2 First-order linear equations

Last time we saw how some simple PDEs can be reduced to ODEs, and subsequently solved using ODE methods. For example, the equation

$$u_x = 0 \tag{1}$$

has "constant in x" as its general solution, and hence u depends only on y, thus $u(x, y) = f(y)$ is the general solution, with f an arbitrary function of a single variable. Today we will see that any linear first order PDE can be reduced to an ordinary differential equation, which will then allow as to tackle it with already familiar methods from ODEs.

Let us start with a simple example. Consider the following constant coefficient PDE

$$au_x + bu_y = 0. \tag{2}$$

Here a and b are constants, such that $a^2 + b^2 \neq 0$, i.e. at least one of the coefficients is nonzero (otherwise this would not be a differential equation). Using the inner (scalar or dot) product in $\mathbb{R}^2$, we can rewrite the left hand side of (2) as

$$(a, b) \cdot (u_x, u_y) = 0, \qquad \text{or} \qquad (a, b) \cdot \nabla u = 0.$$

Denoting the vector $(a, b) = \mathbf{v}$, we see that the left hand side of the above equation is exactly $D_{\mathbf{v}}u(x, y)$, the directional derivative of u in the direction of the vector $\mathbf{v}$. Thus, the solution to (2) must be constant in the direction of the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$.

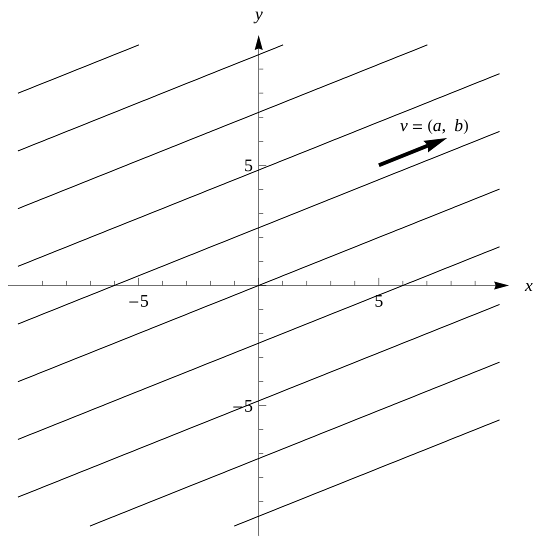

Figure 1: Characteristic lines $bx - ay = c$.

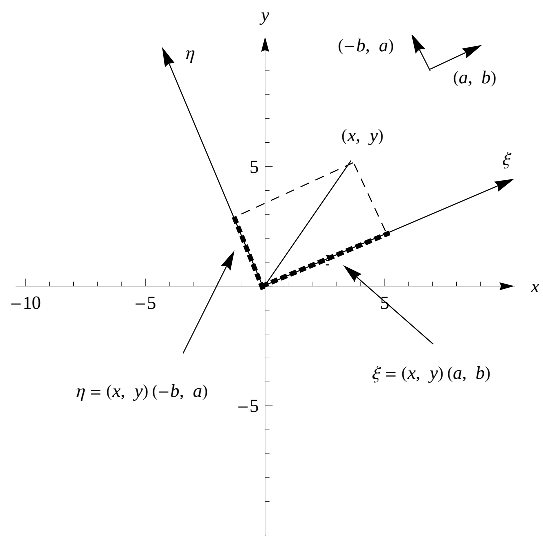

Figure 2: Change of coordinates.

The lines parallel to the vector $\mathbf{v}$ have the equation

$$bx - ay = c, \tag{3}$$

since the vector $(b, -a)$ is orthogonal to $\mathbf{v}$, and as such is a normal vector to the lines parallel to $\mathbf{v}$. In equation (3) c is an arbitrary constant, which uniquely determines the particular line in this family of parallel lines, called *characteristic lines* for the equation (2).

As we saw above, $u(x, y)$ is constant in the direction of $\mathbf{v}$, hence also along the lines (3). The line containing the point (x, y) is determined by $c = bx - ay$, thus u will depend only on $bx - ay$, that is

$$u(x, y) = f(bx - ay), \tag{4}$$

where f is an arbitrary function. One can then check that this is the correct solution by plugging it into the equation. Indeed,

$$a\partial_x f(bx-ay)+b\partial_y f(bx-ay)=abf'(bx-ay)-baf'(bx-ay)=0.$$

The geometric viewpoint that we used to arrive at the solution is akin to solving equation (1) simply by recognizing that a function with a vanishing derivative must be constant. However one can approach equation (2) from another perspective, by trying to reduce it to an ODE.

2.1 The method of characteristics

To have an ODE, we need to eliminate one of the partial derivatives in the equation. But we know that the directional derivative vanishes in the direction of the vector (a,b). Let us then make a change of the coordinate system to one that has its "x-axis" parallel to this vector, as in Figure 2. In this coordinate system

$$(\xi,\eta)=((x,y)\cdot(a,b),(x,y)\cdot(b,-a))=(ax+by,bx-ay).$$

So the change of coordinates is

$$\begin{cases}\xi=ax+by,\\ \eta=bx-ay.\end{cases}\tag{5}$$

To rewrite the equation (2) in this coordinates, notice that

$$\begin{aligned}u_x&=u_\xi\frac{\partial\xi}{\partial x}+u_\eta\frac{\partial\eta}{\partial x}=au_\xi+bu_\eta,\\ u_y&=u_\xi\frac{\partial\xi}{\partial y}+u_\eta\frac{\partial\eta}{\partial y}=bu_\xi-au_\eta.\end{aligned}$$

Thus,

$$0=au_x+bu_y=a(au_\xi+bu_\eta)+b(bu_\xi-au_\eta)=(a^2+b^2)u_\xi.$$

Now, since $a^2+b^2\neq 0$, then, as we anticipated,

$$u_\xi=0,$$

which is an ODE. We can solve this last equation just as we did in the case of equation (1), arriving at the solution

$$u(\xi,\eta)=f(\eta).$$

Changing back to the original coordinates gives $u(x,y)=f(bx-ay)$. This is the same solution that we derived with the geometric deduction. This method of reducing the PDE to an ODE is called *the method of characteristics*, and the coordinates (ξ,η) given by formulas (5) are called *characteristic coordinates.*

Example 2.1. Find the solution of the equation $3u_x-5u_y=0$ satisfying the condition $u(0,y)=\sin y$.

From the above discussion we know that u will depend only on $\eta=-5x-3y$, so $u(x,y)=f(-5x-3y)$. The solution also has to satisfy the additional condition (called initial condition), which we verify by plugging in $x=0$ into the general solution.

$$\sin y=u(0,y)=f(-3y).$$

So $f(z)=\sin(-\frac{z}{3})$, and hence $u(x,y)=\sin\left(\frac{5x+3y}{3}\right)$, which one can verify by substituting into the equation and the initial condition. □

2.2 General constant coefficient equations

We can easily adapt the method of characteristics to general constant coefficient linear first-order equations

$$au_x + bu_y + cu = g(x, y). \tag{6}$$

Recall that to find the general solution of this equation it is enough to find the general solution of the homogeneous equation

$$au_x + bu_y + cu = 0, \tag{7}$$

and add to this a particular solution of the inhomogeneous equation (6). Notice that in the characteristic coordinates (5), equation (7) will take the form

$$(a^2 + b^2)u_\xi + cu = 0, \qquad \text{or} \qquad u_\xi + \frac{c}{a^2 + b^2}u = 0,$$

which can be treated as an ODE in ξ. The solution to this ODE has the form

$$u_h(\xi, \eta) = e^{-\frac{c}{a^2+b^2}\xi} f(\eta),$$

with f again being an arbitrary single-variable function. Changing the coordinates back to the original (x, y), we will obtain the general solution to the homogeneous equation

$$u_h(x, y) = e^{-\frac{c(ax+by)}{a^2+b^2}} f(bx - ay).$$

You should verify that this indeed solves equation (7).

To find a particular solution of (6), we can use the characteristic coordinates to reduce it to the inhomogeneous ODE

$$(a^2 + b^2)u_\xi + cu = g(\xi, \eta), \qquad \text{or} \qquad u_\xi + \frac{c}{a^2 + b^2}u = \frac{g(\xi, \eta)}{a^2 + b^2}.$$

Having found the solution to the homogeneous ODE, we can find the solution to this inhomogeneous equation by e.g. variation of parameters. So the particular solution will be

$$u_p = e^{-\frac{c}{a^2+b^2}\xi} \int \frac{g(\xi, \eta)}{a^2 + b^2} e^{\frac{c}{a^2+b^2}\xi}\, d\xi.$$

The general solution of (6) is then

$$u(\xi, \eta) = u_h + u_p = e^{-\frac{c}{a^2+b^2}\xi} \left(f(\eta) + \int \frac{g(\xi, \eta)}{a^2 + b^2} e^{\frac{c}{a^2+b^2}\xi}\, d\xi \right).$$

To find the solution in terms of (x, y), one needs to first carry out the integration in ξ in the above formula, then replace ξ and η by their expressions in terms of x and y.

Example 2.2. Find the general solution of $-2u_x + 4u_y + 5u = e^{x+3y}$.

The characteristic change of coordinates for this equation is given by

$$\begin{cases} \xi = -2x + 4y, \\ \eta = 4x + 2y. \end{cases}$$

From these we can also find the expressions of x and y in terms of (ξ, η). In particular notice that $x + 3y = \frac{\xi+\eta}{2}$. In the characteristic coordinates the equation reduces to

$$20u_\xi + 5u = e^{\frac{\xi+\eta}{2}}.$$

The general solution of the homogeneous equation associated with the above equation is

$$u_h = e^{-\frac{1}{4}\xi} f(\eta),$$

and the particular solution will be

$$u_p = e^{-\frac{1}{4}\xi} \int \frac{e^{\frac{\xi+\eta}{2}}}{20} e^{\frac{1}{4}\xi}\, d\xi = e^{-\frac{1}{4}\xi} \cdot \frac{1}{15} e^{\frac{\eta}{2}} e^{\frac{3}{4}\xi} = e^{-\frac{1}{4}\xi} \cdot \frac{1}{15} e^{\frac{1}{4}(3\xi+2\eta)}.$$

Adding these two will give the general solution to the inhomogeneous equation

$$u(\xi,\eta) = e^{-\frac{1}{4}\xi} \left(f(\eta) + \frac{1}{15} e^{\frac{1}{4}(3\xi+2\eta)} \right).$$

Finally, substituting the expressions for ξ and η in terms of (x, y), we will obtain the solution

$$u(x,y) = e^{-\frac{1}{4}(2x-4y)} \left(f(4x+2y) + \frac{1}{15} e^{\frac{1}{4}(2x+16y)} \right).$$

You should check that this indeed solves the differential equation. □

2.3 Variable coefficient equations

The method of characteristics can be generalized to variable coefficient first-order linear PDEs as well, albeit the change of variables may no longer be orthogonal. The general variable coefficient linear first-order equations is

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y). \tag{8}$$

Let us first consider the following simple particular case

$$u_x + y u_y = 0. \tag{9}$$

Using our geometric intuition from the constant coefficient equations, we see that the directional derivative of u in the direction of the vector $\mathbf{v} = (1, y)$ is constant. Notice that the vector $\mathbf{v}$ itself is no longer constant, and varies with y. The curves that have $\mathbf{v}$ as their tangent vector have slope $\frac{y}{1}$, and thus satisfy

$$\frac{dy}{dx} = \frac{y}{1}.$$

We can solve this equation as an ODE, and obtain the general solution

$$y = Ce^x, \qquad \text{or} \qquad e^{-x}y = C. \tag{10}$$

As in the case of the constant coefficients, the solution to the equation (9) will be constant along these curves, called *characteristic curves*. This family of non-intersecting curves fills the entire coordinate plane, and the curve containing the point (x, y) is uniquely determined by $C = e^{-x}y$, which implies that the general solution to (9) is

$$u(x,y) = f(C) = f(e^{-x}y).$$

As always, we can check this by substitution.

$$u_x + y u_y = -f'(e^{-x}y)e^{-x}y + y f'(e^{-x}y)e^{-x} = 0.$$

Let us now try to generalize the method of characteristics to the equation

$$a(x,y)u_x + b(x,y)u_y = 0. \tag{11}$$

The idea is again to introduce new coordinates (ξ, η), which will reduce (11) to an ODE. Suppose

$$\begin{cases} \xi = \xi(x,y), \\ \eta = \eta(x,y) \end{cases} \tag{12}$$

gives such a change of variables. To rewrite the equation in these coordinates, we compute

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x, \\ u_y &= u_\xi \xi_y + u_\eta \eta_y, \end{aligned}$$

and substitute these into equation (11) to get

$$(a\xi_x + b\xi_y)u_\xi + (a\eta_x + b\eta_y)u_\eta = 0.$$

Since we would like this to give us an ODE, say in ξ, we require the coefficient of u_η to be zero,

$$a\eta_x + b\eta_y = 0.$$

Without loss of generality, we may assume that $a \neq 0$ (locally). Notice that for curves $y(x)$ that have the slope $\frac{dy}{dx} = \frac{b}{a}$ we have

$$\frac{d}{dx}\eta(x, y(x)) = \eta_x + \eta_y \frac{dy}{dx} = \eta_x + \frac{b}{a}\eta_y = 0.$$

So the characteristic curves, just as before, are given by

$$\frac{dy}{dx} = \frac{b}{a}. \tag{13}$$

The general solution to this ODE will be $\eta(x,y) = C$, with $\eta_y \neq 0$ (otherwise $\eta_x = 0$ as well, and this will not be a solution). This is how we find the new variable η, for which our PDE reduces to an ODE. We choose $\xi(x,y) = x$ as the other variable. For this change of coordinates the Jacobian determinant is

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \eta_y \neq 0.$$

Thus, (12) constitutes a non-degenerate change of coordinates. In the new variables equation (11) reduces to

$$a(\xi,\eta)u_\xi = 0, \quad \text{hence} \quad u_\xi = 0,$$

which has the solution

$$u = f(\eta).$$

The general variable coefficient equation (8) in these coordinates will reduce to

$$a(\xi,\eta)u_\xi + c(\xi,\eta)u = d(\xi,\eta),$$

which is called the *canonical form* of equation (8). This equation, as in previous cases, can be solved by standard ODE methods.

Example 2.3. Find the general solution of the equation

$$xu_x - yu_y + y^2 u = y^2, \qquad x, y \neq 0.$$

Condition (13) in this case is $\frac{dy}{dx} = -\frac{y}{x}$. This is a separable ODE, which can be solved to obtain the general solution $xy = C$. Thus, our change of coordinates will be

$$\begin{cases} \xi = x, \\ \eta = xy. \end{cases}$$

In these coordinates the equation takes the form

$$\xi u_\xi + \frac{\eta^2}{\xi^2} u = \frac{\eta^2}{\xi^2}, \quad \text{or} \quad u_\xi + \frac{\eta^2}{\xi^3} u = \frac{\eta^2}{\xi^3}.$$

Using the integrating factor

$$e^{\int \frac{\eta^2}{\xi^3}\, d\xi} = e^{-\frac{\eta^2}{2\xi^2}},$$

the above equation can be written as

$$\left(e^{-\frac{\eta^2}{2\xi^2}} u\right)_\xi = e^{-\frac{\eta^2}{2\xi^2}} \frac{\eta^2}{\xi^3}.$$

Integrating both sides in ξ, we arrive at

$$e^{-\frac{\eta^2}{2\xi^2}} u = \int e^{-\frac{\eta^2}{2\xi^2}} \frac{\eta^2}{\xi^3}\, d\xi = e^{-\frac{\eta^2}{2\xi^2}} + f(\eta).$$

Thus, the general solution will be given by

$$u(\xi, \eta) = e^{\frac{\eta^2}{2\xi^2}} \left(f(\eta) + e^{-\frac{\eta^2}{2\xi^2}} \right) = e^{\frac{\eta^2}{2\xi^2}} f(\eta) + 1.$$

Finally, substituting the expressions of ξ and η in terms of (x, y) into the solution, we obtain

$$u(x, y) = f(xy) e^{\frac{y^2}{2}} + 1.$$

One should again check by substitution that this is indeed a solution to the PDE. $\square$

2.4 Conclusion

The method of characteristics is a powerful method that allows one to reduce *any* first-order linear PDE to an ODE, which can be subsequently solved using ODE techniques. We will see in later lectures that a subclass of second order PDEs – second order hyperbolic equations - can be also treated with a similar characteristic method.