## 18 Separation of variables: Neumann conditions

The same method of separation of variables that we discussed last time for boundary problems with Dirichlet conditions can be applied to problems with Neumann, and more generally, Robin boundary conditions. We illustrate this in the case of Neumann conditions for the wave and heat equations on the finite interval.

Substituting the separated solution $u(x,t) = X(x)T(t)$ into the wave Neumann problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & 0 < x < l, \\ u(x,0) = \phi(x), & u_t(x,0) = \psi(x), \\ u_x(0,t) = u_x(l,t) = 0, \end{cases} \tag{1}$$

gives the same equations for $X$ and $T$ as in the Dirichlet case,

$$-X'' = \lambda X, \qquad \text{and} \qquad -T'' = c\lambda^2 T.$$

However, the boundary conditions now imply

$$X'(0)T(t) = X'(l)T(t) = 0, \quad \forall t \qquad \Rightarrow \qquad X'(0) = X'(l) = 0.$$

To find all the separated solutions, we need to find all the eigenvalues and eigenfunctions satisfying these boundary conditions. To do this, we need to consider the cases $\lambda = 0$, $\lambda < 0$ and $\lambda > 0$ separately.

Assume $\lambda = 0$, then the equation for $X$ is $X'' = 0$, which has the solution $X(x) = C + Dx$. The derivative is then $X'(x) = D$, and the boundary conditions imply that $D = 0$. So every constant function, $X(x) = C$, is an eigenfunction for the eigenvalue $\lambda_0 = 0$.

Next, we assume that $\lambda = -\gamma^2 < 0$, in which case the equation for $X$ takes the form

$$X'' = \gamma^2 X.$$

The solution to this equation is $X(x) = Ce^{\gamma x} + De^{\gamma x}$, so $X'(x) = C\gamma e^{\gamma x} - D\gamma e^{-\gamma x}$. Checking the boundary conditions gives

$$\begin{cases} C\gamma - D\gamma = 0 \\ C\gamma e^{\gamma l} - D\gamma e^{-\gamma l} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} C = D \\ C\gamma(e^{2\gamma l} - 1) = 0 \end{cases} \quad \Rightarrow \quad C = D = 0,$$

since $\gamma \neq 0$, and hence, also $e^{2\gamma l} - 1 \neq 0$. This leads to the identically zero solution $X(x) \equiv 0$, which means that there are no negative eigenvalues.

For the remaining case, $\lambda = \beta^2 > 0$, the equation is $X'' = -\beta^2 X$, which as we saw last time when discussing Dirichlet boundary conditions, has the solution

$$X(x) = C\cos\beta x + D\sin\beta x.$$

The derivative of this function is

$$X'(x) = -C\beta\sin\beta x + D\beta\cos\beta x,$$

so the boundary conditions give

$$X'(0) = D\beta = 0, \qquad \text{and} \qquad X'(l) = -C\beta\sin\beta l = 0.$$

Since $\beta \neq 0$, and $C$ and $D$ cannot be both zero, we have $\sin\beta l = 0$, which implies that $\beta l = n\pi$ for $n = 1, 2, \ldots$. Then the eigenvalues and the corresponding eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \qquad X_n(x) = \cos\frac{n\pi x}{l}, \qquad \text{for } n = 0, 1, 2, \ldots \tag{2}$$

Notice that we have also included $n = 0$, which gives the zero eigenvalue $\lambda_0 = 0$ with the eigenfunction $X_0 = 1$. The set of all eigenvalues, called the *spectrum*, for the Neumann conditions differs from that for the Dirichlet conditions by this additional eigenvalue.

In the case of $\lambda_0 = 0$ the $T$ equation becomes $T'' = 0$, which has the solution $T(t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t$. The factors of $\frac{1}{2}$ are included for future convenience (to have a single formula for the Fourier coefficients).

The solutions $T_n$ corresponding to $\lambda_n = (n\pi/l)^2$ for $n = 1, 2, \dots$, were found in the last lecture to be

$$T_n(t) = A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}.$$

Putting everything together gives the following series expansion for the solution of problem (1),

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right)\cos\frac{n\pi x}{l}, \tag{3}$$

as long as the initial data can be expanded into cosine Fourier series

$$\begin{aligned} \phi(x) &= \frac{1}{2}A_0 + \sum_n A_n \cos\frac{n\pi x}{l}, \\ \psi(x) &= \frac{1}{2}B_0 + \sum_n \frac{n\pi c}{l} B_n \cos\frac{n\pi x}{l}. \end{aligned} \tag{4}$$

These series for the data come from plugging in $t = 0$ into the solution formula (3), and its derivative with respect to $t$. We notice that in the case of the Neumann conditions we end up with cosine Fourier series for the data, while in the Dirichlet case we had sine Fourier series. This is in agreement with the reflection method, since one needs to take the odd extensions of the data in the case of Dirichlet conditions, and even extensions in the case of Neumann conditions. But odd functions (extended data) have only sines in their Fourier expansions, while even functions have only cosines.

**Example 18.1.** Solve the following Neumann problem for the wave equation by separation of variables.

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & \text{for } 0 < x < \pi, \\ u(x,0) = 3\cos x, & u_t(x,0) = 1 - \cos 4x \\ u_x(0,t) = u_x(\pi,t) = 0, \end{cases}$$

In this problem $l = \pi$, so $\beta_n = n$. Notice also that $c = 2$, and we can write the initial data in the form (4) as follows,

$$\begin{aligned} 3\cos x &= \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx, \\ 1 - \cos 4x &= \frac{1}{2}B_0 + \sum_{n=1}^{\infty} 2nB_n \cos nx. \end{aligned}$$

These identities imply that $A_1 = 3$, and $A_n = 0$ for $n \neq 1$, and $B_0 = 2, B_4 = -\frac{1}{8}$, and $B_n = 0$ for $n \neq 0, 4$. Then solution (3) will take the form

$$u(x,t) = t + 3\cos 2t \cos x - \frac{1}{8}\sin 8t \cos 4x.$$

□

### 18.1 Heat equation

For the Neumann heat problem on the finite interval,

$$\begin{cases} u_t - k u_{xx} = 0, & \text{for } 0 < x < l, \\ u(x,0) = \phi(x), \\ u_x(0,t) = u_x(l,t) = 0, \end{cases} \tag{5}$$

the equations for $X$ and $T$ factors of the separated solution $u(x,t) = X(x)T(t)$ are

$$X'' = -\lambda X, \qquad \text{and} \qquad T' = -\lambda k T.$$

The boundary conditions are the same as in the wave problem (1), so one gets the same eigenvalues and eigenfunctions (2). For the eigenvalue $\lambda_0 = 0$, the $T$ equation is $T' = 0$, so $T_0(t) = \frac{1}{2}A_0$. For the positive eigenvalues we found the solutions for $T$ in the last lecture to be

$$T_n(t) = A_n e^{-(n\pi/l)^2 kt}.$$

Thus, the solution to the heat Neumann problem is given by the series

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \cos \frac{n\pi x}{l},$$

as long as the initial data can be expanded into the cosine Fourier series

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}.$$

### 18.2 Mixed boundary conditions

Sometimes one needs to consider problems with mixed Dirichlet-Neumann boundary conditions, i.e. Dirichlet conditions at one end of the finite interval, and Neumann conditions at the other. Examples of such problems are vibrations of a finite string with one free and one fixed end, and the heat conduction in a finite rod with one insulated end, and the other end kept at a constant zero temperature.

In such cases the method of separation of variables leads to the eigenvalue problem

$$\begin{cases} X'' = -\lambda X, \\ X(0) = X'(l) = 0, \end{cases} \qquad \text{or} \qquad \begin{cases} X'' = -\lambda X, \\ X'(0) = X(l) = 0. \end{cases}$$

One can then show that the eigenvalues are $\lambda_n = [(n + \frac{1}{2})\pi/l]^2$, and the corresponding eigenfunctions for the respective problems are

$$X_n(x) = \sin \frac{(n+\frac{1}{2})\pi x}{l}, \qquad \text{and} \qquad X_n(x) = \cos \frac{(n+\frac{1}{2})\pi x}{l}, \qquad \text{for } n = 0, 1, 2, \ldots,$$

which is left as a homework exercise.

### 18.3 Conclusion

Similar to the case of the Dirichlet problems for heat and wave equations, the method of separation of variables applied to the Neumann problems on a finite interval leads to an eigenvalue problem for the $X(x)$ factor of the separated solution. In this case, however, we discovered a new eigenvalue $\lambda = 0$ in addition to the eigenvalues found for the Dirichlet problems. Then the general solutions of the Neumann problems for wave and heat equations can be written in series forms, as (infinite) linear combinations of all separated solutions, as long as the initial data can be expanded in cosine Fourier series. We will discuss in detail the questions on whether and how a given function can be expanded into Fourier series next quarter.