3 Method of characteristics revisited

3.1 Transport equation

A particular example of a first order constant coefficient linear equation is the transport, or advection equation $u_t + cu_x = 0$, which describes motions with constant speed. One way to derive the transport equation is to consider the dynamics of the concentration of a pollutant in a stream of water flowing through a thin tube at a constant speed c.

Let $u(t, x)$ denote the concentration of the pollutant in gr/cm (unit mass per unit length) at time t. The amount of pollutant in the interval $[a, b]$ at time t is then

$$\int_a^b u(x,t)\,dx.$$

Due to conservation of mass, the above quantity must be equal to the amount of the pollutant after some time h. After the time h, the pollutant would have flown to the interval $[a + ch, b + ch]$, thus the conservation of mass gives

$$\int_a^b u(x,t)\,dx = \int_{a+ch}^{b+ch} u(x,t+h)\,dx.$$

To derive the dynamics of the concentration $u(x, t)$, differentiate the above identity with respect to b to get

$$u(b,t) = u(b+ch, t+h).$$

Notice that this equation asserts that the concentration at the point b at time t is equal to the concentration at the point $b + ch$ at time $t + h$, which is to be expected, due to the fact that the water containing the pollutant particles flows with a constant speed. Since b is arbitrary in the last equation, we replace it with x. Now differentiate both sides of the equation with respect to h, and set h equal to zero to obtain the following differential equation for $u(x, t)$.

$$0 = cu_x(x,t) + u_t(x,t),$$

or

$$u_t + cu_x = 0. \tag{1}$$

Since equation (1) is a first order linear PDE with constant coefficients, we can solve it by the method of characteristics. First, we rewrite the equation as

$$(1, c) \cdot \nabla u = 0,$$

which implies that the slope of the characteristic lines is given by

$$\frac{dx}{dt} = \frac{c}{1}.$$

Integrating this equation, one arrives at the equation for the characteristic lines

$$x = ct + x(0), \tag{2}$$

where $x(0)$ is the coordinate of the point at which the characteristic line intersects the x-axis. The solution to the PDE (1) can then be written as

$$u(t,x) = f(x - ct) \tag{3}$$

for any arbitrary single-variable function f.

Let us now consider a particular initial condition for $u(t,x)$

$$u(0,x) = \begin{cases} x & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

According to (3), $u(0,x) = f(x)$, which determines the function f. Having found the function from the initial condition, we can now evaluate the solution $u(t,x)$ of the transport equation from (3). Indeed

$$u(t,x) = f(x - ct) = \begin{cases} x - ct & 0 < x - ct < 1 \\ 0 & \text{otherwise} \end{cases}$$

Noticing that the inequalities $0 < x - ct < 1$ imply that x is in-between ct and $ct + 1$, we can rewrite the above solution as

$$u(t,x) = \begin{cases} x - ct & ct < x < ct + 1, \\ 0 & \text{otherwise,} \end{cases}$$

which is exactly the initial function $u(0,x)$, given by (4), moved to the right along the x-axis by ct units. Thus, the *initial data* $u(0,x)$ travels from left to right with constant speed c.

We can alternatively understand the dynamics by looking at the characteristic lines in the xt coordinate plane. From (2), we can rewrite the characteristics as

$$t = \frac{1}{c}(x - x(0)).$$

Along these characteristics the solution remains constant, and one can obtain the value of the solution at any point (t,x) by tracing it back to the x-axis:

$$u(t,x) = u(t - t, x - ct) = u(0, x(0)).$$

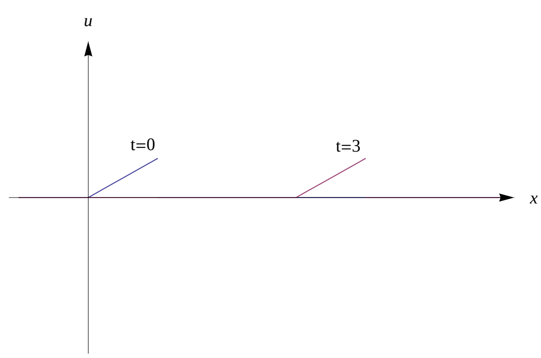

Figure 1: $u(t,x)$ at two different times t.

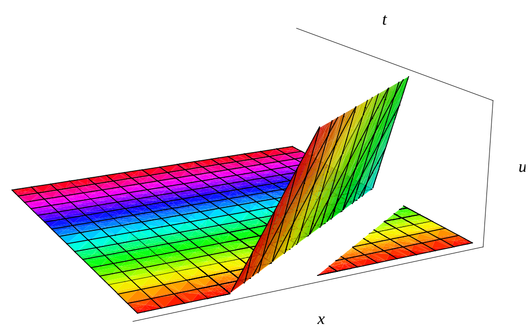

Figure 2: The graph of $u(t,x)$ colored with respect to time t.

Figure 1 gives the graphs of $u(t,x)$ at two different times, while Figure 3.1 gives the three dimensional graph of $u(t,x)$ as a function of two variables.

3.2 Quasilinear equations

We next look at a simple nonlinear equation, for which the method of characteristics can be applied as well. The general first order quasilinear equation has the following form

$$a(x,y,u)u_x + b(x,y,u)u_y = g(x,y,u).$$

We can see that the highest order derivatives, in this case the first order derivatives, enter the equation linearly, although the coefficients depend on the unknown u. A very particular example of first order quasilinear equations is the inviscid Burger's equation

$$u_t + uu_x = 0. \tag{5}$$

As before, we can rewrite this equation in the form of a dot product, which is a vanishing condition for a certain directional derivative

$$(1, u) \cdot (u_t, u_x) = 0, \qquad \text{or} \qquad (1, u) \cdot \nabla u = 0.$$

This shows that the tangent vector of the characteristic curves, $\mathbf{v} = (1, u)$, depends on the unknown function u.

3.3 Rarefaction

Let as now look at a particular initial condition, and try to construct solutions along the characteristic curves. Suppose

$$u(0, x) = \begin{cases} 1 & \text{if } x < 0, \\ 2 & \text{if } x > 0. \end{cases} \tag{6}$$

The slope of the characteristic curves satisfies

$$\frac{dx}{dt} = u(t, x(t)) = u(0, x(0)).$$

Here we used the fact that the directional derivative of the solution vanishes in the direction of the tangent vector of the characteristic curves. This implies that the solution remains constant along the characteristics, i.e. $u(t, x(t))$ remains constant for all values of t. We can find the equation of the characteristics by integrating the above equation, which gives

$$x(t) = u(0, x(0))t + x(0). \tag{7}$$

Using the initial condition (6), this equation will become

$$x(t) = \begin{cases} t + x(0) & \text{if } x(0) < 0, \\ 2t + x(0) & \text{if } x(0) > 0. \end{cases}$$

Thus, the characteristics have different slopes depending on whether they intersect the x axis at a positive, or negative intercept $x(0)$. We can express the characteristic lines to give t as a function of x, so that the initial condition is defined along the horizontal x axis.

$$t = \begin{cases} x - x(0) & \text{if } x(0) < 0, \\ \frac{1}{2}(x - x(0)) & \text{if } x(0) > 0. \end{cases} \tag{8}$$

Some of the characteristic lines corresponding to the initial condition (6) are sketched in Figure 3 below. The solid lines are the two families of characteristics with different slopes.

Notice that in this case the waves originating at $x(0) > 0$ move to the right faster than the waves originating at points $x(0) < 0$. Thus an increasing gap is formed between the faster moving wave front and the slower one. One can also see from Figure 3, that there are no characteristic lines from either of the two families given by (8) passing through the origin, since there is a jump discontinuity at $x = 0$ in the initial condition (6). In fact, in this case we can imagine that there are infinitely many characteristics originating from the origin with slopes ranging between $\frac{1}{2}$ and 1 (the dotted lines in Figure 3). The proper way to see this is to notice that in the case of $x(0) = 0$, (7) implies that

$$u = \frac{x}{t}, \qquad \text{if} \qquad t < x < 2t.$$

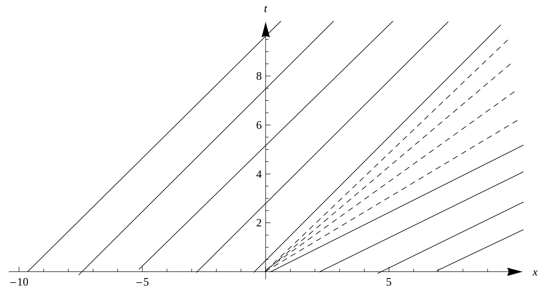

Figure 3: Characteristic lines illustrating rarefaction.

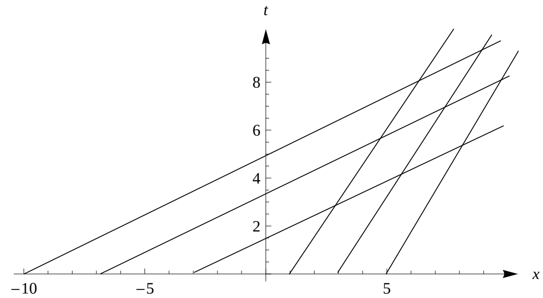

Figure 4: Characteristic lines illustrating shock wave formation.

This type of waves, which arise from decompression, or *rarefaction* of the medium due to the increasing gap formed between the wave fronts traveling at different speeds are called *rarefaction waves*. Putting all the pieces together, we can write the solution to equation (5) satisfying initial condition (6) as follows.

$$u(t,x) = \begin{cases} 1 & \text{if } x < t, \\ \frac{x}{t} & \text{if } t < x < 2t, \\ 2 & \text{if } x > 2t. \end{cases}$$

3.4 Shock waves

A completely opposite phenomenon to rarefaction is seen when one has a faster wave moving from left to right catching up to a slower wave. To illustrate this, let us consider the following initial condition for the Burger's inviscid equation

$$u(0,x) = \begin{cases} 2 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases} \tag{9}$$

Then the characteristic lines (7) will take the form

$$x(t) = \begin{cases} 2t + x(0) & \text{if } x(0) < 0, \\ t + x(0) & \text{if } x(0) > 0. \end{cases}$$

Or expressing t in terms of x, we can write the equations as

$$t = \begin{cases} \frac{1}{2}(x - x(0)) & \text{if } x(0) < 0, \\ x - x(0) & \text{if } x(0) > 0. \end{cases} \tag{10}$$

Thus, the characteristics origination from $x(0) < 0$ have smaller slope (corresponding to faster speed), than the characteristics originating from $x(0) > 0$. In this case the characteristics from the two families will intersect eventually, as seen in Figure 4. At the intersection points the solution u becomes multivalued, since the point can be traced back along either of the characteristics to an initial value of either 1, or 2, given by the initial condition (9). This phenomenon is known as *shock waves*, since the faster moving wave catches up to the slower moving wave to form a multivalued (multicrest) wave.

3.5 Conclusion

We saw that the method of characteristics can be generalized to quasilinear equations as well. Using the method of characteristics for the inviscid Burger's equations, we discovered that in the case of nonlinear equations one may encounter characteristics that diverge from each other to give rise to an unexpected solution in the widening region in-between, as well as intersecting characteristics, leading to multivalued solutions. These are nonlinear phenomena, and do not arise in the study of linear PDEs.