{S}(t)\phi = \int_{-\infty}^{\infty} S(x-y,t)\phi(t)\, dy,$$

which again maps the initial data ϕ to the solution of the homogeneous equation at later times. Using the heat propagator, we can rewrite formula (6) in exactly the same form as (9).

We now show that (6) indeed solves problem (1) by a direct substitution. Since we have solved the homogeneous equation before, it suffices to show that u^p given by (5) solves problem (3). Differentiating (5) with respect to t gives

$$\partial_t u^p = \int_{-\infty}^{\infty} S(x-y,0) f(y,t)\, dy + \int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial t} S(x-y,t-s) f(y,s)\, dy\, ds.$$

Recall that the heat kernel solves the heat equation and has the Dirac delta function as its initial data, i.e. $S_t = kS_{xx}$, and $S(x-y,0) = \delta(x-y)$. Hence,

$$\begin{aligned} \partial_t u^p &= \int_{-\infty}^{\infty} \delta(x-y) f(y,t)\, dy + \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2}{\partial x^2} S(x-y,t-s) f(y,s)\, dy\, ds \\ &= f(x,t) + k\frac{\partial^2}{\partial x^2} \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y,s)\, dy\, ds = f(x,t) + ku^p_{xx}, \end{aligned}$$

which shows that u^p solves the inhomogeneous heat equation. It is also clear that

$$\lim_{t\to 0} u^p(x,t) = \lim_{t\to 0} \int_0^t \int_{-\infty}^{\infty} S(x-y, t-s) f(y,s)\, dy\, ds = 0.$$

Thus, u^p, given by (5), indeed solves problem (3), which finishes the proof that (6) solves the original IVP (1).

Example 15.1. Find the solution of the inhomogeneous heat equation with the source $f(x,t) = \delta(x-2)\delta(t-1)$ and zero initial data.

Using formula (6), and substituting the expression for $f(x,t)$, and $\phi(x) = 0$, we get

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} S(x-y, t-s)\delta(y-2)\delta(s-1)\, dy\, ds = \int_0^t S(x-2, t-s)\delta(s-1)\, ds.$$

For the last integral, notice that if $t < 1$, then $\delta(s-1) = 0$ for all $s \in [0,t]$, and if $t > 1$, then the delta function will act on the heat kernel by assigning its value at $s = 1$. Hence,

$$u(x,t) = \begin{cases} 0 & \text{for } 0 < t < 1, \\ S(x-2, t-1) & \text{for } t > 1. \end{cases}$$

This, of course, coincides with our intuition of heat conduction, since the external heat source in this case gives an instantaneous temperature boost to the point $x = 1$ at time $t = 1$. Henceforth, the temperature in the rod will remain zero till the time $t = 1$, and afterward the heat will transfer exactly as in the case of the homogeneous heat equation with data given at time $t = 1$ as $u(x,1) = \delta(x-2)$. □

15.1 Source on the half-line

We will use the reflection method to solve the inhomogeneous heat equation on the half-line. Consider the Dirichlet heat problem

$$\begin{cases} v_t - kv_{xx} = f(x,t), & \text{for } 0 < x < \infty, \\ v(x,0) = \phi(x), \\ v(0,t) = h(t). \end{cases} \tag{10}$$

Notice that in the above problem not only the equation is inhomogeneous, but the boundary data is given by an arbitrary function $h(t)$. In this case the Dirichlet condition is called inhomogeneous. We can reduce the above problem to one with zero initial data by the following subtraction method. Defining the new quantity

$$V(x,t) = v(x,t) - h(t), \tag{11}$$

we have that

$$\begin{aligned} &V_t - kV_{xx} = v_t - h'(t) - kv_{xx} = f(x,t) - h'(t), \\ &V(x,0) = v(x,0) - h(0) = \phi(x) - h(0), \\ &V(0,t) = v(0,t) - h(t) = h(t) - h(t) = 0. \end{aligned}$$

Thus, $v(x,t)$ solves problem (10) if and only if $V(x,t)$ solves the Dirichlet problem

$$\begin{cases} V_t - kV_{xx} = f(x,t) - h'(t), & \text{for } 0 < x < \infty, \\ V(x,0) = \phi(x) - h(0), \\ V(0,t) = 0. \end{cases} \tag{12}$$

With this procedure, we essentially combined the heat source given as the boundary data at the endpoint $x = 0$ with the external heat source $f(x,t)$. Notice that $h(t)$ has units of temperature, so its derivative will have units of heat flux, which matches the units of $f(x,t)$. We will denote the combined source in the last problem by $F(x,t) = f(x,t) - h'(t)$, and the initial data by $\Phi(x) = \phi(x) - h(0)$. Since the

Dirichlet boundary condition for V is homogeneous, we can extend $F(x,t)$ and $\Phi(x,t)$ to the whole line in an odd fashion, and use the reflection method to solve (12). The extensions are

$$\Phi_{\text{odd}}(x) = \begin{cases} \phi(x) - h(0) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -\phi(-x) + h(0) & \text{for } x < 0, \end{cases} \qquad F_{\text{odd}}(x,t) = \begin{cases} f(x,t) - h'(t) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -f(-x,t) + h'(t) & \text{for } x < 0. \end{cases}$$

Clearly, the solution to the problem

$$\begin{cases} U_t - kU_{xx} = F_{\text{odd}}(x,t), & \text{for } -\infty < x < \infty, \\ U(x,0) = \Phi_{\text{odd}}(x), \end{cases}$$

is odd, since $U(x,t) + U(-x,t)$ will solve the homogeneous heat equation with zero initial data. Then $U(0,t) = 0$, and the restriction to $x \geq 0$ will solve the Dirichlet problem (12) on the half-line. Thus, for $x > 0$,

$$V(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\Phi_{\text{odd}}(y)\,dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)F_{\text{odd}}(y,s)\,dy\,ds.$$

Proceeding exactly as in the case of the (homogeneous) heat equation on the half-line, we will get

$$\begin{aligned} V(x,t) &= \int_0^{\infty} [S(x-y,t) - S(x+y,t)](\phi(y) - h(0))\,dy \\ &\quad + \int_0^t \int_0^{\infty} [S(x-y,t-s) - S(x+y,t-s)](f(y,s) - h'(s))\,dy\,ds. \end{aligned}$$

Finally, using that $v(x,t) = V(x,t) + h(t)$, we have

$$\begin{aligned} v(x,t) = h(t) &+ \int_0^{\infty} [S(x-y,t) - S(x+y,t)](\phi(y) - h(0))\,dy \\ &+ \int_0^t \int_0^{\infty} [S(x-y,t-s) - S(x+y,t-s)](f(y,s) - h'(s))\,dy\,ds. \end{aligned}$$

15.2 Conclusion

Using our intuition of heat conduction as an averaging process with the weight given by the heat kernel, we guessed formula (6) for the solution of the inhomogeneous heat equation, treating the inhomogeneity as an external heat source. Employing the propagator operator, this formula coincided exactly with the solution formula for the analogous inhomogeneous ODE, which further hinted at the correctness of the formula. However, to obtain a rigorous proof that formula (6) indeed gives the unique solution, we verified that this function satisfies the equation and the initial condition by a direct substitution. We then used this formula along with the reflection method to also find the solution for the inhomogeneous heat equation on the half-line.