5 Classification of second order linear PDEs

Last time we derived the wave and heat equations from physical principles. We also saw that Laplace's equation describes the steady physical state of the wave and heat conduction phenomena. Today we will consider the general second order linear PDE and will reduce it to one of three distinct types of equations that have the wave, heat and Laplace's equations as their canonical forms. Knowing the type of the equation allows one to use relevant methods for studying it, which are quite different depending on the type of the equation. One should compare this to the conic sections, which arise as different types of second order algebraic equations (quadrics). Since the hyperbola, given by the equation $x^2 - y^2 = 1$, has very different properties from the parabola $x^2 - y = 0$, it is expected that the same holds true for the wave and heat equations as well. For conic sections, one uses change of variables to reduce the general second order equation to a simpler form, which are then classified according to the form of the reduced equation. We will see that a similar procedure works for second order PDEs as well.

The general second order linear PDE has the following form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

where the coefficients A, B, C, D, F and the free term G are in general functions of the independent variables x, y, but do not depend on the unknown function u. The classification of second order equations depends on the form of the leading part of the equations consisting of the second order terms. So, for simplicity of notation, we combine the lower order terms and rewrite the above equation in the following form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + I(x, y, u, u_x, u_y) = 0. \tag{1}$$

As we will see, the type of the above equation depends on the sign of the quantity

$$\Delta(x, y) = B^2(x, y) - 4A(x, y)C(x, y), \tag{2}$$

which is called the *discriminant* for (1). The classification of second order linear PDEs is given by the following.

Definition 5.1. At the point (x_0, y_0) the second order linear PDE (1) is called

i) *hyperbolic*, if $\Delta(x_0, y_0) > 0$

ii) *parabolic*, if $\Delta(x_0, y_0) = 0$

ii) *elliptic*, if $\Delta(x_0, y_0) < 0$

Notice that in general a second order equation may be of one type at a specific point, and of another type at some other point. In order to illustrate the significance of the discriminant $\Delta = B^2 - 4AC$, we next describe how one reduces equation (1) to a canonical form. Similar to the second order algebraic equations, we use change of coordinates to reduce the equation to a simpler form. Define the new variables as

$$\begin{cases} \xi = \xi(x, y), \\ \eta = \eta(x, y), \end{cases} \qquad \text{with} \qquad J = \det\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0. \tag{3}$$

We then use the chain rule to compute the terms of the equation (1) in these new variables.

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x, \\ u_y &= u_\xi \xi_y + u_\eta \eta_y. \end{aligned}$$

To express the second order derivatives in terms of the (ξ, η) variables, differentiate the above expressions for the first derivatives using the chain rule again.

$$\begin{aligned} u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + \text{l.o.t}, \\ u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \eta_x\xi_y) + u_{\eta\eta}\eta_x\eta_y + \text{l.o.t}, \\ u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + \text{l.o.t}. \end{aligned}$$

Here *l.o.t.* stands for the low order terms, which contain only one derivative of the unknown u. Using these expressions for the second order derivatives of u, we can rewrite equation (1) in these variables as

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + I^*(\xi, \eta, u, u_\xi, u_\eta) = 0, \tag{4}$$

where the new coefficients of the higher order terms A^*, B^* and C^* are expressed via the original coefficients and the change of variables formulas as follows.

$$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2, \tag{5}$$
$$B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \eta_x\xi_y) + 2C\xi_y\eta_y, \tag{6}$$
$$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2. \tag{7}$$

One can form the discriminant for the equation in the new variables via the new coefficients in the obvious way,

$$\Delta^* = (B^*)^2 - 4A^*C^*.$$

We need to guarantee that the reduced equation will have the same type as the original equation. Otherwise, the classification given by Definition 5.1 is meaningless, since in that case the same physical phenomenon will be described by equations of different types, depending on the particular coordinate system in which one chooses to view them. The following statement provides such a guarantee.

Theorem 5.2. *The discriminant of the equation in the new variables can be expressed in terms of the discriminant of the original equation* (1) *as follows*

$$\Delta^* = J^2\Delta,$$

where J is the Jacobian determinant of the change of variables given by (3).

As a simple corollary, the type of the equation is invariant under nondegenerate coordinate transformations, since the signs of Δ and Δ^* coincide.

This theorem can be proved by a straightforward, although somewhat messy, calculation to express Δ^* in terms of the coefficients of the original equation (1). The bottom line of the theorem is that we can perform any nondegenerate change of variables to reduce the equation, while the type remains unchanged. Let us now try to construct such transformations, which will make one, or possibly two of the coefficients of the leading second order terms of equation (4) vanish, thus reducing the equation to a simpler form.

For simplicity, we assume that the coefficients A, B and C are constant. The material of this lecture can be extended to the variable coefficient case with minor changes, but we will not study variable coefficient second order PDEs in this class.

Notice that the expressions for A^* and C^* in (5), respectively (7) have the same form, with the only difference being in that the first equation contains the variable ξ, while the second one has η. Due to this, we can try to chose a transformation, which will make both A^* and C^* vanish. This is equivalent to the following equation

$$A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2 = 0. \tag{8}$$

We use ζ (*zeta*) in place of both ξ and η. The solutions to this equation are called *characteristic curves* for the second order PDE (1) (compare this to the characteristic curves for first order PDEs, where the idea was again to reduce the equation to a simpler form, in which only one of the first order derivatives appears). We divide both sides of the above equation by ζ_y^2 to get

$$A\left(\frac{\zeta_x}{\zeta_y}\right)^2 + B\left(\frac{\zeta_x}{\zeta_y}\right) + C = 0. \tag{9}$$

Without loss of generality we can assume that $A \neq 0$. Indeed, if $A = 0$, but $C \neq 0$, one can proceed in a similar way, by considering the ratio ζ_y/ζ_x instead of ζ_x/ζ_y. Otherwise, if both $A = 0$, and $C = 0$, then the equation is already in the reduced form, and there is nothing to do. Now recall that we are trying to find change of variables formulas, which are given as curves $\zeta(x, y) = \text{const}$ (fix the new variable, e.g. $\xi(x, y) = \xi_0$). Along such curves we have

$$d\zeta = \zeta_x dx + \zeta_y dy = 0.$$

Hence, the slope of the characteristic curve is given by

$$\frac{dy}{dx} = -\frac{\zeta_x}{\zeta_y}.$$

Substituting this into equation (9), we arrive at the following equation for the slope of the characteristic curve

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0.$$

Since the above is a quadratic equation, it has 2, 1, or 0 real solutions, depending on the sign of the discriminant, $B^2 - 4AC$, and the solutions are given by the quadratic formulas

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{10}$$

5.1 Hyperbolic equations

If the discriminant $\Delta > 0$, then the quadratic formulas (10) give two distinct families of characteristic curves, which will define the change of variables (3). To derive these change of variables formulas, integrate (10) to get

$$y = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} x + c, \qquad \text{or} \qquad \frac{B \pm \sqrt{B^2 - 4AC}}{2A} x - y = c.$$

These equations give the following change of variables

$$\begin{cases} \xi = \frac{B+\sqrt{B^2-4AC}}{2A} x - y \\ \eta = \frac{B-\sqrt{B^2-4AC}}{2A} x - y \end{cases} \tag{11}$$

In these new variables $A^* = C^* = 0$, while for B^* we have from (6)

$$B^* = 2A\left(\frac{B^2 - (B^2 - 4AC)}{4A^2}\right) + B\left(-\frac{B}{2A} - \frac{B}{2A}\right) + 2C = 4C - \frac{B^2}{A} = -\frac{\Delta}{A} \neq 0.$$

One can then divide equation (4) by B^*, to arrive at the reduced equation

$$u_{\xi\eta} + \cdots = 0. \tag{12}$$

This is called the first canonical form for hyperbolic equations. Under the orthogonal transformations

$$\begin{cases} x' = \xi + \eta \\ y' = \xi - \eta \end{cases}$$

equation (12) becomes

$$u_{x'x'} - u_{y'y'} + \cdots = 0,$$

which is the second canonical form for hyperbolic equations. Notice that the last equation has exactly the same form in its leading terms as the wave equation (with $c = 1$).

5.2 Parabolic equations

In the case of parabolic equations $\Delta = B^2 - 4AC = 0$, and the quadratic formulas (10) give only one family of characteristic curves. This means that there is no change of variables that makes both A^* and C^* vanish. However we can make one of this vanish, for example A^*, by choosing ξ to be the unique solution of equation (10). We can then chose η arbitrarily, as long as the change of coordinates formulas (3) define a nondegenerate transformation. Notice that in such a case, according to Theorem 5.2, $\Delta^* = (B^*)^2 - 4A^*C^* = 0$. But then we have $B^* = \pm 2\sqrt{A^*C^*} = 0$.

We solve equation (10) by integration to get ξ, and set $\eta = x$ to arrive at the following change of variables formulas

$$\begin{cases} \xi = \frac{B}{2A}x - y \\ \eta = x \end{cases} \tag{13}$$

The Jacobian determinant of this transformation is

$$J = \begin{vmatrix} B/(2A) & -1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0.$$

Thus, the transformation (13) is indeed nondegenerate, and reduces equation (1) to the following form (after division by C^*)

$$u_{\eta\eta} + \cdots = 0,$$

which is the canonical form for parabolic PDEs. Notice that this equation has the same leading terms as the heat equation $u_{xx} - u_t = 0$.

5.3 Elliptic equations

In the case of elliptic equations $\Delta = B^2 - 4AC < 0$, and the quadratic formulas (10) give two complex conjugate solutions. We can formally solve for ξ similar to the hyperbolic case, and arrive at the formula

$$\xi = \left(\frac{B}{2A} + \frac{\sqrt{B^2 - 4AC}}{2A}i\right)x - y.$$

We define new variables (α, β) by taking respectively the real and imaginary parts of ξ.

$$\begin{cases} \alpha = \frac{B}{2A}x - y \\ \beta = \frac{\sqrt{B^2-4AC}}{2A}x \end{cases} \tag{14}$$

In these variables equation (1) has the form

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + I^{**}(\alpha, \beta, u, u_\alpha, u_\beta) = 0, \tag{15}$$

in which the coefficients will be given by formulas similar to (5)-(7) with ξ replaced by α, and η replaced by β. Computing these new coefficients we get

$$\begin{aligned} A^{**} &= A\left(\frac{B}{2A}\right)^2 - \frac{B^2}{2A} + C = \frac{4AC - B^2}{4A}, \\ B^{**} &= 2A\frac{B}{2A}\frac{\sqrt{4AC - B^2}}{2A} - B\frac{\sqrt{4AC - B^2}}{2A} = 0, \\ C^{**} &= A\frac{4AC - B^2}{2A^2} = \frac{4AC - B^2}{4A}. \end{aligned}$$

As we can see, $A^{**} = C^{**}$, and $B^{**} = 0$. This is a direct consequence of the fact that $\xi = \alpha + \beta i$ and $\eta = \alpha - \beta i$ solve equation (8). One can then divide both sides of equation (4) by $A^{**} = C^{**} \neq 0$, to arrive at the reduced equation

$$u_{\alpha\alpha} + u_{\beta\beta} + \cdots = 0,$$

which is the canonical form for elliptic PDEs. Notice that this equation has the same leading terms as the Laplace equation.

Example 5.1. Determine the regions in the xy plane where the following equation is hyperbolic, parabolic, or elliptic.

$$u_{xx} + yu_{yy} + \frac{1}{2}u_y = 0.$$

The coefficients of the leading terms in this equation are

$$A = 1, B = 0, C = y.$$

The discriminant is then $\Delta = B^2 - 4AC = -4y$. Hence the equation is hyperbolic when $y < 0$, parabolic when $y = 0$, and elliptic when $y > 0$. □

5.4 Conclusion

The second order linear PDEs can be classified into three types, which are invariant under changes of variables. The types are determined by the sign of the discriminant. This exactly corresponds to the different cases for the quadratic equation satisfied by the slope of the characteristic curves. We saw that hyperbolic equations have two distinct families of (real) characteristic curves, parabolic equations have a single family of characteristic curves, and the elliptic equations have none. All the three types of equations can be reduced to canonical forms. Hyperbolic equations reduce to a form coinciding with the wave equation in the leading terms, the parabolic equations reduce to a form modeled by the heat equation, and Laplace's equation models the canonical form of elliptic equations. Thus, the wave, heat and Laplace's equations serve as canonical models for all second order constant coefficient PDEs. We will spend the rest of the quarter studying the solutions to the wave, heat and Laplace's equations.