16 Waves with a source, appendix

In the lecture we used the method of characteristics to solve the initial value problem for the inhomogeneous wave equation,

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & -\infty < x < \infty, t > 0, \\ u(x,0) = \phi(x), & u_t(x,0) = \psi(x), \end{cases} \tag{1}$$

and obtained the formula

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds. \tag{2}$$

Another way to derive the above solution formula is to integrate both sides of the inhomogeneous wave equation over the triangle of dependence and use Green's theorem.

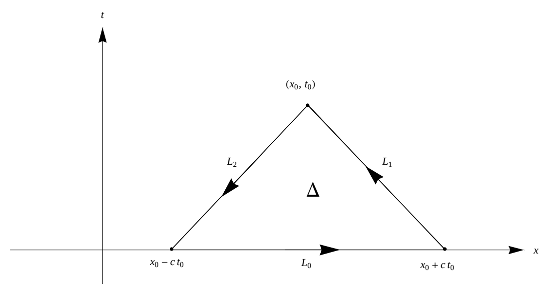

Figure 1: The triangle of dependence of the point (x_0, t_0).

Fix a point (x_0, t_0), and integrate both sides of the equation in (1) over the triangle of dependence for this point.

$$\iint_\triangle (u_{tt} - c^2 u_{xx})\,dx\,dt = \iint_\triangle f(x,t)\,dx\,dt. \tag{3}$$

Recall that by Green's theorem

$$\iint_D (Q_x - P_t)\,dx\,dt = \oint_{\partial D} P\,dx + Q\,dt,$$

where ∂D is the boundary of the region D with counterclockwise orientation. We thus have

$$\iint_\triangle (u_{tt} - c^2 u_{xx})\,dx\,dt = \iint_\triangle (-c^2 u_x)_x - (-u_t)_t\,dx\,dt = \oint_{\partial\triangle} -u_t\,dx - c^2 u_x\,dt.$$

The boundary of the triangle of dependence consists of three sides, $\partial\triangle = L_0 + L_1 + L_2$, as can be seen in Figure 1, so

$$\iint_\triangle (u_{tt} - c^2 u_{xx})\,dx\,dt = \int_{L_0+L_1+L_2} -u_t\,dx - c^2 u_x\,dt,$$

and we have the following relations on each of the sides

$$\begin{aligned} L_0 &: \quad dt = 0 \\ L_1 &: \quad dx = -cdt \\ L_2 &: \quad dx = cdt \end{aligned}$$

Using these, we get

$$\int_{L_0} -c^2 u_x\,dt - u_t\,dx = -\int_{x_0-ct_0}^{x_0+ct_0} u_t(x,0)\,dx = -\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx,$$
$$\int_{L_1} -c^2 u_x\,dt - u_t\,dx = c\int_{L_1} du = c[u(x_0,t_0) - u(0,x_0+ct_0)] = cu(x_0,t_0) - c\phi(x_0+ct_0),$$
$$\int_{L_2} -c^2 u_x\,dt - u_t\,dx = -c\int_{L_2} du = -c[u(0,x_0-ct_0) - u(x_0,t_0)] = cu(x_0,t_0) - c\phi(x_0-ct_0).$$

Putting all the sides together gives

$$\iint_{\triangle} (u_{tt} - c^2 u_{xx})\,dx\,dt = 2cu(x_0,t_0) - c[\phi(x_0+ct_0) + \phi(x_0-ct_0)] - \int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx,$$

and using (3), we obtain

$$u(x_0,t_0) = \frac{1}{2}[\phi(x_0+ct_0) + \phi(x_0-ct_0)] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx + \frac{1}{2c}\iint_{\triangle} f(x,t)\,dx\,dt,$$

which is equivalent to formula (2).

16.1 The operator method

For the inhomogeneous heat equation we interpreted the solution formula in terms of the heat propagator, which also showed the parallels between the heat equation and the analogous ODE. We would like to obtain such a description for the solution formula (2) as well. For this, consider the ODE analog of the wave equation with the associated initial conditions

$$\begin{cases} \frac{d^2u}{dt^2} + A^2 u = f(t), \\ u(0) = \phi, \quad u'(0) = \psi, \end{cases} \tag{4}$$

where A is a constant (a matrix, if we allow u to be vector valued). To find the solution of the inhomogeneous ODE, we need to first solve the homogeneous equation, and then use variation of parameters to find a particular solution of the inhomogeneous equation. The solution of the homogeneous equation is

$$u^h(t) = c_1\cos(At) + c_2\sin(At),$$

and the initial conditions imply that $c_1 = \phi$, and $c_2 = A^{-1}\psi$. To obtain a particular solution of the inhomogeneous equation we assume that c_1 and c_2 depend on t,

$$u^p(t) = c_1(t)\cos(At) + c_2(t)\sin(At),$$

and substitute u^p into the equation to solve for $c_1(t)$ and $c_2(t)$. This procedure leads to

$$c_1(t) = -\int_0^t A^{-1}\sin(As)f(s)\,dt, \qquad c_2(t) = \int_0^t A^{-1}\cos(As)f(s)\,ds.$$

Putting everything together, the solution to (4) will be

$$u(t) = \cos(At)\phi + A^{-1}\sin(At)\psi + \int_0^t A^{-1}\sin(A(t-s))f(s)\,ds.$$

If we now define the propagator

$$\mathscr{S}(t)\psi = A^{-1}\sin(At)\psi,$$

then the solution to (4) can be written as

$$u(t) = \mathscr{S}'(t)\phi + \mathscr{S}(t)\psi + \int_0^t \mathscr{S}(t-s)f(s)\,ds. \tag{5}$$

For the wave equation, similarly denoting the operator acting on ψ from d'Alambert's formula by

$$\mathscr{S}(t)\psi = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy,$$

we can rewrite formula (2) in exactly the same form as (4). The moral of this story is that having solved the homogeneous equation and found the propagator, we have effectively derived the solution of the inhomogeneous equation as well. The rigorous connection between the solution of the homogeneous equation and that of the inhomogeneous wave equation is contained in the following statement.

Duhamel's Principle. Consider the following 1-parameter family of wave IVPs

$$\begin{cases} u_{tt}(x,t;s) - c^2 u_{xx}(x,t;s) = 0, \\ u(x,s;s) = 0, \quad u_t(x,s;s) = f(x,s), \end{cases} \tag{6}$$

then the function

$$v(x,t) = \int_0^t u(x,t;s)\,ds$$

solves the following inhomogeneous wave equation with vanishing data

$$\begin{cases} v_{tt}(x,t) - c^2 v_{xx}(x,t) = f(x,t), \\ v(x,0) = 0, \quad v_t(x,0) = 0. \end{cases}$$

Note that the initial conditions of the s-IVP (6) are given at time $t = s$, and the initial velocity is $\psi(x;s) = f(x,s)$. Duhamel's principle has the physical description of replacing the external force by its effect on the velocity. From Newton's second law, the force is responsible for acceleration, or change in velocity per unit time. So if we can account for the effect of the external force on the instantaneous velocity, then the solution of the equation with the external force can be found by solving the homogeneous equations with the effected velocities, namely (6), and "summing" these solutions over the instances $t = s$.

We prove Duhamel's principle by direct substitution. The derivatives of v are

$$\begin{aligned} v_t(x,t) &= u(x,t;t) + \int_0^t u_t(x,t;s)\,ds = \int_0^t u_t(x,t;s)\,ds, \\ v_{tt}(x,t) &= u_t(x,t;t) + \int_0^t u_{tt}(x,t;s)\,ds = f(x,t) + \int_0^t u_{tt}(x,t;s)\,ds, \\ v_{xx}(x,t) &= \int_0^t u_{xx}(x,t;s)\,ds, \end{aligned}$$

where we used the initial conditions of (6). Substituting this into the wave equation gives

$$(\partial_t^2 - c^2\partial_x^2)v = \int_0^t [u_{tt}(x,t;s) - c^2 u_{xx}(x,t;s)]\,ds + f(x,t) = f(x,t),$$

and v indeed solves the inhomogeneous wave equation. It is also clear that v has vanishing initial data.

Duhamel's principle gives an alternative way of proving that (2) solves the inhomogeneous wave equation. Indeed, from d'Alambert's formula for (6) and a time shift $t \mapsto t - s$, we have

$$u(x,t;s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy.$$

Thus, the solution of the inhomogeneous wave equation with zero initial data is

$$v(x,t) = \int_0^t u(x,t;s)\,ds = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds.$$

16.2 Conclusion

We defined the wave propagator as the operator that maps the initial velocity to the solution of the homogeneous wave equation with zero initial displacement. Using this operator, the solution of the inhomogeneous wave equation can be written in exactly the same form as the solution of the analogous inhomogeneous ODE in terms of its propagator. The significance of this observation is in the connection between the solution of the homogeneous and that of the inhomogeneous wave equations, which is the substance of Duhamel's principle. Hence, to solve the inhomogeneous wave equation, all one needs is to find the propagator operator for the homogeneous equation.