9 Heat equation: solution

Equipped with the uniqueness property for the solutions of the heat equation with appropriate auxiliary conditions, we will next present a way of deriving the solution to the heat equation

$$u_t - ku_{xx} = 0. \tag{1}$$

Considering the equation on the entire real line $x \in \mathbb{R}$ simplifies the problem by eliminating the effects of the boundaries. We will first concentrate on this case, which corresponds to the dynamics of the temperature in a rod of infinite length. We want to solve the IVP

$$\begin{cases} u_t - ku_{xx} = 0 \quad (-\infty < x < \infty, 0 < t < \infty), \\ u(x,0) = \phi(x). \end{cases} \tag{2}$$

Since the solution to the above IVP is not easy to derive directly, unlike the case of the wave IVP, we will first derive a particular solution for a special simple initial data, and try to produce solutions satisfying all other initial conditions by exploiting the invariance properties of the heat equation.

9.1 Invariance properties of the heat equation

The heat equation (1) is invariant under the following transformations

(a) Spatial translations: If $u(x,t)$ is a solution of (1), then so is the function $u(x-y,t)$ for any fixed y.

(b) Differentiation: If u is a solution of (1), then so are u_x, u_t, u_{xx} and so on.

(c) Linear combinations: If $u_1, u_2, \dots, u_n$ are solutions of (1), then so is $u = c_1u_1 + c_2u_2 + \cdots + c_nu_n$ for any constants $c_1, c_2, \dots, c_n$.

(d) Integration: If $S(x,t)$ is a solution of (1), then so is the integral

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)\,dy$$

for any function $g(y)$, as long as the improper integral converges (we will ignore the issue of the convergence for the time being).

(e) Dilation (scaling): If $u(x,t)$ is a solution of (1), then so is the dilated function $v(x,t) = u(\sqrt{a}x, at)$ for any constant $a > 0$ (compare this to the scaling property of the wave equation, which is invariant under the dilation $u(x,t) \mapsto u(ax,at)$ for all $a \in \mathbb{R}$).

Properties (a), (b) and (c) are trivial (check by substitution), while property (d) is the limiting case of property (c). Indeed, if we use the notation $u^y(x,t) = S(x-y,t)$, and $c^y = g(y)\Delta y$, then u^y is also a solution by property (a), and we have the formal limit

$$\int_{-\infty}^{\infty} S(x-y,t)g(y)\,dy = \lim_{\Delta y \to 0} \sum_y c^y u^y.$$

To make this precise, we need to consider a finite interval of integration, which is partitioned by points $\{y_i\}_{i=1}^n$ into subintervals of length Δy, and use the definition of the integral as the limit of the corresponding Riemann sum to write

$$\int_{-\infty}^{\infty} S(x-y,t)g(y)\,dy = \lim_{b\to\infty} \int_{-b}^{b} S(x-y,t)g(y)\,dy = \lim_{b\to\infty}\lim_{n\to\infty}\sum_{i=1}^{n} S(x-y_i)g(y_i)\Delta y,$$

where $-b = y_1 < y_2 < \dots y_n = b$ is a partition of the interval $[-b,b]$.

Finally, property (e) can be checked by direct substitution as well. Notice that we cannot formally reverse the time by dilating with the factor $a = -1$, as was the case for the wave equation, since the $\sqrt{a}$ factor in front of the x argument would make the dilated function complex, which is not allowed in the theory of real PDEs (what is the meaning of complex valued temperature?!). We will later see that the heat equation is indeed time irreversible.

9.2 Solving a particular IVP

As a special initial data we take the following function

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \tag{3}$$

which is called the Heaviside step function. We consider the IVP

$$\begin{cases} Q_t - kQ_{xx} = 0 \quad (-\infty < x < \infty, 0 < t < \infty), \\ Q(x,0) = H(x), \end{cases} \tag{4}$$

which we solve in successive steps.

Step 1: Reduction to an ODE. Notice that the Heaviside function (3) is invariant under the dilation $x \mapsto \sqrt{a}x$, i.e. $H(\sqrt{a}x) = H(x)$. From the dilation property of the heat equation, we know that $Q(\sqrt{a}x, at)$ also solves the heat equation. But $Q(\sqrt{a}x, 0) = H(\sqrt{a}x) = H(x)$, thus $Q(\sqrt{a}x, at)$ and $Q(x,t)$ both solve the IVP (4). The uniqueness of solutions then implies that $Q(\sqrt{a}x, at) = Q(x,t)$ for all $x \in \mathbb{R}, t > 0$, so Q is invariant under the dilation $(x,t) \mapsto (\sqrt{a}x, at)$ as well.

Due to this invariance, Q can depend only on the ratio $\frac{x}{\sqrt{t}}$, that is $Q(x,t) = q(\frac{x}{\sqrt{t}})$. To see this, define the function q in the following way $q(z) = Q(z, 1)$. But then for fixed (x,t), we have

$$Q(x,t) = Q\left(\frac{1}{\sqrt{t}}x, \frac{1}{t}t\right) = Q\left(\frac{x}{\sqrt{t}}, 1\right) = q\left(\frac{x}{\sqrt{t}}\right).$$

Thus Q is completely determined by the function of one variable q.

For convenience of future calculations we pass to the function $g(z) = q(\sqrt{4k}z)$, so that

$$Q(x,t) = q\left(\frac{x}{\sqrt{t}}\right) = g\left(\frac{x}{\sqrt{4kt}}\right) = g(p),$$

where we used the notation $p = x/\sqrt{4kt}$. We next compute the derivatives of Q in terms of g, and substitute them into the heat equation in order to obtain an ODE for g. Using the chain rule, one gets

$$\begin{aligned} Q_t &= \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{4k}{2}\frac{x}{(\sqrt{4kt})^3}g'(p) = -\frac{1}{2t}\frac{x}{\sqrt{4kt}}g'(p), \\ Q_x &= \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}}g'(p), \\ Q_{xx} &= \frac{dQ_x}{dp}\frac{\partial p}{\partial t} = \frac{1}{4kt}g''(p). \end{aligned}$$

The heat equation then implies

$$0 = Q_t - kQ_{xx} = \frac{1}{4t}[-2pg'(p) - g''(p)],$$

which gives the following equation for g

$$g'' + 2pg' = 0. \tag{5}$$

Step 2: Solving the ODE. Using the integrating factor $\exp(\int 2p\, dp) = e^{p^2}$, the ODE (5) reduces to

$$[e^{p^2}g'(p)]' = 0.$$

Thus, we have

$$e^{p^2} g'(p) = c_1.$$

Solving for $g'(p)$, and integrating, we obtain

$$g(p) = c_1 \int e^{-p^2}\, dp + c_2.$$

Step 3: Checking the initial condition. Recalling that $Q(x,t) = g(p)$, where $p = x/\sqrt{4kt}$, we obtain the following explicit formula for Q

$$Q(x,t) = c_1 \int_0^{x/\sqrt{4kt}} e^{-p^2}\, dp + c_2. \tag{6}$$

Notice that we chose a particular antiderivative, which we are free to do due to the presence of the arbitrary constants. Also note that the above formula is only valid for $t > 0$, so to check the initial condition, we need to take the limit $t \to 0+$. Recalling the initial condition from (4), we have that,

$$\text{if} \quad x > 0, \qquad 1 = \lim_{t\to 0+} Q(x,t) = c_1 \int_0^{+\infty} e^{-p^2}\, dp + c_2 = c_1 \frac{\sqrt{\pi}}{2} + c_2,$$
$$\text{if} \quad x < 0, \qquad 0 = \lim_{t\to 0+} Q(x,t) = c_1 \int_0^{-\infty} e^{-p^2}\, dp + c_2 = -c_1 \frac{\sqrt{\pi}}{2} + c_2,$$

where we used the fact that $\int_0^\infty e^{-p^2}\, dp = \sqrt{\pi}/2$ to compute the improper integrals. The above identities give

$$c_1 \frac{\sqrt{\pi}}{2} + c_2 = 1, \qquad -c_1 \frac{\sqrt{\pi}}{2} + c_2 = 0.$$

Solving for c_1 and c_2, we get $c_1 = 1/\sqrt{\pi}$ and $c_2 = 1/2$. Substituting these into (6) gives the unique solution of the IVP (4),

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2}\, dp, \qquad \text{for} \quad t > 0. \tag{7}$$

9.3 Solving the general IVP

Returning to the general IVP (2), we would like to derive a solution formula, which will express the solution to the IVP in terms of the initial data (similar to d'Alambert's solution for the wave equation).

We first define the function

$$S(x,t) = \frac{\partial Q}{\partial x}(x,t), \tag{8}$$

where $Q(x,t)$ is the solution to the particular IVP (4), and is given by (7). Then, by the invariance properties of the heat equation, $S(x,t)$ also solves the heat equation, and so does

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\, dy, \qquad \text{for} \quad t > 0. \tag{9}$$

We claim that this u is the unique solution of the IVP (2). To verify this claim one only needs to check the initial condition of (2). Notice that using $S = Q_x$, we can rewrite u as follows

$$u(x,t) = \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\phi(y)\, dy = -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}[Q(x-y,t)]\phi(y)\, dy$$

Integrating by parts in the last integral, we get

$$u(x,t) = -Q(x-y,t)\phi(y)\Big|_{y=-\infty}^{y=\infty} + \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)\,dy.$$

We assume that the boundary terms vanish, which can be guaranteed for example by assuming that $\phi(y)$ vanishes for large $|y|$ (this is not strictly necessary, since $S(x-y,t)$ decays rapidly as $|y-x|$ becomes large, as we will shortly see). Now plugging in $t=0$, and using that Q has the Heaviside function (3) as its initial data, we have

$$u(x,0) = \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)\,dy = \int_{-\infty}^{x} \phi'(y)\,dy = \phi(y)\Big|_{y=-\infty}^{y=x} = \phi(x).$$

So $u(x,t)$ indeed satisfies the initial condition of (2).

We can compute $S(x,t)$ from (8), which will give

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}, \tag{10}$$

Using this expression of $S(x,t)$, we can now rewrite the solution given by (9) in the following explicit form

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt}\phi(y)\,dy, \qquad \text{for} \quad t>0. \tag{11}$$

The function $S(x,t)$ is known as the *heat kernel*, *fundamental solution*, *source function*, *Green's function*, or *propagator* of the heat equation. Notice that it gives a way of *propagating* the initial data ϕ to later times, giving the solution at any time $t>0$.

It is clear that formula (11) does not make sense for $t=0$, although one can compute the limit of $u(t,x)$ as $t \to 0+$ in that formula, which will give an alternate way of checking the initial condition of (2).

9.4 Conclusion

We derived the solution to the heat equation by first looking at a particular initial data, which was invariant under dilation. This guaranteed that the solution corresponding to this initial data is also dilation invariant, which reduced the heat equation to an ODE. After solving this ODE, and obtaining the solution, we saw that the solution to the general heat IVP can be written in an integral form using this particular solution. Next time we will explore the solution given by formula (11), and will study its qualitative behavior.