12 Green's first identity

Having studied Laplace's equation in regions with simple geometry, we now start developing some tools, which will lead to representation formulas for harmonic functions in general regions.

The fundamental principle that we will use throughout is the *Divergence theorem*, which states that

$$\iiint_D \operatorname{div}\mathbf{F}\,d\mathbf{x} = \iint_{\partial D} \mathbf{F}\cdot\mathbf{n}\,dS \tag{1}$$

for a vector field $\mathbf{F}$ defined in the closure $\overline{D}$ of an open solid region D. In the above identity ∂D stands for the boundary of the region D, $\mathbf{n}$ is the outward normal vector to this boundary, and the integral on the right hand side is taken with respect to the area element of the boundary. One can think of the divergence theorem as the generalization of the Fundamental Theorem of Calculus to higher dimensions. Indeed, it states the equality of the integral of the derivative of $\mathbf{F}$ inside the region D and the values of $\mathbf{F}$ on the boundary ∂D. We will routinely write $\nabla\cdot\mathbf{F}$ for the divergence.

We consider two functions, u, v, both of two or three (or more) variables. From the product rule, we have

$$(vu_x)_x = v_xu_x + vu_{xx}, \quad (vu_y)_y = v_yu_y + vu_{yy},$$

and similarly for the other partial derivatives. Adding up these identities gives

$$\nabla\cdot(v\nabla u) = \nabla u\cdot\nabla v + v\Delta u.$$

Integrating both sides of the above identity over the region D, and using the divergence theorem for the integral on the left hand side, we get

$$\iint_{\partial D} v\frac{\partial u}{\partial n}\,dS = \iiint_D \nabla u\cdot\nabla v\,d\mathbf{x} + \iiint_D v\Delta u\,d\mathbf{x}, \tag{2}$$

since $\nabla u\cdot\mathbf{n} = \partial u/\partial n$. This is Green's first identity. Rewriting (2) as

$$\iiint_D v\Delta u\,d\mathbf{x} = \iint_{\partial D} v\frac{\partial u}{\partial n}\,dS - \iiint_D \nabla u\cdot\nabla v\,d\mathbf{x},$$

we can think of this identity as the generalization of integration by parts, in the sense that one derivative is transferred from the function u to the function v under the integral, which results in a switched sign and a boundary term, much like the integration by parts for functions of single variable.

In particular, we can take $v\equiv 1$ in (2), which will give

$$\iint_{\partial D} \frac{\partial u}{\partial n}\,dS = \iiint_D \Delta u\,d\mathbf{x}. \tag{3}$$

The last identity implies a necessary condition for the Neumann problem to be well posed. Indeed, applying (3) to the solution of the Neumann problem

$$\begin{cases} \Delta u = f(\mathbf{x}) & \text{in } D,\\ \dfrac{\partial u}{\partial n} = h(\mathbf{x}) & \text{on } \partial D, \end{cases} \tag{4}$$

we have that necessarily

$$\iint_{\partial D} h(\mathbf{x})\,dS = \iiint_D f(\mathbf{x})\,d\mathbf{x}. \tag{5}$$

This means that unless h and f satisfy (5), the Neumann problem for the Poisson's equation cannot have a solution. For nice enough functions f and h, one can show that the solution to the Neumann problem exists, if (5) is satisfied. However uniqueness of the solution to the Neumann problem holds only up to a constant, since adding a constant to any solution of (4) will give another solution.

12.1 Mean value property

We previously used Poisson's formula for the solution of the Dirichlet problem on the circle to prove the mean value property of harmonic functions in two dimensions. A similar property holds in three and higher dimensions, and we next prove this property in three dimensions using Green's first identity (2). The mean value property in three dimensions states that the average value of a harmonic function over any sphere is equal to its value at the center.

Without loss of generality, we can assume that the sphere is centered at the origin, since we can use translation invariance of Laplace's equation to translate the coordinate system so that the center of the sphere coincides with the origin. Let $D = \{|\mathbf{x}| = a\}$ be a sphere centered at the origin with radius a. The outward normal to the sphere has the direction of $\mathbf{x}$, and hence,

$$\mathbf{n} = \frac{\mathbf{x}}{|\mathbf{x}|} = \frac{\mathbf{x}}{r}, \qquad \text{and} \qquad \frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n} = \nabla u \cdot \frac{\mathbf{x}}{r} = \frac{\partial u}{\partial r},$$

since $r = \sqrt{x^2 + y^2 + z^2}$ and the chain rule imply that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{x}{r}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{y}{r}, \quad \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{z}{r},$$

and subsequently

$$\nabla u \cdot \frac{\mathbf{x}}{r} = \frac{\partial u}{\partial x}\frac{x}{r} + \frac{\partial u}{\partial y}\frac{y}{r} + \frac{\partial u}{\partial z}\frac{z}{r} = \frac{\partial u}{\partial r}\frac{x^2 + y^2 + z^2}{r^2} = \frac{\partial u}{\partial r}.$$

Using the identity $\partial u/\partial n = \partial u/\partial r$, and the fact that u is harmonic, (3) can be written as

$$\iint_{\partial D} \frac{\partial u}{\partial r}\, dS = 0.$$

Converting to spherical coordinates, the above integral over the sphere $r = a$ takes the form

$$\int_0^{2\pi}\int_0^{\pi} u_r(a, \theta, \phi) a^2 \sin\theta\, d\theta d\phi = 0.$$

We divide both sides of the above identity by the surface area of the sphere, $4\pi a^2$, and notice that, since the integration is with respect to ϕ and θ, we can pull out the derivative with respect to the radial variable r outside of the integral,

$$\left(\frac{\partial}{\partial r}\left[\frac{1}{4\pi a^2}\int_0^{2\pi}\int_0^{\pi} u(r, \theta, \phi) a^2 \sin\theta\, d\theta d\phi\right]\right)\Bigg|_{r=a} = 0.$$

Since the radius a is arbitrary, the above derivative must vanish for every value of a, that is,

$$\frac{\partial}{\partial r}\left[\frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi} u(r, \theta, \phi) r^2 \sin\theta\, d\theta d\phi\right] = 0$$

for all values of $r > 0$. But the expression in the square brackets is exactly the average of the harmonic function u over the sphere $\{|\mathbf{x}| = r\}$, so it is constant with respect to the radius r, and hence the average of u over concentric circles centered at the origin is independent of the radii of the circles. In particular passing to the limit $r \to 0$ gives:

$$\text{Average of } u \text{ over the sphere } S = \{|\mathbf{x}| = r\} = \frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\phi) r^2 \sin\theta \, d\theta d\phi$$
$$= \lim_{r\to 0}\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\phi)\sin\theta \, d\theta d\phi = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(\mathbf{0})\sin\theta \, d\theta d\phi = u(\mathbf{0}),$$

since

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin\theta \, d\theta d\phi = 1$$

as the average of the constant function 1 over a unit sphere.

The mean value property works in higher dimensions as well, and can be proved by exactly the same method, making use of the Green's first identity in higher dimensions.

Just as we showed in two dimensions, the mean value property implies the maximum principle for harmonic functions, i.e. that the maximum of a harmonic function u is assumed on the boundary of open connected domains, and cannot be assumed inside the domain D, unless u is everywhere constant. The idea is again based on the fact that, if the harmonic function assumes a maximum at an inner point, then it must be constant in any disk centered at this point, which lies entirely inside D, due to the mean value property.

12.2 Uniqueness of Dirichlet's and Neumann's problems

Previously we saw that the maximum principle implies uniqueness of solutions to the Dirichlet problem for Laplace's, and more generally Poisson's equations. Today we will see that an energy method, similar to ones used for the wave and heat equations, also implies the uniqueness for the Dirichlet, as well as the Neumann problem for Poisson's equation.

Let us consider Poisson's equation with either Dirichlet or Neumann boundary conditions,

$$\begin{cases} \Delta u = f(\mathbf{x}) & \text{in } D \\ u\Big|_{\partial D} = h, \quad \text{or} \quad \dfrac{\partial u}{\partial n}\Big|_{\partial D} = g. \end{cases}$$

Assuming that the above problem has two solutions, u_1 and u_2,their difference, $w = u_1 - u_2$, will solve Laplace's equation, i.e. will be harmonic, in D, and will have vanishing Dirichlet (Neumann) data on the boundary ∂D. But then multiplying both sides of the equation $\Delta w = 0$ by w, and using Green's first identity (2), we get

$$\iiint_D w\Delta w \, d\mathbf{x} = \iint_{\partial D} w\frac{\partial w}{\partial n}\, dS - \iiint_D |\nabla w|^2 \, d\mathbf{x} = 0.$$

The integral over the boundary is zero due to the vanishing Dirichlet or Neumann boundary data of w, and consequently,

$$\iiint_D |\nabla w|^2 \, d\mathbf{x} = 0.$$

The function $|\nabla w|^2$ is nonnegative, and the only way the above integral can be zero, is if $\nabla w \equiv 0$ in D. Hence, $w \equiv$ constant in D.

In the case of the Dirichlet condition, we have $w \equiv 0$ on ∂D, and hence $u_1 - u_2 = w \equiv 0$ in D as well. In the case of Neumann condition, all we can assert is that $u_1 - u_2 = w \equiv$ constant in D, that is, the solution to the Neumann problem is unique up to a constant.

12.3 Dirichlet's principle

The quantity encountered in the above energy method has a physical meaning of potential energy. More specifically, we define the energy of a function w defined in D to be

$$E[w] = \frac{1}{2} \iiint_D |\nabla w|^2 \, d\mathbf{x}. \tag{6}$$

Recall that Laplace's equation describes steady states of oscillatory (wave) or heat conduction processes, and hence we expect that the solutions of Laplace's equation will have the smallest possible energy as the preferred equilibrium states. Mathematically, this can be stated as the following.

Dirichlet's principle. Among all functions $w(\mathbf{x})$ defined in D that satisfy the boundary condition $w = h$ on ∂D, the function that has the least energy is the unique harmonic function with Dirichlet data h on ∂D.

That is, if u is a harmonic function, then $E[w] \geq E[u]$ for any function w that has the same values as u on the boundary ∂D, $w = u = h$ on ∂D.

The proof of Dirichlet's principle follows from Green's first identity. Let u be harmonic in D, and $w = u = h$ on ∂D, then the function $v = u-w$ vanishes on the boundary ∂D. Writing $w = u-v$, we have

$$\begin{aligned} E[w] &= \frac{1}{2} \iiint_D |\nabla(u-v)|^2 \, d\mathbf{x} = \frac{1}{2} \iiint_D (\nabla u - \nabla v) \cdot (\nabla u - \nabla v) \, d\mathbf{x} \\ &= \frac{1}{2} \iiint_D (|\nabla u|^2 - 2\nabla u \cdot \nabla v + |\nabla v|^2) \, d\mathbf{x} = E[u] - \iiint_D \nabla u \cdot \nabla v \, d\mathbf{x} + E[v]. \end{aligned}$$

From (2), we see that

$$\iiint_D \nabla u \cdot \nabla v \, d\mathbf{x} = -\iiint_D v \Delta u \, d\mathbf{x} + \iint_{\partial D} v \frac{\partial u}{\partial n} \, dS = 0,$$

since the first integral on the right vanishes due to u being harmonic, and the second integral vanishes due to v vanishing on the boundary. But then

$$E[w] = E[u] + E[v] \geq E[u],$$

since $E[v] \geq 0$ as an integral of a nonnegative function.

One can prove that the converse of the Dirichlet's principle is true as well, i.e. that the function minimizing the energy must be harmonic. In this sense harmonic functions are solutions to the variational problem associated with the energy (6), and Laplace's equation is exactly the Euler-Lagrange equation for this Lagrangian.

12.4 Conclusion

Starting from the divergence theorem we derived Green's first identity (2), which can be thought of as integration by parts in higher dimensions. Using this identity, we proved several properties of harmonic functions in higher dimensions, namely, the mean value property, which implies the maximum principle; uniqueness of the Dirichlet and Neumann problems for Poisson's equation; and Dirichlet's principle. The mean value property obtained in this way generalized the mean value property in two dimensions, previously proved by Poisson's formula for the solution of Dirichlet's problem for the circle.

Green's first identity will be used next time to derive Green's second identity, which will lead to representation formulas for harmonic functions.