

10 Poisson's formula

Let us consider the Dirichlet problem for the circle $x^2 + y^2 = a^2$,

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{in } x^2 + y^2 < a^2, \\ u = h(\theta) & \text{on } x^2 + y^2 = a^2. \end{cases} \quad (1)$$

Notice that we can write the disk as a rectangle in the polar coordinates

$$\{x^2 + y^2 < a^2\} = [0, a) \times [0, 2\pi),$$

so it makes sense to solve the Dirichlet problem (1) by separation of variables in polar coordinates. Recall that Laplace's equation in polar variables has the form

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

So for the separated solution $u(r, \theta) = R(r)\Theta(\theta)$, the equation will reduce to

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0.$$

Dividing this equation by $R\Theta$, and multiplying by r^2 , and separating the variables on different sides, we get

$$r^2 \frac{R''}{R} + r \cdot \frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

Thus, we have the following ODEs for R and Θ .

$$r^2 R'' + rR' - \lambda R = 0, \quad \Theta'' = -\lambda\Theta. \quad (2)$$

Notice that the Θ component satisfies periodic boundary conditions $\Theta(\theta + 2\pi) = \Theta(\theta)$ due to the nature of polar coordinates. This gives the eigenvalue problem

$$\begin{cases} \Theta'' = -\lambda\Theta, \\ \Theta(\theta + 2\pi) = \Theta(\theta). \end{cases}$$

The eigenvalues and eigenfunctions of this problem are

$$\lambda_n = n^2, \quad \Theta_n(\theta) = A \cos n\theta + B \sin n\theta, \quad n = 0, 1, \dots$$

Using these values of λ , we can solve the R equation.

$$\text{For } \lambda_0 = 0, \quad r^2 R'' + rR' = 0 \quad \Rightarrow \quad R(r) = C_0 + D_0 \log r.$$

For the positive values of λ , the equation for R is of Euler's type, so it has power solutions $R(r) = r^\alpha$. Substituting this into the equation gives

$$r^2 \alpha(\alpha - 1)r^{\alpha-2} + r\alpha r^{\alpha-1} - \lambda r^\alpha = 0 \quad \Rightarrow \quad (\alpha^2 - \lambda)r^\alpha = 0,$$

so we must have $\alpha = \pm\sqrt{\lambda}$. For $\lambda = n^2$, we have

$$R_n(r) = C_n r^n + D_n r^{-n}.$$

We observe that the $D_0 \log r$ term of $R_0(r)$, and $D_n r^{-n}$ term of $R_n(r)$ are not defined at the origin, so they can not be parts of the solution in the entire disk $r < a$. Thus, we must have $D_n = 0$ for all

$n = 0, 1, \dots$

Using the components $R_n(r)$ and $\Theta_n(\theta)$, we can write the series solution to Laplace's equation as

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta), \quad (3)$$

where we combined the constants C_n into the coefficients A_n and B_n . The coefficients in the above series will be determined by the (inhomogeneous) Dirichlet boundary conditions of (1). Indeed,

$$u(a, \theta) = h(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta),$$

so we have

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \cos n\phi \, d\phi, \quad \text{and} \quad B_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\phi) \sin n\phi \, d\phi. \quad (4)$$

Thus, the solution of the Dirichlet problem (1) is given by (3), where the coefficients are determined from (4). Amazingly, if the coefficients (4) are substituted into the solution (3), the resulting series can be summed explicitly. Indeed, we have

$$\begin{aligned} u(r, \theta) &= \int_0^{2\pi} h(\phi) \frac{d\phi}{2\pi} + \sum_{n=1}^{\infty} \frac{r^n}{\pi a^n} \int_0^{2\pi} h(\phi) [\cos n\phi \cos n\theta + \sin n\phi \sin n\theta] \, d\phi \\ &= \int_0^{2\pi} h(\phi) \left[1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) \right] \frac{d\phi}{2\pi}. \end{aligned}$$

Using Euler's formula to express $\cos \alpha = (e^{i\alpha} + e^{-i\alpha})/2$, we can rewrite the term in the square brackets above as

$$1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) = 1 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{in(\theta - \phi)} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{-in(\theta - \phi)}$$

Noticing that the above series are geometric series with ratios $\left(\frac{r}{a}\right)^n e^{i(\theta - \phi)}$, and $\left(\frac{r}{a}\right)^n e^{-i(\theta - \phi)}$ respectively, both of which have absolute values less than 1, we can sum them to arrive at

$$\begin{aligned} 1 + 2 \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) &= 1 + \frac{re^{i(\theta - \phi)}}{a - re^{i(\theta - \phi)}} + \frac{re^{-i(\theta - \phi)}}{a - re^{-i(\theta - \phi)}} \\ &= \frac{a^2 - ar(e^{i(\theta - \phi)} + e^{-i(\theta - \phi)}) + r^2 + ar(e^{i(\theta - \phi)} + e^{-i(\theta - \phi)}) - 2r^2}{a^2 - ar(e^{i(\theta - \phi)} + e^{-i(\theta - \phi)}) + r^2} \\ &= \frac{a^2 - r^2}{a^2 - 2ar \cos(\theta - \phi) + r^2}, \end{aligned}$$

where we used Euler's formula again in the last step. The solution then can be written as

$$u(r, \theta) = (a^2 - r^2) \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar \cos(\theta - \phi) + r^2} \frac{d\phi}{2\pi}. \quad (5)$$

This is Poisson's formula. It expresses any harmonic function in a disk in terms of its values on the boundary circle.

If we use the Cartesian variables \mathbf{x}' for the point (a, ϕ) on the circle, and \mathbf{x} for any point (r, θ) inside the disk, then the denominator of the integrand in (5), according to the law of cosines, is exactly the

square of the length of the vector $\mathbf{x} - \mathbf{x}'$, i.e.

$$a^2 - 2ar \cos(\theta - \phi) + r^2 = |\mathbf{x} - \mathbf{x}'|^2.$$

But then we can rewrite Poisson's formula (5) in terms of Cartesian coordinates as follows.

$$u(\mathbf{x}) = \frac{a^2 - |\mathbf{x}|^2}{2\pi a} \int_{|\mathbf{x}'|=a} \frac{u(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^2} ds', \quad (6)$$

where the integration is with respect to the arc length element of the circle $|\mathbf{x}'| = a$, i.e. $ds' = a d\phi$.

One can prove that the function u given by (6) is harmonic, differentiable to all orders inside the disk $|\mathbf{x}| < a$, continuous on the closed disk $|\mathbf{x}| \leq a$, and that $\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} u(\mathbf{x}) = h(\mathbf{x}_0)$ for every point \mathbf{x}_0 on the boundary of the disk $|\mathbf{x}'| = a$.

10.1 Mean value property

Using Poisson's formula (6), one can show the following important property of harmonic functions.

Mean value property. Let u be a harmonic function in a disk D , and continuous in its closure \overline{D} , then the value of u at the center of the disk is equal to the average of u on its circumference. That is,

$$u(\mathbf{x}_0) = \frac{1}{2\pi a} \int_{|\mathbf{x}' - \mathbf{x}_0|=a} u(\mathbf{x}') ds', \quad (7)$$

where \mathbf{x}_0 is the center of the disk D , and a is its radius.

To prove (7), we can shift the coordinate system so that the origin coincides with the center of the disk, then (7) will take the form

$$u(\mathbf{0}) = \frac{1}{2\pi a} \int_{|\mathbf{x}'|=a} u(\mathbf{x}') ds'.$$

But this follows immediately from Poisson's formula (6) by substituting $\mathbf{x} = \mathbf{0}$, and noticing that $|\mathbf{0} - \mathbf{x}'| = |\mathbf{x}'| = a$.

Using the mean value property, one can prove the strong maximum principle for harmonic functions, which states that a nonconstant harmonic function in an open connected bounded domain D which is also continuous on the closure $\overline{D} = D \cup \partial D$, must attain its maximum on the boundary ∂D , and not inside D . To see this, let us assume that the harmonic function u attains its maximum value M at an internal point $\mathbf{x}_0 \in D$. So $u(\mathbf{x}) \leq u(\mathbf{x}_0) = M$ at any point $\mathbf{x} \in D$. Then for any circle C which entirely lies in D , and is centered at the point \mathbf{x}_0 , the mean value property implies that the average of u on the circle is equal to M . But since the values on the circle can not be greater than M , the only way their average can be equal to M is for all the values to be exactly M . Thus u will be constant on any of such concentric circles, and hence on the maximal disk that lies in D and is centered at \mathbf{x}_0 . Repeating this logic for all the points of this maximal disk, and then the points of the resulting maximal disks, and so on, we see that u is constant on the entire set D . Thus, a nonconstant harmonic function can not attain its maximum at an internal point. But then the maximum of the continuous function u on the closed set D must be attained on the boundary ∂D .

10.2 Conclusion

Using separation of variables in polar coordinates we found a series solution for the Dirichlet problem on the circle. Using the Dirichlet conditions, we found the coefficients in the series in terms of the Dirichlet data. In this case we were able to explicitly sum the series, arriving at Poisson's formula (5).

Using Poisson's formula, we also proved the mean value property of harmonic functions, as a corollary of which we obtained the strong maximum principle for harmonic functions.

Separation of variables in polar coordinates can be also applied to any other polar rectangle, such as a wedge, annulus, or the exterior of a circle, to solve boundary value problems for Laplace's equation as a series. We will do this in detail in the next lecture.