

4 Vibrations and heat flow

In this lecture we will derive the wave and heat equations from physical principles. These are second order constant coefficient linear PDEs, which model mechanical vibrations and thermal flow respectively.

4.1 Vibrating string

Consider a thin string of length l , which undergoes relatively small transverse vibrations (think of a string of a musical instrument, say a violin string). Let ρ be the linear density of the string, measured in units of mass per unit of length. We will assume that the string is made of homogeneous material and its density is constant along the entire length of the string. The displacement of the string from its equilibrium state at time t and position x will be denoted by $u(t, x)$. We will position the string on the x -axis with its left endpoint coinciding with the origin of the xu coordinate system.

Considering the motion of a small portion of the string sitting atop the interval $[a, b]$, which has mass $\rho(b - a)$, and acceleration u_{tt} , we can write Newton's second law of motion (balance of forces) as follows

$$\rho(b - a)u_{tt} = \text{Total force.} \quad (1)$$

Having a thin string with negligible mass, we can ignore the effect of gravity on the string, as well as air resistance, and other external forces. The only force acting on the string is then the tension force. Assuming that the string is perfectly flexible, the tension force will have the direction of the tangent vector along the string. At a fixed time t the position of the string is given by the parametric equations

$$\begin{cases} x = x, \\ u = u(x, t), \end{cases}$$

where x plays the role of a parameter. The tangent vector is then $(1, u_x)$, and the unit tangent vector will be $\left(\frac{1}{\sqrt{1+u_x^2}}, \frac{u_x}{\sqrt{1+u_x^2}}\right)$. Thus, we can write the tension force as

$$\mathbf{T}(x, t) = T(x, t) \left(\frac{1}{\sqrt{1+u_x^2}}, \frac{u_x}{\sqrt{1+u_x^2}} \right) = \frac{1}{\sqrt{1+u_x^2}}(T, Tu_x), \quad (2)$$

where $T(t, x)$ is the magnitude of the tension. Since we consider only small vibrations, it is safe to assume that u_x is small, and the following approximation via the Taylor's expansion can be used

$$\sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + O(u_x^4) \approx 1.$$

Substituting this approximation into (2), we arrive at the following form of the tension force

$$\mathbf{T} = (T, Tu_x).$$

With our previous assumption that the motion is transverse, i.e. there is no longitudinal displacement (along the x -axis), we arrive at the following identities for the balance of forces (1) in the x , respectively u directions

$$\begin{aligned} 0 &= T(b, t) - T(a, t) \\ \rho(b - a)u_{tt} &= T(b, t)u_x(b, t) - T(a, t)u_x(a, t). \end{aligned}$$

The first equation above merely states that in the x direction the tensions from the two edges of the small portion of the string balance each other out (no longitudinal motion). From this we can also see that the tension force is independent of the position on the string. Then the second equation can be rewritten as

$$\rho u_{tt} = T \frac{u_x(b, t) - u_x(a, t)}{b - a}.$$

Passing to the limit $b \rightarrow a$, we arrive at the wave equation

$$\rho u_{tt} = T u_{xx}, \quad \text{or} \quad u_{tt} - c^2 u_{xx} = 0,$$

where we used the notation $c^2 = T/\rho$. As we will see later, c will be the speed of wave propagation, similar to the constant appearing in the transport equation (we assume that ρ and T are independent of time, which is justified by the smallness of the vibrations).

One can generalize the wave equation to incorporate effects of other forces. Some examples follow.

a) With air resistance: force is proportional to the speed u_t

$$u_{tt} - c^2 u_{xx} + r u_t = 0, \quad r > 0.$$

b) With transverse elastic force: force is proportional to the displacement u

$$u_{tt} - c^2 u_{xx} + k u = 0, \quad k > 0.$$

c) With an externally applied force

$$u_{tt} - c^2 u_{xx} = f(x, t) \quad (\text{inhomogeneous}).$$

4.2 Vibrating drumhead

Similar to the vibrating string, one can consider a vibrating drumhead (elastic membrane), and look at the dynamic of the displacement $u(x, y, t)$, which now depends on the two spatial variables (x, y) denoting the point on the 2-dimensional space of the equilibrium state. Taking a small region on the drumhead, the tension force will again be directed tangentially to the surface, in this case along the normal vector to the boundary of the region. Its vertical component will be $T \frac{\partial u}{\partial \mathbf{n}}$, where $\frac{\partial u}{\partial \mathbf{n}}$ denotes the derivative of u in the normal direction to the boundary of the region. The vertical component of the cumulative tension force will then be

$$T_{\text{vert}} = \int_{\partial D} T \frac{\partial u}{\partial \mathbf{n}} ds = \int_{\partial D} T \nabla u \cdot \mathbf{n} ds,$$

where ∂D denotes the boundary of the region D , and the integral is taken with respect to the length element along the boundary of D . The second law of motion will then take the form

$$\int_{\partial D} T \nabla u \cdot \mathbf{n} ds = \iint_D \rho u_{tt} dx dy.$$

Using Green's theorem, we can convert the line integral on the left hand side to a two dimensional integral, and arrive at the following identity

$$\iint_D \nabla \cdot (T \nabla u) dx dy = \iint_D \rho u_{tt} dx dy.$$

Since the region D was taken arbitrarily, the integrands on both sides must be the same, resulting in

$$\rho u_{tt} = T(u_{xx} + u_{yy}), \quad \text{or} \quad u_{tt} - c^2(u_{xx} + u_{yy}) = 0.$$

This is the wave equation in two spatial dimensions. Three dimensional vibrations can be treated in much the same way, leading to the three dimensional wave equation

$$u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0.$$

Often one makes use of the operator notation

$$\Delta = \nabla \cdot \nabla = \partial_x^2 + \partial_y^2 + \dots$$

to write the wave equation in any dimension as

$$u_{tt} - c^2 \Delta u = 0.$$

The Δ operator is called Laplace's operator or the *Laplacian*.

4.3 Heat flow

Let $u(x, t)$ denote the temperature at time t at the point x in some thin insulated rod. Again, think of the rod as positioned along the x -axis in the xu coordinate system, where the vertical axis will measure the temperature. The heat, or thermal energy of a small portion of the rod situated at the interval $[a, b]$ is given by

$$H(x, t) = \int_a^b c\rho u \, dx,$$

where ρ is the linear density of the rod (mass per unit of length), and c denotes the *specific heat capacity* of the material of the rod. The instantaneous change of the heat with respect to time will be the time derivative of the above expression

$$\frac{dH}{dt} = \int_a^b c\rho u_t \, dx.$$

Since the heat cannot be lost or gained in the absence of an external heat source, the change in the heat must be balanced by the heat flux across the cross-section of the cylindrical piece around the interval $[a, b]$ (we assume that the lateral boundary of the rod is perfectly insulated). According to Fourier's law, the heat flux across the boundary will be proportional to the derivative of the temperature in the direction of the outward normal to the boundary, in this case the x -derivative.

$$\text{Heat flux} = \kappa u_x,$$

where κ denotes the thermal conductivity. Using this expression for the heat flux, and noting that the change in the internal heat of the portion of the rod is equal to the combined flux through the two ends of this portion, we have

$$\int_a^b c\rho u_t \, dx = \kappa(u_x(b, t) - u_x(a, t)).$$

Differentiating this identity with respect to b , we arrive at the heat equation

$$c\rho u_t = \kappa u_{xx}, \quad \text{or} \quad u_t - k u_{xx} = 0,$$

where we denoted $k = \frac{\kappa}{c\rho}$.

As in the case of the wave equation, one can consider higher dimensional heat flows (heat flow in a two dimensional plate, or a three dimensional solid) to arrive at the general heat equation

$$u_t - k\Delta u = 0.$$

We also note that diffusion phenomena lead to an equation which has the same form as the heat equation (cf. Strauss for the actual derivation, where instead of Fourier's law of heat conduction one uses Fick's law of diffusion).

4.4 Stationary waves and heat distribution

If one looks at vibrations or heat flows where the physical state does not change with time, then $u_t = u_{tt} = 0$, and both the wave and the heat equations reduce to

$$\Delta u = 0. \tag{3}$$

This equation is called the *Laplace equation*. Notice that in the one dimensional case (3) reduces to

$$u_{xx} = 0,$$

which has the general solution $u(x, t) = c_1 x + c_2$ (remember that u is independent of t). The solutions to the Laplace equation are called *harmonic functions*, and we will see later in the course that one encounters nontrivial harmonic function in higher dimensions.

4.5 Boundary conditions

We saw in previous lectures that PDEs generally have infinitely many solutions. One then imposes some auxiliary conditions to single out relevant solutions. Since the equations that we study come from physical considerations, the auxiliary conditions must be physically relevant as well. These conditions come in two different varieties: initial conditions and boundary value conditions.

The initial condition, familiar from the theory of ODEs, specifies the physical state at some particular time t_0 . For example for the heat equation one would specify the initial temperature, which in general will be different at different points of the rod,

$$u(x, 0) = \phi(x). \quad (4)$$

For the vibrating string, one needs to specify both the initial position of (each point of) the string, and the initial velocity, since the equation is of second order in time,

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x). \quad (5)$$

In the physical examples that we considered at the beginning of this lecture, it is clear that there is a domain on which the solutions must live. For example in the case of the vibrating string of length l , we only look at the amplitude $u(t, x)$, where $0 \leq x \leq l$. Similarly, for the heat conduction in an insulated rod. In higher dimensions the domain is bounded by curves (2d), surfaces (3d) or higher dimensional geometric shapes. The boundary of this domain is where the system interacts with the external factors, and one needs to impose boundary conditions to account for these interactions.

There are three important kinds of boundary conditions:

- (D) the value (on the boundary) of u is specified (*Dirichlet condition*)
- (N) the value of the normal derivative $\partial u / \partial n$ is specified (*Neumann condition*)
- (R) the value of $\partial u / \partial n + au$ is specified (*Robin condition*, a is a function of x, y, z, \dots and t)

These conditions are usually written as equations, for example the Dirichlet condition

$$u(\mathbf{x}, t)|_{\partial D} = f(\mathbf{x}, t),$$

or the Robin condition

$$\frac{\partial u}{\partial \mathbf{n}} + au|_{\partial D} = h(\mathbf{x}, t).$$

The condition is called *homogeneous*, if the right hand side vanishes for all values of the variables.

In one dimensional case, the domain D is an interval ($0 < x < l$), hence the boundary consists of just two points $x = 0$, and $x = l$. The boundary conditions then take the simple form

- (D) $u(0, t) = g(t)$ and $u(l, t) = h(t)$
- (N) $u_x(0, t) = g(t)$ and $u_x(l, t) = h(t)$
- (R) $u(0, t) + a(t)u_x(0, t) = g(t)$ and $u(l, t) + b(t)u_x(l, t) = h(t)$

4.6 Examples of physical boundary conditions

In the case of a vibrating string, one can impose the condition that the endpoints of the string remain fixed (the case for strings of musical instruments). This gives the Dirichlet conditions $u(0, t) = u(l, t) = 0$.

If one end is allowed to move freely in the transverse direction, then the lack of any tension force at this endpoint can be expressed as the Neumann condition $u_x(l, 0) = 0$.

A Robin condition will correspond to the case when one endpoint is allowed to move transversely, but the motion is restricted by a force proportional to the amplitude u (think of a coiled spring attached to the endpoint).

In the case of heat conduction in a rod, the perfect insulation of the rod surface leads to the Neumann condition $\partial u / \partial n = 0$ (no exchange of heat with the ambient space).

If the endpoints of the rod are kept in a thermal balance with the ambient temperature $g(t)$, then one has the Dirichlet condition $u = g(t)$ on the boundary.

If one allows thermal exchange between the rod and the ambient space obeying Newton's law of cooling, then the boundary condition takes the form of a Robin condition $u_x(l, t) = -a(u(l, t) - g(t))$.

4.7 Conclusion

We derived the wave and heat equations from physical principles, identifying the unknown function with the amplitude of a vibrating string in the first case and the temperature in a rod in the second case. Understanding the physical significance of these PDEs will help us better grasp the qualitative behavior of their solutions, which will be derived by purely mathematical techniques in the subsequent lectures. The physicality of the initial and boundary conditions will also help us immediately rule out solutions that do not conform to the physical laws behind the appropriate problems.