17 Separation of variables: Dirichlet conditions

Earlier in the course we solved the Dirichlet problem for the wave equation on the finite interval $0 < x < l$ using the reflection method. This required separating the domain $(x, t) \in (0, l) \times (0, \infty)$ into different regions according to the number of reflections that the backward characteristic originating in the regions undergo before reaching the x axis. In each of these regions the solution was given by a different expression, which is impractical in applications, and the method does not generalize to higher dimensions or other equations. We now study a different method of solving the boundary value problems on the finite interval, called *separation of variables.*

Let us start by considering the wave equation on the finite interval with homogeneous Dirichlet conditions.

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < l, \\ u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x), \\ u(0, t) = u(l, t) = 0. \end{cases} \tag{1}$$

The idea of the separation of variables method is to find the solution of the boundary value problem as a linear combination of simpler solutions (compare this to finding the simpler solution $S(x, t)$ of the heat equation, and then expressing any other solution in terms of the heat kernel). The building blocks in this case will be the *separated solutions*, which are the solutions that can be written as a product of two functions, one of which depends only on x, and the other only on t, i.e.

$$u(x, t) = X(x)T(t). \tag{2}$$

Let us try to find all the separated solutions of the wave equation. Substituting (2) into the equation gives

$$X(x)T''(t) = c^2 X''(x)T(t).$$

Dividing both sides of these identity by $-c^2 X(x)T(t)$, we get

$$-\frac{X''(x)}{X(x)} = -\frac{T''(t)}{c^2 T(t)} = \lambda. \tag{3}$$

Clearly λ is a constant, since it is independent of x from $\lambda = -T''/(c^2 T)$, and is independent of t from $\lambda = -X''/X$. We will shortly see that the boundary conditions force λ to be positive, so let $\lambda = \beta^2$, for some $\beta > 0$. One can then rewrite (3) as a pair of separate ODEs for $X(x)$ and $T(t)$

$$T'' + c^2\beta^2 T = 0, \qquad \text{and} \qquad X'' + \beta^2 X = 0.$$

The solutions of these ODEs are

$$T(t) = A\cos\beta ct + B\sin\beta ct, \qquad \text{and} \qquad X(x) = C\cos\beta x + D\sin\beta x, \tag{4}$$

where A, B, C and D are arbitrary constants. From the boundary conditions in (1), we have

$$X(0)T(t) = X(l)T(t) = 0, \quad \forall t \qquad \Rightarrow \qquad X(0) = X(l) = 0,$$

since $T(t) \equiv 0$ would result in the trivial solution $u(x, t) \equiv 0$ (our goal is to find all separated solutions). With this boundary condition for $X(x)$, we have from (4)

$$X(0) = C = 0, \qquad \text{and} \qquad X(l) = D\sin\beta l = 0.$$

The solution with $D = 0$ will again lead to the trivial zero solution, so we consider the case when $\sin\beta l = 0$. But this implies that $\beta l = n\pi$ for $n = 1, 2, \dots$, and

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \qquad X_n(x) = \sin\frac{n\pi x}{l} \quad \text{for } n = 1, 2, \dots$$

These formulas give distinct solutions for $X(x)$, and multiplying these by the $T(t)$ corresponding to λ_n, we find infinitely many separated solutions

$$u_n(x,t) = \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l} \quad \text{for } n = 1, 2, \ldots,$$

where A_n, B_n are arbitrary constants as before. Since a linear combination of solutions of the wave equation is also a solution, any finite sum

$$u(x,t) = \sum_n \left(A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}\right) \sin \frac{n\pi x}{l}, \tag{5}$$

will also solve the wave equation.

Returning to our boundary value problem (1), we would like to find the solution as a linear combination of separated solutions. However, finite sums in the form (5) are very special, since not every function is a finite sum of sines and cosines. Checking the initial conditions, we have

$$\begin{aligned} \phi(x) &= \sum_n A_n \sin \frac{n\pi x}{l}, \\ \psi(x) &= \sum_n \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l}. \end{aligned} \tag{6}$$

Obviously, not all initial data ϕ, ψ can be written as finite sums of sine functions. So instead of restricting ourselves to finite sums, we allow infinite sums, and ask the question whether any functions ϕ, ψ can be written as infinite sums of sine functions. This question was first studied by Fourier, and these infinite sums have the name of *Fourier series* (Fourier sine series in this case). It turns out that practically any function defined on $0 < x < l$ can be expressed in the form (6). Leaving the question of convergence of such sums, we see that if the initial data can be expressed in the form (6), then the solution is given by (5).

The coefficients of t inside the series (5), $\frac{n\pi c}{l}$, are called the frequencies. For a violin string of length l, we had $c^2 = \frac{T}{\rho}$, so the frequencies are

$$\frac{n\pi\sqrt{T}}{l\sqrt{\rho}} \quad n = 1, 2, \ldots$$

The smallest frequency, $\frac{\pi\sqrt{T}}{l\sqrt{\rho}}$, is the fundamental note, while the double, triple, and so on of the fundamental note are the overtones. Notice that by shortening the length l of the vibrating portion of the string with a finger, a violinist produces notes of higher frequency.

17.1 Heat equation

For the Dirichlet heat problem on the finite interval,

$$\begin{cases} u_t - ku_{xx} = 0, & \text{for } 0 < x < l, \\ u(x,0) = \phi(x), \\ u(0,t) = u(l,t) = 0, \end{cases} \tag{7}$$

we similarly search for all the separated solutions in the form $u(x,t) = X(x)T(t)$. In this case the equation gives

$$-\frac{X''}{X} = -\frac{T'}{kt} = \beta^2,$$

and the resulting ODEs are

$$T' = -\beta^2 kT, \qquad \text{and} \qquad X'' + \beta^2 X = 0.$$

The solution for the T equation is then $T(t) = Ae^{-\beta^2 kt}$, while the function $X(x)$ satisfies the same equation and boundary conditions as before. This yields the same values $\beta_n = n\pi/l$. We thus have that

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \sin \frac{n\pi x}{l} \tag{8}$$

is the solution to problem (7), provided that the initial data is given as

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}. \tag{9}$$

Notice that as t grows, all the terms in the series (8) decay exponentially, so the solution itself will decay, which makes sense in terms of heat conduction, since in the absence of a heat source, the temperatures in the rod will equalize with the zero temperature of the environment.

Example 17.1. Solve the following Dirichlet problem for the heat equation by separation of variables.

$$\begin{cases} u_t - ku_{xx} = 0, \quad \text{for } 0 < x < \pi/2, \\ u(x,0) = 3\sin 4x, \\ u(0,t) = u(\frac{\pi}{2},t) = 0, \end{cases}$$

In this problem $l = \pi/2$, so $\beta_n = 2n$. We can write the initial data in the form (9),

$$3\sin 4x = \sum_{n=1}^{\infty} A_n \sin 2nx,$$

which implies that $A_2 = 3$, and $A_n = 0$ for $n \neq 2$. But then from (8) the solution will be

$$u(x,t) = 3e^{-16kt} \sin 4x.$$

□

17.2 Eigenvalues

The numbers $\lambda = \left(\frac{n\pi}{l}\right)^2$ are called eigenvalues, and the functions $X_n(x) = \sin \frac{n\pi x}{l}$ are called eigenfunctions. Notice that we can think of the equation $-X'' = \lambda X$ as an eigenvalue problem for the operator $-\frac{d^2}{dx^2}$ in the space of functions that satisfy the Dirichlet conditions $X(0) = X(l) = 0$. An eigenfunction is then a solution of the equation which is not identically zero, i.e. $X(x) \not\equiv 0$.

However, unlike the operators in linear algebra, which have finitely many eigenvalues, in our case we have an infinite number of eigenvalues. This is due to the fact that the space of functions is infinite dimensional.

We return to the question of the sign of the eigenvalues. Suppose $\lambda = 0$, then we would have $X'' = 0$, which leads to $X(x) = C + Dx$. The boundary conditions then imply that $C = 0$, and $Dl = 0$, giving $X(x) \equiv 0$.

If, on the other hand, we assume that $\lambda < 0$, and write $\lambda = -\gamma^2$ for some $\gamma > 0$, then the equation for X becomes $X'' = \gamma^2 X$, which has the solution

$$X(x) = Ce^{\gamma x} + De^{-\gamma x}.$$

The boundary conditions then give

$$\begin{cases} C + D = 0 \\ Ce^{\gamma l} + De^{-\gamma l} = 0 \end{cases} \Rightarrow \begin{cases} C = -D \\ Ce^{2\gamma l} = C \end{cases} \Rightarrow C = D = 0,$$

which again results in the identically zero solution $X(x) \equiv 0$. So there are no nonpositive eigenvalues.

17.3 Conclusion

Returning to the Dirichlet problems for the wave and heat equations on a finite interval, we solved them with the method of separation of variables. That is, we looked for the solution in the form of an infinite linear combination of separated solutions. This lead to infinite series, which solve the appropriate initial-boundary value problems, as long as the initial data can be expanded in corresponding series, which in the case of Dirichlet conditions are the Fourier sine series. The question of convergence of such series will be discussed next quarter, while the case of Neumann conditions will be considered next time.