6 Completeness

Recall that we defined the dot product of two (real) functions on the interval (a, b) as

$$(f, g) = \int_a^b f(x)g(x)\, dx.$$

We can then define the L^2-*norm* of the function f as

$$\|f\| = (f, f)^{\frac{1}{2}} = \left[\int_a^b |f(x)|^2\, dx\right]^{\frac{1}{2}}.$$

This notion corresponds to the length of vectors in linear algebra, $|\mathbf{v}| = (\mathbf{v}, \mathbf{v})^{1/2} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$. Analogously, we define the "distance" between two function f, g as

$$\|f - g\| = \left[\int_a^b |f(x) - g(x)|^2\, dx\right]^{\frac{1}{2}}.$$

Thus, the norm gives a natural metric in L^2, or a way of measuring "distances" between functions. The L^2 convergence theorem of the Fourier series discussed in the last lecture can then be restated as

$$\left\|f - \sum_{n=1}^{N} A_n X_n\right\| \to 0, \quad \text{as } N \to \infty, \quad \text{if } f \text{ is square integrable } (f \in L^2).$$

Recall that the Fourier coefficients A_n were found by taking the dot product of the function $f(x)$ with the eigenfunctions X_n, which form an orthogonal set. It is not surprising then, that one has the following property.

Theorem 6.1. *Let $\{X_n\}$ be any orthogonal set of functions, and $\|f\| < \infty$. With N fixed, among all the possible choices of constants $c_1, c_2, \ldots c_N$, the choice that minimizes the quantity*

$$\left\|f - \sum_{n=1}^{N} c_n X_n\right\|,$$

is $c_1 = A_1$, $c_2 = A_2, \ldots, c_N = A_N$, where A_n's are the Fourier coefficients of f with respect to the orthogonal set $\{X_n\}$.

This theorem asserts that the least square approximation of f by the functions $X_1, X_2, \ldots, X_N$ is exactly given by the Fourier combination. In other words, the closest function to f in the span of the orthogonal set $\{X_n\}$ is its orthogonal projection onto the space spanned by this set.

The proof of the above theorem requires estimation of the square of the approximation error,

$$\begin{aligned} E_N = \left\|f - \sum_{n=1}^{N} c_n X_n\right\|^2 &= \int_a^b \left|f(x) - \sum_{n=1}^{N} c_n X_n(x)\right|^2 dx \\ &= \int_a^b |f(x)|^2\, dx - 2\sum_{n=1}^{N} c_n \int_a^b f(x)X_n(x)\, dx + \sum_{n,m} c_n c_m \int_a^b X_n(x)X_m(x)\, dx, \end{aligned}$$

where we simply expanded the square under the integral. Since X_n's are pairwise orthogonal, only the summands with $m = n$ survive in the last term. So we get

$$E_N = \|f\|^2 - 2\sum_{n=1}^{N} c_n(f, X_n) + \sum_{n=1}^{N} c_n^2\|X_n\|^2. \tag{1}$$

Completing the squares for the last two terms, we obtain

$$E_N = \sum_{n=1}^{N} \|X_n\|^2 \left[c_n - \frac{(f, X_n)}{\|X_n\|^2}\right]^2 + \|f\|^2 - \sum_{n=1}^{N} \frac{(f, X_n)^2}{\|X_n\|^2}.$$

The only term that depends on c_n above is the first term, which will be smallest when

$$c_n = \frac{(f, X_n)}{\|X_n\|^2} = A_n,$$

as we wanted to show.

Notice also that we have

$$0 \leq E_N \leq \|f\|^2 - \sum_{n=1}^{N} \frac{(f, X_n)^2}{\|X_n\|^2} = \|f\|^2 - \sum_{n=1}^{N} A_n^2\|X_n\|^2,$$

which implies that

$$\sum_{n=1}^{N} A_n^2\|X_n\|^2 \leq \|f\|^2, \quad \text{or} \quad \sum_{n=1}^{N} A_n^2 \int_a^b |X_n(x)|^2\, dx \leq \int_a^b |f(x)|^2\, dx.$$

So the partial sums of a positive series $\sum_n A_n^2\|X_n\|^2$ are bounded, hence, it must converge, and we have

$$\sum_{n=1}^{\infty} A_n^2\|X_n\|^2 \leq \|f\|^2.$$

This is Bessel's inequality, which asserts that the norm (length) of the projection of f onto the span of $\{X_n\}$ is less than the norm (length) of f itself.

We observe that for the series $\sum_{n=1}^{\infty} A_n X_n$ to converge to f in L^2 sense, the error E_N must converge to zero as $N \to \infty$. But then taking $c_n = A_n = (f, X_n)/\|X_n\|^2$ in (1), we have

$$E_N = \|f\|^2 - 2\sum_{n=1}^{N} A_n^2\|X_n\|^2 + \sum_{n=1}^{N} A_n^2\|X_n\|^2 \to 0, \quad \text{or}$$

$$\sum_{n=1}^{\infty} A_n^2\|X_n\|^2 = \|f\|^2.$$

This is Parseval's equality, and it is the analogue of the Pythagorean theorem for L^2 functions. Thus, the convergence of Fourier series in the sense of L^2 is equivalent to Parseval's identity.

We now define the notion of completeness of an orthogonal set, followed by the statement of equivalence of completeness and Parseval's identity.

Definition 6.2 (Completeness)**.** The infinite orthogonal set of functions $\{X_1, X_2, \dots\}$ is called complete in L^2, if any function $f \in L^2$ can be uniquely expanded into a series in terms of these functions.

Theorem 6.3. *The following are equivalent* *(i)* $\{X_1, X_2, \dots\}$ *is complete in* L^2*.*

(ii) If $f \perp X_n$*, for all* $n = 1, 2, \dots$*, then* $\|f\| = 0$*.*

(iii) Parseval's identity holds for any $f \in L^2$*.*

The proof of this equivalence is rather easy, and is left as an exercise for the curious student.

Finally, using the theorem of L^2 convergence of the Fourier series, which as we pointed above is equivalent to Parseval's identity, we see that the set of eigenfunctions of $-X'' = \lambda X$ with the associated symmetric conditions is complete in L^2. In particular, the set $\{1, \cos nx, \sin nx : n = 1, 2, \dots\}$ is complete in the space of square integrable functions over the interval $(-\pi, \pi)$ $(L^2(-\pi, \pi))$.

Example 6.1. Let us see how Parseval's identity can be applied to Fourier series of particular functions to yield the sums of some curious series. We consider the Fourier sine series of the function x on the interval $(0, l)$,

$$x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2l}{n\pi} \sin \frac{n\pi x}{l}.$$

First, notice that the squares of the norms of the functions $X_n(x) = \sin(n\pi x/l)$ are

$$\|X_n\|^2 = \int_0^l \sin^2 \frac{n\pi x}{l}\, dx = \frac{l}{2}.$$

Then Parseval's identity implies that

$$\sum_{n=1}^{\infty} \left(\frac{2l}{n\pi}\right)^2 \frac{l}{2} = \int_0^l x^2\, dx = \frac{l^3}{3}.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$\square$

6.1 Proof of uniform convergence

We now turn to the proof of uniform convergence of Fourier series, provided the conditions of the uniform convergence theorem of last lecture are met. We use the interval $[-\pi, \pi]$, and assume periodic boundary conditions. The Dirichlet and Neumann cases are similar.

We need to show that the series

$$f(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$$

converges uniformly on $[0, \pi]$. Notice that the piecewise continuity of $f'(x)$ implies that $\int_{-\pi}^{\pi} |f'(x)|^2\, dx < \infty$. But then for the Fourier series of $f'(x)$,

$$f'(x) \sim \sum_{n=1}^{\infty} A'_n \cos nx + B'_n \sin nx,$$

we have by Bessel's inequality

$$\sum_{n=1}^{\infty} |A'_n|^2 + |B'_n|^2 \le \frac{2}{l} \|f\|^2 < \infty.$$

As we found last time when discussing differentiation of Fourier series, the Fourier coefficients of $f(x)$ are related to the Fourier coefficients of $f'(x)$ as follows

$$A_n = -\frac{B'_n}{n}, \quad \text{and} \quad B_n = \frac{A'_n}{n}, \quad \text{for } n \neq 0. \tag{2}$$

Hence,

$$\begin{aligned}
\sum_{n=1}^{\infty} |A_n \cos nx| + |B_n \sin nx| \leq \sum_{n=1}^{\infty} |A_n| + |B_n| &= \sum_{n=1}^{\infty} \frac{1}{n} \left(|B'_n| + |A'_n|\right) \\
&\leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} (|A'_n| + |B'_n|)^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} 2(|A'_n|^2 + |B'_n|^2)\right)^{\frac{1}{2}} < \infty,
\end{aligned}$$

where we used Schwarz's inequality for infinite series in the next to last step,

$$\sum a_n b_n \leq \left(\sum a_n^2\right)^{1/2} \left(\sum b_n^2\right)^{1/2},$$

and the algebraic inequality $(a+b)^2 \leq 2(a^2+b^2)$ in the last step. So the Fourier series of $f(x)$ converges absolutely. But then

$$\max |f(x) - S_N(x)| \leq \max \sum_{n=N+1}^{\infty} |A_n \cos nx + B_n \sin nx| \leq \sum_{n=N+1}^{\infty} (|A_n| + |B_n|) \to 0, \quad \text{as } N \to 0,$$

as the tail of an absolutely convergent series.

In the above argument the 2π periodicity is needed for the relations (2) to hold, without which the proof falls apart. As we saw in the example of the sine series of the constant function $f(x) \equiv 1$, the condition that $f(x)$ satisfies the boundary conditions is necessary for the uniform convergence of the Fourier series.

6.2 Conclusion

In this lecture we defined the norm of square integrable functions, which induces a notion of distance between functions in L^2. Then we showed that the least square approximation of a function by an orthogonal set in L^2 exactly coincides with the Fourier combination in terms of this set. All these ideas relied on the notion or orthogonality, with which the space L^2 is naturally equipped. Extending these ideas, we arrived at an analog of the Pythagorean theorem in the space of functions, which was also seen to be equivalent to mean-square convergence of the Fourier series. Using these tools, we were also able to show that, provided a function is continuous with piecewise continuous derivative, and satisfies the associated boundary conditions, its Fourier series converges absolutely, and hence also uniformly, thus proving the uniform convergence theorem for Fourier series.