5 Convergence of Fourier series

Starting from the method of separation of variables for the homogeneous Dirichlet and Neumann boundary value problems, we studied the eigenvalue problem $-X'' = \lambda X$ with the associated boundary conditions. This lead to the sine and cosine Fourier series respectively, and then the full Fourier series, which corresponds to the periodic boundary conditions. Using the pairwise orthogonality of the eigenfunctions in each of these cases, we were able to derive formulas for the Fourier coefficients. Finally, in the last lecture we demonstrated that these ideas survive for general boundary conditions for the interval (a, b), provided the boundary conditions are symmetric (hermitian). We showed that in this case the eigenvalues are all real, and the eigenfunctions can be chosen to be real valued and pairwise orthogonal.

One can also show that the eigenvalues form a sequence $\lambda_n \to \infty$, as $n \to \infty$ for the general symmetric boundary conditions. Notice that for the eigenvalues of the classical Fourier series, which we computed explicitly to be $\lambda_n = (n\pi/l)^2$, this property holds. Then for the eigenvalues listed as

$$\lambda_1 \leq \lambda_2 \leq \cdots \to \infty,$$

we will have the corresponding eigenfunctions $X_1, X_2, \ldots$, which are real valued and pairwise orthogonal. We are interested in expanding any function $f(x)$ defined on the interval (a, b) in terms of these eigenfunctions. Formally writing

$$f(x) = \sum_{n=1}^{\infty} A_n X_n, \tag{1}$$

we found the coefficients A_n using the pairwise orthogonality of the eigenfunctions to be

$$A_n = \frac{(f, X_n)}{(X_n, X_n)} = \frac{\int_a^b f(x) X_n(x)\,dx}{\int_a^b X_n^2(x)\,dx}.$$

If we now use these formulas to compute the coefficients, and form the series $\sum_n A_n X_n$, then one should make sure that this series converges for the equality (1) to make any sense. Since this is a series of functions, there are different ways in which the convergence may be understood. We next define three different notions of convergence, followed by criteria for convergence of the Fourier series for each of the three notions.

5.1 Notions of convergence

The convergence of the series of functions $\sum_{n=1}^{\infty} f_n(x)$ is equivalent to the convergence of the partial sums of the series, $S_1, S_2, \ldots$, where

$$S_N = \sum_{n=1}^{N} f_n(x).$$

Conversely, the convergence of a sequence of functions $F_1(x), F_2(x), \ldots$ is equivalent to the convergence of the telescoping series $\sum_{n=1}^{\infty} f_n(x)$, where $f_1(x) = F_1(x)$, and $f_n(x) = F_n(x) - F_{n-1}(x)$, for $n = 2, 3, \ldots$, since the functions $F_N(x)$ are the partial sums of this telescoping series.

Definition 5.1 (Convergence)**.** We say that

(i) $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ *pointwise* in (a, b), if for each fixed $x \in (a, b)$,

$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0 \quad \text{as } N \to \infty \qquad (\text{equivalently } |f(x) - S_N(x)| \to 0).$$

That is, for each fixed $a < x < b$ the numeric sequence $\sum_{n=1}^{\infty} f_n(x)$ converges to the number $f(x)$.

(ii) $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ *uniformly* in $[a, b]$, if

$$\max_{a\le x\le b}\left|f(x)-\sum_{n=1}^{N} f_n(x)\right| \to 0 \quad \text{as } N\to\infty \qquad (\text{equivalently } \max_{a\le x\le b}|f(x)-S_N(x)|\to 0).$$

That is, the overall "distance" between the function $f(x)$ and the partial sums $S_N(x)$ converges to zero. Notice that the endpoints of the interval are included in the definition.

(iii) $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ *in the mean-square (or L^2 sense)* in (a, b), if

$$\int_a^b \left|f(x)-\sum_{n=1}^{N} f_n(x)\right|^2 dx \to 0 \quad \text{as } N\to\infty \qquad (\text{equivalently } \int_a^b |f(x)-S_N(x)|^2\,dx\to 0).$$

That is, the the "distance" between $f(x)$ and the partial sums $S_N(x)$ in the mean-square sense converges to zero.

It is obvious from the definition that uniform convergence is the strongest notion of the three, since uniformly convergent series will clearly converge pointwise, as well as in L^2 sense (for finite intervals). The converse is not true, since not every pointwise or L^2 convergent series is uniformly convergent. An example is the telescoping series $\sum_{n=1}^{\infty}(x^{n-1}-x^n)$ in the interval $(0, 1)$ (check that it converges pointwise and in L^2 sense, but not uniformly).

Between the pointwise and L^2 convergence, neither is stronger than the other, since there are series that converge pointwise, but not in L^2, and vice versa.

An example of a pointwise convergent series that fails to be L^2 convergent is the telescoping series $\sum_{n=1}^{\infty} f(x) = \sum_{n=1}^{\infty}(g_n(x) - g_{n-1}(x))$ in $(0, 1)$, where

$$g_n(x) = \begin{cases} n & 0 < x < \frac{1}{n} \\ 0 & \frac{1}{n} \le x < 1, \end{cases} \qquad \text{for } n = 1, 2, \ldots, \qquad \text{and } g_0(x) \equiv 0.$$

Check that $\sum_{n=1}^{N} f_n(x) \to 0$ pointwise, but $\int_0^1 |0 - S_N(x)|^2\,dx = \int_0^1 g_n^2(x)\,dx \to \infty$ as $N \to \infty$. In this example the functions in the series are discontinuous, but one can cook up a similar example with continuous, and even differentiable (smooth) functions.

Finally, an example of a series which converges in L^2, but not pointwise, is the telescoping series $\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty}(g_n(x) - g_{n-1}(x))$ in $(0, 1)$, where

$$g_n(x) = \begin{cases} (-1)^n & x = \frac{1}{2} \\ 0 & \text{otherwise}, \end{cases} \qquad \text{for } n = 0, 1, 2, \ldots, \qquad \text{and } g_0(x) \equiv 0.$$

Clearly, $\sum_{n=1}^{N} f_n(x) \to 0$ in the L^2 sense, since $\int_0^1 |0 - S_N(x)|^2\,dx = \int_0^1 g_n^2(x)\,dx = 0$. But the numeric series $\sum_n f_n(1/2)$ diverges, so the series $\sum_{n=1}^{\infty} f_n(x)$ does not converge pointwise. Notice that again the functions in the series are discontinuous. In this case this is necessary, since for series of continuous functions L^2 convergence implies pointwise convergence, the proof of which is left as a simple exercise.

5.2 Convergence theorems

We will next list criteria for convergence of the Fourier series of a function $f(x)$

$$f(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx. \tag{2}$$

For convenience, we will write the series as $\sum_{n=1}^{\infty} A_n X_n(x)$, which will include the full, as well as sine and cosine Fourier series.

Definition 5.2. A function $f(x)$ is called piecewise continuous on an interval $[a, b]$, if

(i) $f(x)$ is continuous on $[a, b]$, except for at most finitely many points $x_1, x_2, \dots, x_k$.

(ii) at each of the points $x_1, x_2, \dots, x_k$, both the left-hand and right-hand limits of $f(x)$ exist,

$$f(x_i-) = \lim_{x \to x_i-} f(x), \qquad f(x_i+) = \lim_{x \to x_i+} f(x).$$

We say that the function has a jump discontinuity at such a point with the jump equal to $f(x_i+) - f(x_i-)$. At the points $x = a, b$ the continuity and limits are understood to be one-sided.

Next we state the convergence theorems for the Fourier series (2). We begin with a criteria for mean-square convergence

Theorem 5.3 (L^2 convergence)**.** *The Fourier series $\sum_n A_n X_n$ converges to $f(x)$ in the mean-square sense in (a, b), if $f(x)$ is square integrable over (a, b), that is*

$$\int_a^b |f(x)|^2 \, dx < \infty.$$

We remark that this very weak condition on the function $f(x)$ can be made even weaker, by replacing the Riemann integral above with the Lebesgue integral. The condition then simply states that $f \in L^2$, where L^2 is the space of square-integrable functions in the Lebesgue sense. The above theorem also holds for generalized Fourier series originating from the eigenvalue problem $-X'' = \lambda X$ with symmetric boundary conditions.

The next criteria is for uniform convergence.

Theorem 5.4 (Uniform convergence)**.** *The Fourier series converges to $f(x)$ uniformly in $[a, b]$, if*

(i) $f(x)$ is continuous, and $f'(x)$ is piecewise continuous on $[a, b]$.

(ii) $f(x)$ satisfies the associated boundary conditions.

The boundary conditions for the classical Fourier series will be the Dirichlet conditions for the sine series, Neumann for the cosine, and periodic for the full Fourier series. As the L^2 convergence theorem above, the uniform convergence theorem can be extended to hold for the generalized Fourier series, in which case one needs to add the condition that $f''(x)$ be piecewise continuous on $[a, b]$ as well.

Finally, we give the criteria for pointwise convergence.

Theorem 5.5 (Pointwise convergence)**.** *(i) The Fourier series converges to $f(x)$ pointwise in (a, b), if $f(x)$ is continuous, and $f'(x)$ is piecewise continuous on $[a, b]$.*

(ii) More generally, if $f(x)$ is only piecewise continuous on (a, b), as is $f'(x)$, then the Fourier series converges at every point x in the interval (a, b), and we have

$$\sum_{n=1}^{\infty} A_n X_n(x) = \frac{1}{2}[f(x+) + f(x-)], \quad \textit{for all } a < x < b.$$

Thus, if a function has a jump discontinuity, then its Fourier series converges to the average of the one-sided limits.

In the above theorems we used the interval (a, b), which in the case of the classical Fourier series is either $(0, l)$, or $(-l, l)$. It is clear that the above convergence theorems will hold for the periodic extension of the function to the entire real line as well.

Example 5.1. We have seen many examples of Fourier series that converge pointwise, but fail to be uniformly convergent. One such example is the sine Fourier series of the function $f(x) \equiv 1$ on the interval $(0, \pi)$,

$$1 = \sum_{n-\text{odd}} \frac{4}{n\pi} \sin nx.$$

This function satisfies the criteria for pointwise convergence, however, the series vanishes at $x = 0, \pi$, but the function is 1 in the neighborhoods of these points. So the series cannot converge uniformly. Notice that the second condition of the uniform convergence is not satisfied in this case, since the function 1 does not satisfy the Dirichlet conditions $X(0) = X(\pi) = 0$. □

5.3 Integration and differentiation of Fourier series

We are interested in using Fourier series to solve boundary value problems for PDEs, so we would like to know under which conditions one can differentiate or integrate the Fourier series of a function. The following theorems give these necessary conditions, which we state for $2l$-periodic functions. It is obvious how the statements will change for the sine and cosine series.

Theorem 5.6 (Integration of Fourier series)**.** *Suppose f is a piecewise continuous function with the Fourier coefficients a_n, b_n,*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}.$$

Let $F(x) = \int_0^x f(y)\,dy$. If $a_0 = 0$, then for all x in (a, b) we have

$$F(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{n\pi/l} \sin\frac{n\pi x}{l} - \frac{b_n}{n\pi/l} \cos\frac{n\pi x}{l}, \tag{3}$$

where $A_0 = \frac{1}{l}\int_{-l}^{l} F(x)\,dx$. If $a_0 \neq 0$, then the sum of the series on the right of (3) *is $F(x) - \frac{a_0}{2}x$.*

The series (3) is obtained by formally integrating the series of $f(x)$ term by term, irrespective of whether this series converges or not.

Notice that if $a_0 = 0$, then

$$F(x + 2l) - F(x) = \int_x^{x+2l} f(y)\,dy = \int_{-l}^{l} f(y)\,dy = la_0 = 0,$$

so $F(x)$ is $2l$-periodic. It is also continuous, with piecewise continuous derivative $F'(x) = f(x)$. But then its Fourier series will converge uniformly, and the coefficients in (3) can be obtained by integration by parts as follows.

$$A_n = \frac{1}{l}\int_{-l}^{l} F(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\cdot\frac{1}{n\pi/l}F(x)\sin\frac{n\pi x}{l}\bigg|_{-l}^{l} - \frac{1}{l}\cdot\frac{1}{n\pi/l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = -\frac{b_n}{n\pi/l},$$

since $F(x)\sin(n\pi x/l)$ is $2l$-periodic. We can compute B_n similarly to get $B_n = a_n/(n\pi/l)$.

If $a_0 \neq 0$, then the above argument can be applied to the function $f(x) - \frac{a_0}{2}$.

Theorem 5.7 (Differentiation of Fourier series)**.** *If f is $2l$-periodic, continuous, with continuous derivative $f'(x)$, and piecewise continuous second order derivative $f''(x)$, and has the Fourier series*

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l},$$

then we have

$$f'(x) = \sum_{n=1}^{\infty} -\frac{n\pi}{l}A_n \sin\frac{n\pi x}{l} + \frac{n\pi}{l}B_n \cos\frac{n\pi x}{l},$$

and this series converges uniformly.

Notice that the series of $f'(x)$ is the term by term derivative of $f(x)$. Indeed, if A_n', B_n' are the Fourier coefficients of $f'(x)$, then

$$A_n' = \frac{1}{l}\int_{-l}^{l} f'(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}f(x)\cos\frac{n\pi x}{l}\bigg|_{-l}^{l} + \frac{1}{l}\cdot\frac{n\pi}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{n\pi}{l}B_n,$$

where we used the $2l$-periodicity of $f(x)$, which is crucial in this case. The coefficients B_n' can be computed similarly, and we have $B_n' = -(n\pi/l)A_n$. The uniform convergence then follows from Theorem (5.4), since $f'(x)$ satisfies the criteria for uniform convergence.

5.4 Conclusion

After defining three notions of convergence – uniform, pointwise and in mean-square sense – we gave the criteria for convergence of the Fourier series for each of these notions. We will see in the next lecture that the space of square integrable functions (L^2) is the natural space to consider for the Fourier series. For such functions approximation by the Fourier series can be interpreted in terms of orthogonal projections, and expected properties, such as an analog of the Pythagorean theorem, hold. Using these tools we will give the proof of the uniform convergence theorem (5.4). The proof of the L^2 convergence requires some techniques from the measure theory, and is beyond the scope of this class. The proof of the pointwise convergence can be found in the textbook.