7 The energy method

7.1 Energy for the wave equation

Let us consider an infinite string with constant linear density ρ and tension magnitude T. The wave equation describing the vibrations of the string is then

$$\rho u_{tt} = T u_{xx}, \qquad -\infty < x < \infty. \tag{1}$$

Since this equation describes the mechanical motion of a vibrating string, we can compute the kinetic energy associated with the motion of the string. Recall that the kinetic energy is $\frac{1}{2}mv^2$. In this case the string is infinite, and the speed differs for different points on the string. However, we can still compute the energy of small pieces of the string, add them together, and pass to a limit in which the lengths of the pieces go to zero. This will result in the following integral

$$KE = \frac{1}{2}\int_{-\infty}^{\infty} \rho u_t^2\,dx.$$

We will assume that the initial data vanishes outside of a large interval $|x| \leq R$, so that the above integral is convergent due to the finite speed of propagation. We would like to see if the kinetic energy KE is conserved in time. For this, we differentiate the above integral with respect to time to see whether it is zero, as is expected for a constant function, or whether it is different from zero.

$$\frac{d}{dt}KE = \frac{1}{2}\rho\int_{-\infty}^{\infty} 2u_t u_{tt}\,dx = \int_{-\infty}^{\infty} \rho u_t u_{tt}\,dx.$$

Using the wave equation (1), we can replace the ρu_{tt} by Tu_{xx}, obtaining

$$\frac{d}{dt}KE = T\int_{-\infty}^{\infty} u_t u_{xx}\,dx.$$

The last quantity does not seem to be zero in general, thus the next best thing we can hope for, is to convert the last integral into a full derivative in time. In that case the difference of the kinetic energy and some other quantity will be conserved. To see this, we perform an integration by parts in the last integral

$$\frac{d}{dt}KE = Tu_t u_x\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} Tu_{xt}u_x\,dx.$$

Due to the finite speed of propagation, the endpoint terms vanish. The last integral is a full derivative, thus we have

$$\frac{d}{dt}KE = -\int_{-\infty}^{\infty} Tu_{xt}u_x\,dx = -\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} Tu_x^2\,dx\right).$$

Defining

$$PE = \frac{1}{2}T\int_{-\infty}^{\infty} u_x^2\,dx,$$

we see that

$$\frac{d}{dt}KE = -\frac{d}{dt}PE, \qquad \text{or} \qquad \frac{d}{dt}(KE + PE) = 0.$$

The quantity $E = KE + PE$ is then conserved, which is the total energy of the string undergoing vibrations. Notice that PE plays the role of the potential energy of a stretched string, and the conservation of energy implies conversion of the kinetic energy into the potential energy and back without a loss.

Another way to see that the energy

$$E = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + Tu_x^2)\,dx \tag{2}$$

is conserved, is to multiply equation (1) by u_t and integrate with respect to x over the real line.

$$0 = \int_{-\infty}^{\infty} \rho u_{tt} u_t \, dx - \int_{-\infty}^{\infty} T u_{xx} u_t \, dx.$$

The first integral above is a full derivative in time. Integrating by parts in the second term, and realizing that the subsequent integral is a full derivative as well, while the boundary terms vanish, we obtain the identity

$$\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} \rho u_t^2 + T u_x^2 \, dx\right) = 0,$$

which is exactly the conservation of total energy.

The conservation of energy provides a straightforward way of showing that the solution to an IVP associated with the linear equation is unique. We demonstrate this for the wave equation next, while a similar procedure will be applied to establish uniqueness of solutions for the heat IVP in the next section.

Example 7.1. Show that the initial value problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & \text{for} - \infty < x < +\infty, \\ u(x,0) = \phi(x), & u_t(x,0) = \psi(x), \end{cases} \tag{3}$$

has a unique solution.

Arguing from the inverse, let as assume that the IVP (3) has two distinct solutions, u and v. But then their difference $w = u - v$ will solve the homogeneous wave equation, and will have the initial data

$$\begin{aligned} w(x,0) = u(x,0) - v(x,0) = \phi(x) - \phi(x) \equiv 0, \\ w_t(x,0) = u_t(x,0) - v_t(x,0) = \psi(x) - \psi(x) \equiv 0. \end{aligned}$$

Hence the energy associated with the solution w at time $t = 0$ is

$$E[w](0) = \frac{1}{2}\int_{-\infty}^{\infty} [(w_t(x,0))^2 + c^2(w_x(x,0))^2] \, dx = 0$$

This differs from the energy defined above by a constant factor of $1/\rho$ (recall that $T/\rho = c^2$), and is thus still a conserved quantity. It will subsequently be zero at any later time as well. Thus,

$$E[w](t) = \frac{1}{2}\int_{-\infty}^{\infty} [(w_t(x,t))^2 + c^2(w_x(x,t))^2] \, dx = 0, \qquad \forall t.$$

But since the integrand in the expression of the energy is nonnegative, the only way the integral can be zero, is if the integrand is uniformly zero. That is,

$$\nabla w(t,x) = (w_t(x,t), w_x(x,t)) = 0, \qquad \forall x, t.$$

This implies that w is constant for all values of x and t, but since $w(x,0) \equiv 0$, the constant value must be zero. Thus,

$$u(x,t) - v(x,t) = w(x,t) \equiv 0,$$

which is in contradiction with our initial assumption that u and v are different. This implies that the solution to the IVP (3) is unique. □

The procedure used in the last example, called the *energy method*, is quite general, and works for other linear evolution equations possessing a conserved (or decaying) positive definite energy. The heat equation, considered next, is one such case.

7.2 Energy for the heat equation

We next consider the (inhomogeneous) heat equation with some auxiliary conditions, and use the energy method to show that the solution satisfying those conditions must be unique. Consider the following mixed initial-boundary value problem, which is called the *Dirichlet problem for the heat equation*

$$\begin{cases} u_t - ku_{xx} = f(x,t) \qquad \text{for} \quad 0 \le x \le l, \quad t > 0 \\ u(x,0) = \phi(x), \\ u(0,t) = g(t), \quad u(l,t) = h(t), \end{cases} \tag{4}$$

for given functions f, ϕ, g, h.

Example 7.2. Show that there is at most one solution to the Dirichlet problem (4).

Just as in the case of the wave equation, we argue from the inverse by assuming that there are two functions, u, and v, that both solve the inhomogeneous heat equation and satisfy the initial and Dirichlet boundary conditions of (4). Then their difference, $w = u - v$, satisfies the homogeneous heat equation with zero initial-boundary conditions, i.e.

$$\begin{cases} w_t - kw_{xx} = 0 \qquad \text{for} \quad 0 \le x \le l, \quad t > 0 \\ w(x,0) = 0, \\ u(0,t) = 0, \quad u(l,t) = 0, \end{cases} \tag{5}$$

Now define the following "energy"

$$E[w](t) = \frac{1}{2}\int_0^l [w(x,t)]^2 \, dx, \tag{6}$$

which is always positive, and decreasing, if w solves the heat equation. Indeed, differentiating the energy with respect to time, and using the heat equation we get

$$\frac{d}{dt}E = \int_0^l ww_t \, dx = k\int_0^l ww_{xx} \, dx.$$

Integrating by parts in the last integral gives

$$\frac{d}{dt}E = kww_x\Big|_0^l - \int_0^l w_x^2 \, dx \le 0,$$

since the boundary terms vanish due to the boundary conditions in (5), and the integrand in the last term is nonnegative.

Due to the initial condition in (5), the energy at time $t = 0$ is zero. But then using the fact that the energy is a nonnegative decreasing quantity, we get

$$0 \le E[w](t) \le E[w](0) = 0.$$

Hence,

$$\frac{1}{2}\int_0^l [w(x,t)]^2 \, dx = 0, \quad \text{for all } t \ge 0,$$

which implies that the nonnegative continuous integrand must be identically zero over the integration interval, i.e $w \equiv 0$, for all $x \in [0,l], t > 0$. Hence also

$$u_1 \equiv u_2,$$

which finishes the proof of uniqueness. □

The energy (6) arises if one multiplies the heat equation by w and integrates in x over the interval $[0,l]$. Then realizing that the first term will be the time derivative of the energy, and performing the same integration by parts on the second term as above, we can reprove that this energy is decreasing.

Notice that all of the above arguments hold for the case of the infinite interval $-\infty < x < \infty$ as well. In this case one ignores the effect of the infinitely far endpoints and considers an IVP.

7.3 Conclusion

Using the energy motivated by the vibrating string model behind the wave equation, we derived a conserved quantity, which corresponds to the total energy of motion for the infinite string. This positive definite quantity was then used to prove uniqueness of solution to the wave IVP via the energy method, which essentially asserts that zero initial total energy precludes any (nonzero) dynamics. A similar approach was used to prove uniqueness for the heat IBVP, concluding that zero initial heat implies steady zero temperatures at later times.