13 Green's second identity, Green's functions

Last time we derived Green's first identity for the pair of functions (u, v), which in three dimensions can be written as

$$\iiint_D v\Delta u\, d\mathbf{x} = \iint_{\partial D} v\frac{\partial u}{\partial n}\, dS - \iiint_D \nabla u\cdot\nabla v\, d\mathbf{x}. \tag{1}$$

Interchanging u and v, we can also write the Green's first identity for the pair (v, u),

$$\iiint_D u\Delta v\, d\mathbf{x} = \iint_{\partial D} u\frac{\partial v}{\partial n}\, dS - \iiint_D \nabla v\cdot\nabla u\, d\mathbf{x}. \tag{2}$$

Notice that the last terms of (1) and (2) are identical, thus subtracting the first identity from the second one, we obtain

$$\iiint_D (u\Delta v - v\Delta u)\, d\mathbf{x} = \iint_{\partial D}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS. \tag{3}$$

This is Green's second identity for the pair of functions (u, v).

Similar to the notion of symmetric boundary conditions for the heat and wave equations, one can define symmetric boundary conditions for Laplace's equation, by requiring that the right hand side of (3) vanish for any functions u, v satisfying the boundary conditions. It is not hard to see that (homogeneous) Dirichlet, Neumann and Robin boundary conditions are all symmetric.

13.1 Representation formula

Green's second identity (3) leads to the following representation formula for the solution of the Dirichlet problem in a domain D. If $\Delta u = 0$ in D, then for any point $\mathbf{x}_0 \in D$,

$$u(\mathbf{x}_0) = \iint_{\partial D}\left[-u(\mathbf{x})\frac{\partial}{\partial n}\left(\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|}\right) + \frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|}\frac{\partial u}{\partial n}\right] dS. \tag{4}$$

Indeed, the above formula is essentially a direct application of (3) to the pair of functions $u(\mathbf{x})$ and

$$v(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|}, \tag{5}$$

the last function being a multiple of the radial solution $|\mathbf{x}|^{-1}$ translated by the vector $\mathbf{x}_0$. The particular choice of the factor $-1/(4\pi)$ will be apparent from the proof of (4).

The only issue with the application of (3) to the pair u and v arises from the singularity of $v(\mathbf{x})$ at the point $\mathbf{x}_0$. To circumvent this, we will apply (3) in the region D with a ball centered at $\mathbf{x}_0$ of radius ϵ cut out, where ϵ will be made to approach zero in the end. Defining $B_\epsilon = \{\mathbf{x} : |\mathbf{x}-\mathbf{x}_0| < \epsilon\}$ to be a ball of radius ϵ centered at $\mathbf{x}_0$, we observe that for ϵ small enough, B_ϵ will entirely lie inside D. In the region $D_\epsilon = D \setminus B_\epsilon$, where $\setminus$ stands for the difference of sets, both $u(\mathbf{x})$ and $v(\mathbf{x})$ are well-defined and harmonic, so (3) implies that

$$\iint_{\partial D_\epsilon}\left[u(\mathbf{x})\frac{\partial v}{\partial n} - v(\mathbf{x})\frac{\partial u}{\partial n}\right] dS = 0,$$

where ∂D_ϵ is the boundary of D_ϵ. But ∂D_ϵ consists of two pieces: ∂D, and ∂B_ϵ, which have opposite orientations, in the sense that the outward normals of ∂D and ∂D_ϵ coincide on their common boundary, while those of ∂D_ϵ and ∂B_ϵ have opposite directions. Thus,

$$0 = \iint_{\partial D_\epsilon}\left[u(\mathbf{x})\frac{\partial v}{\partial n} - v(\mathbf{x})\frac{\partial u}{\partial n}\right] dS = \iint_{\partial D}\left[u(\mathbf{x})\frac{\partial v}{\partial n} - v(\mathbf{x})\frac{\partial u}{\partial n}\right] dS - \iint_{\partial B_\epsilon}\left[u(\mathbf{x})\frac{\partial v}{\partial n} - v(\mathbf{x})\frac{\partial u}{\partial n}\right] dS.$$

The first term on the right hand side is exactly the right hand side of (4), so to prove (4) it suffices to show that

$$u(\mathbf{x}_0) = \lim_{\epsilon \to 0} \iint_{\partial B_\epsilon} \left[u(\mathbf{x}) \frac{\partial v}{\partial n} - v(\mathbf{x}) \frac{\partial u}{\partial n} \right] dS. \tag{6}$$

Denoting $r = |\mathbf{x} - \mathbf{x}_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$, we can rewrite $v(\mathbf{x}) = -1/(4\pi r)$, and $\partial B_\epsilon = \{r = \epsilon\}$. Also notice that the outward normal to the boundary of the ball B_ϵ has the direction of the vector $\mathbf{r} = \mathbf{x} - \mathbf{x}_0$, hence $\mathbf{n} = \mathbf{r}/r$. But then, $\partial/\partial n = \partial/\partial r$, and the integral under the limit on the right hand side of (6) can be rewritten as

$$\iint_{\partial B_\epsilon} \left[u(\mathbf{x}) \frac{\partial v}{\partial n} - v(\mathbf{x}) \frac{\partial u}{\partial n} \right] dS = \iint_{r=\epsilon} \left[-u \frac{\partial}{\partial r} \left(\frac{1}{4\pi r} \right) + \frac{\partial u}{\partial r} \cdot \frac{1}{4\pi r} \right] dS.$$

On $\{r = \epsilon\}$, we have

$$\frac{\partial}{\partial r} \left(\frac{1}{r} \right) = -\frac{1}{r^2} = -\frac{1}{\epsilon^2}, \qquad \text{and} \qquad \frac{1}{r} = \frac{1}{\epsilon},$$

which after substitution into the previous identity gives

$$\iint_{\partial B_\epsilon} \left[u(\mathbf{x}) \frac{\partial v}{\partial n} - v(\mathbf{x}) \frac{\partial u}{\partial n} \right] dS = \frac{1}{4\pi\epsilon^2} \iint_{r=\epsilon} u \, dS + \frac{1}{4\pi\epsilon} \iint_{r=\epsilon} \frac{\partial u}{\partial r} \, dS = \overline{u} + \epsilon \overline{\frac{\partial u}{\partial r}},$$

where $\overline{u}$ and $\overline{\partial u/\partial r}$ denote the average values of respectively the functions $u(\mathbf{x})$ and $\partial u/\partial n$ on the sphere $|\mathbf{x} - \mathbf{x}_0| = r = \epsilon$. But as $\epsilon \to 0$, the points on the sphere converge to its center, $\mathbf{x} \to \mathbf{x}_0$, and we have

$$\lim_{\epsilon \to 0} \iint_{\partial B_\epsilon} \left[u(\mathbf{x}) \frac{\partial v}{\partial n} - v(\mathbf{x}) \frac{\partial u}{\partial n} \right] dS = u(\mathbf{x}_0) + 0 \cdot \frac{\partial u}{\partial n}(\mathbf{x}_0),$$

where $(\partial u/\partial n)(\mathbf{x}_0) = \lim_{\epsilon \to 0} \overline{\partial u/\partial n}$, and as such is bounded. Hence, (6) follows, which also finishes the proof of (4).

Notice that for the function $v(\mathbf{x})$ given by (5), we have $\Delta v = 0$ in $D \setminus B_\epsilon$. Then (3) gives

$$\iiint_D w \Delta v \, d\mathbf{x} = \iiint_{B_\epsilon} w \Delta v \, d\mathbf{x} = \iiint_{B_\epsilon} v \Delta w + \iint_{\partial B_\epsilon} \left(w \frac{\partial v}{\partial n} - v \frac{\partial w}{\partial n} \right) dS,$$

for any function w. But then the above arguments will imply that

$$\iiint_D w \Delta v \, d\mathbf{x} = \lim_{\epsilon \to 0} \iiint_{B_\epsilon} w \Delta v \, d\mathbf{x} = \lim_{\epsilon \to 0} \left(\epsilon^2 \widetilde{\Delta w} + \overline{w} + \epsilon \overline{\frac{\partial w}{\partial v}} \right) = w(\mathbf{x}_0),$$

where $\widetilde{\Delta w}$ is bounded by some constant times the maximum of $\Delta w(\mathbf{x})$ in the closed ball $\overline{B}_\epsilon$, and $\overline{w}$ and $\overline{\partial w/\partial n}$ are the averages of respectively $w(\mathbf{x})$ and $\partial w/\partial n$ over the sphere ∂B_ϵ. This shows that for any smooth function w,

$$\iiint_D w \Delta v \, d\mathbf{x} = w(\mathbf{x}_0),$$

and hence for the function $v(\mathbf{x})$ defined by (5) we have

$$\Delta v = \delta(\mathbf{x} - \mathbf{x}_0), \tag{7}$$

with δ being the Dirac delta function. It is easy to see that the representation formula (4) follows directly from Green's second identity and (7).

In general dimensions, the (distributional) solution of the equation

$$\Delta u = \delta(\mathbf{x})$$

is called a *fundamental solution* of Laplace's equation. Comparing this to (7), we can see that in three dimensions the radial function $-1/(4\pi|\mathbf{x}|)$ is a fundamental solution, and $v(\mathbf{x})$ is an $\mathbf{x}_0$ translation of it. Observe that for the representation formula (4) the fundamental solution of Laplace's equation plays a similar role to that of the heat kernel in the context of the heat equation.

In two dimensions the fundamental radial solution of Laplace's equation is

$$v(\mathbf{x}) = \frac{1}{2\pi} \log|\mathbf{x}|,$$

and the corresponding representation formula for the solution of Laplace's equation $\Delta_2 u = 0$ is

$$u(\mathbf{x}_0) = \int_{\partial D} \left[u(\mathbf{x}) \frac{\partial}{\partial n} \left(\frac{1}{2\pi} \log|\mathbf{x} - \mathbf{x}_0| \right) - \frac{1}{2\pi} \log|\mathbf{x} - \mathbf{x}_0| \frac{\partial u}{\partial n} \right] ds. \tag{8}$$

The above integral is a line integral over the bounding curve of a two-dimensional region D, and ds denotes the arc-length element of this boundary.

13.2 Green's functions

The representation formulas (4) and (8) are not very useful for solving boundary value problems for the Laplace equation, since they require data of both u and $\partial u/\partial n$ on the boundary. So by themselves these formulas will not give the solution to Dirichlet or Neumann problems, since we would have the data for either u or $\partial u/\partial n$, but not both.

Notice that to derive the representation formula we used Green's second identity, and relied on the fact that the function $v(\mathbf{x})$ given by (5) has a particular kind of singularity at the point $\mathbf{x}_0$, and is harmonic everywhere else. This is encompassed in equation (7), from which the representation formula follows directly. So if we could find another function with these properties, for which in addition either the first or the second term under the integral in (4) vanishes, then we would have solution formulas for the Dirichlet and Neumann problems.

Definition 13.1 (Green's functions)**.** The function $G(x)$ is called a Green's function for the operator $-\Delta$ in the three dimensional domain D at the point $\mathbf{x}_0 \in D$, if it satisfies the following properties.

(i) $G(\mathbf{x})$ has continuous second derivatives and is harmonic in $D \setminus \{\mathbf{x}_0\}$.

(ii) $G(\mathbf{x}) = 0$ on the boundary of D.

(iii) $G(x) + \dfrac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|}$ is finite at $\mathbf{x}_0$ and is harmonic in all of D.

In two dimensions condition (iii) must be replaced by the requirement that $G(\mathbf{x}) - \frac{1}{2\pi}\log|\mathbf{x} - \mathbf{x}_0|$ is harmonic in all of D. We will use the notation $G(\mathbf{x}, \mathbf{x}_0)$ to indicate that G is the Green's function at the point $\mathbf{x}_0$. It is not hard to see that the definition of the Green's function is equivalent to requiring that $G(\mathbf{x}, \mathbf{x}_0)$ solve the following Dirichlet problem

$$\begin{cases} \Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) & \text{in } D, \\ G(\mathbf{x}, \mathbf{x}_0) = 0 \quad \text{on } \partial D. \end{cases} \tag{9}$$

Indeed, it is clear from (7) that $G(\mathbf{x}, \mathbf{x}_0) - v(\mathbf{x})$ solves Laplace's equation, and is hence harmonic in all of D. It can be shown that a Green's function exists, and must be unique as the solution to the Dirichlet problem (9).

Using Green's function, we can show the following.

Theorem 13.2. *If $G(\mathbf{x}, \mathbf{x}_0)$ is a Green's function in the domain D, then the solution to Dirichlet's problem for Laplace's equation in D is given by*

$$u(\mathbf{x}_0) = \iint_{\partial D} u(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial n}\, dS. \tag{10}$$

The proof of this theorem is a straightforward application of Green's second identity (3) to the pair (u, G). Indeed, from (3) we have

$$\iiint_D (u\Delta G - G\Delta u)\, d\mathbf{x} = \iint_{\partial D} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n} \right) dS.$$

In the above identity the second term on the left hand side vanishes due to u being harmonic, and the second term in the right hand side integral vanishes, since $G(\mathbf{x}, \mathbf{x}_0) = 0$ on ∂D. Also,

$$\iiint_D u\Delta G\, d\mathbf{x} = \iiint_D u(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}_0)\, d\mathbf{x} = u(\mathbf{x}_0),$$

and (10) follows.

Equipped with Theorem 13.2 we can find the solution to the Dirichlet problem on a domain D, provided we have a Green's function in D. In practice, however, it is quite difficult to find an explicit Green's function for general domains D. Next time we will see some examples of Green's functions for domains with simple geometry.

One can use Green's functions to solve Poisson's equation as well.

Theorem 13.3. *If $G(\mathbf{x}, \mathbf{x}_0)$ is a Green's function in the domain D, then the solution to the Dirichlet's problem for Poisson's equation $\Delta u = f(\mathbf{x})$ is given by*

$$u(\mathbf{x}_0) = \iint_{\partial D} u(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial n}\, dS + \iiint_D f(\mathbf{x}) G(\mathbf{x}, \mathbf{x}_0)\, d\mathbf{x}.$$

To find a solution formula for the Neumann problem, condition (ii) in the definition of a Green's function must be replaced by

(ii$_{\mathrm{N}}$) $\dfrac{\partial G(\mathbf{x})}{\partial n} = c$ on the boundary of D for a suitable constant c.

Such a Green's function would solve the Neumann problem

$$\begin{cases} \Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0) & \text{in } D, \\ \dfrac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial n} = c & \text{on } \partial D. \end{cases} \tag{11}$$

The divergence theorem then implies that

$$\iiint_D \Delta G(\mathbf{x}, \mathbf{x}_0)\, d\mathbf{x} = \iint_{\partial D} \frac{\partial G}{\partial n}\, dS.$$

But from (11) we have

$$\iiint_D \Delta G(\mathbf{x}, \mathbf{x}_0)\, d\mathbf{x} = \iiint_D \delta(\mathbf{x} - \mathbf{x}_0)\, d\mathbf{x} = 1, \quad \text{and} \quad \iint_{\partial D} \frac{\partial G}{\partial n}\, dS = \iint_{\partial D} c\, dS = c \cdot \text{Area}(\partial D).$$

Hence, the constant c is given by $c = 1/\text{Area}(\partial D)$.

The solution formula for the Neumann problem is then

$$u(\mathbf{x}_0) = C - \iint_{\partial D} G(\mathbf{x}, \mathbf{x}_0) \frac{\partial u(\mathbf{x})}{\partial n}\, d\mathbf{x},$$

where the Green's function $G(\mathbf{x}, \mathbf{x}_0)$ satisfies the condition (ii$_{\mathrm{N}}$), and C is an arbitrary constant. The proof is left as an exercise for the curious student.

13.3 Conclusion

We derived Green's second identity from Green's first identity, which was subsequently applied to the pair of a harmonic function and the fundamental radial solution of Laplace's equation to arrive at the representation formula (4). To find a harmonic function with this representation formula one needs the boundary values of both the harmonic function and its derivative with respect to the outer normal on the boundary, which is not useful for solving the classical boundary value problems (Dirichlet, Neumann or Robin). To circumvent this, we defined Green's function as a fundamental solution of Laplace's equation that has vanishing Dirichlet data on the boundary of the domain, which leads to a solution formula for the Dirichlet problem. A similar Green's function can be defined to solve Neumann's problem as well.

Finding a Green's function for an arbitrary domain can be quite cumbersome, however. Next time we will consider some examples of domains with simple geometry, for which Green's functions can be found by geometric considerations.