

7 Inhomogeneous boundary value problems

Having studied the theory of Fourier series, with which we successfully solved boundary value problems for the homogeneous heat and wave equations with homogeneous boundary conditions, we would like to turn to inhomogeneous problems, and use the Fourier series in our search for solutions. We start with the following boundary value problem for the inhomogeneous heat equation with homogeneous Dirichlet conditions.

$$\begin{cases} u_t - ku_{xx} = f(x, t), & \text{for } 0 < x < l, t > 0, \\ u(x, 0) = \phi(x), \\ u(0, t) = u(l, t) = 0. \end{cases} \quad (1)$$

A naïve separation of variables will not amount to much, since the source function $f(x, t)$ may not be separable. Instead, we can use the eigenfunctions of the corresponding (homogeneous) eigenvalue problem, and look for a solution in the series form

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{l}. \quad (2)$$

This is, of course, the Fourier sine series of $u(x, t)$ for each fixed $t > 0$, the coefficients of which will vary with the variable t . Such an expansion always exists due to completeness of the set of eigenfunctions $\{\sin(n\pi x/l)\}$. The coefficients will then be given by the Fourier sine coefficients formula

$$u_n(t) = \frac{2}{l} \int_0^l u(x, t) \sin \frac{n\pi x}{l} dx.$$

We can similarly expand the source function,

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{l}, \quad \text{where} \quad f_n(t) = \frac{2}{l} \int_0^l f(x, t) \sin \frac{n\pi x}{l} dx. \quad (3)$$

Now, since we are looking for a twice differentiable function $u(x, t)$ that satisfies the homogeneous Dirichlet boundary conditions, we can differentiate the Fourier series (2) term by term to obtain

$$u_{xx}(x, t) = - \sum_{n=1}^{\infty} u_n(t) \left(\frac{n\pi}{l} \right)^2 \sin \frac{n\pi x}{l}. \quad (4)$$

Notice that to be able to differentiate twice, we need to guarantee that u_x satisfies homogeneous Neumann conditions, which can be arranged by taking the odd extension of u to the interval $(-l, 0)$, and then the periodic extension to the rest of the real line.

We can also differentiate the series (2) with respect to t to obtain

$$u_t(x, t) = \sum_{n=1}^{\infty} u'_n(t) \sin \frac{n\pi x}{l}, \quad (5)$$

since the Fourier coefficients of $u_t(x, t)$ are

$$\frac{2}{l} \int_0^l u_t(x, t) \sin \frac{n\pi x}{l} dx = \frac{\partial}{\partial t} \left[\frac{2}{l} \int_0^l u(x, t) \sin \frac{n\pi x}{l} dx \right] = u'_n(t).$$

Differentiation under the above integral is allowed since the resulting integrand is continuous.

Substituting (5) and (4) into the equation, and using (3), we have

$$\sum_{n=1}^{\infty} \left[u'_n(t) + k \left(\frac{n\pi}{l} \right)^2 u_n(t) \right] \sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{l}.$$

But then, due to the completeness,

$$u'_n + k \left(\frac{n\pi}{l} \right)^2 u_n = f_n(t), \quad \text{for } n = 1, 2, \dots,$$

which are ODEs for the coefficients $u_n(t)$ of the series (2). Using the integrating factor $e^{k(n\pi/l)^2 t}$, we can rewrite the ODE as

$$\left(e^{k(n\pi/l)^2 t} u_n \right)' = e^{k(n\pi/l)^2 t} f_n(t).$$

Hence, we get

$$u_n(t) = u_n(0) e^{-k(n\pi/l)^2 t} + e^{-k(n\pi/l)^2 t} \int_0^t e^{k(n\pi/l)^2 s} f_n(s) ds.$$

But the initial condition implies

$$u(x, 0) = \phi(x) = \sum_{n=1}^{\infty} u_n(0) \sin \frac{n\pi x}{l},$$

thus,

$$u_n(0) = \phi_n = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi x}{l} dx. \quad (6)$$

So the solution can be written in the series form as

$$u(x, t) = \sum_{n=1}^{\infty} \left[\phi_n e^{-k(n\pi/l)^2 t} + \int_0^t e^{k(n\pi/l)^2 (s-t)} f_n(s) ds \right] \sin \frac{n\pi x}{l}, \quad (7)$$

where ϕ_n , and f_n are the Fourier coefficients of the initial data $\phi(x)$ and the source term $f(x, t)$, and can be found from (6) and (3) respectively. Notice that the first coefficient term in the above series comes from the homogeneous heat equation, while the second term can be thought of as coming from the variation of parameters for the inhomogeneous equation.

The case of the Neumann boundary conditions for the inhomogeneous heat equation is similar, with the only difference that one looks for a series solution in terms of cosines, rather than the sine series (2).

The boundary value problem for the inhomogeneous wave equation,

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x, t), & \text{for } 0 < x < l, \\ u(x, 0) = \phi(x), u_t(x, 0) = \psi(x), \\ u(0, t) = u(l, t) = 0, \end{cases} \quad (8)$$

can also be solved by expanding the solution into the series (2). In this case the ODEs for the coefficients will be

$$u''_n + c^2 \left(\frac{n\pi}{l} \right)^2 u_n = f_n(t), \quad \text{for } n = 1, 2, \dots,$$

with the initial conditions

$$u_n(0) = \phi_n, \quad u'_n(0) = \psi_n.$$

These ODEs can be solved explicitly using variation of parameters to obtain the coefficients $u_n(t)$, with which one can form the series solution (2).

7.1 Inhomogeneous boundary conditions

We next study the case of inhomogeneous boundary conditions, and consider the following boundary value problem for the inhomogeneous heat equation.

$$\begin{cases} u_t - ku_{xx} = f(x, t), & \text{for } 0 < x < l, t > 0, \\ u(x, 0) = \phi(x), \\ u(0, t) = h(t), u(l, t) = j(t). \end{cases} \quad (9)$$

We will use the *method of shifting the data* to reduce this problem to one with homogeneous boundary conditions, which will be equivalent to (1). The idea of this method is to subtract a function satisfying the inhomogeneous boundary conditions from the solution to the above problem. Let us define the function

$$U(x, t) = \left(1 - \frac{x}{l}\right) h(t) + \frac{x}{l} j(t),$$

for which trivially $U(0, t) = h(t)$, and $U(l, t) = j(t)$. But then for the new quantity $v(x, t) = u(x, t) - U(x, t)$, we have

$$\begin{aligned} v_t - kv_{xx} &= u_t - ku_{xx} - (U_t - kU_{xx}) = f(x, t) - \left(1 - \frac{x}{l}\right) h'(t) - \frac{x}{l} j'(t), \\ v(x, 0) &= u(x, 0) - U(x, 0) = \phi(x) - \left(1 - \frac{x}{l}\right) h(0) - \frac{x}{l} j(0), \\ v(0, t) &= u(0, t) - U(0, t) = h(t) - h(t) = 0, \\ v(l, t) &= u(l, t) - U(l, t) = j(t) - j(t) = 0. \end{aligned}$$

So $v(x, t)$ satisfies the following boundary value problem with homogeneous boundary conditions,

$$\begin{cases} v_t - kv_{xx} = \tilde{f}(x, t), & \text{for } 0 < x < l, t > 0, \\ v(x, 0) = \tilde{\phi}(x), \\ v(0, t) = v(l, t) = 0, \end{cases} \quad (10)$$

where

$$\begin{aligned} \tilde{f}(x, t) &= f(x, t) - \left(1 - \frac{x}{l}\right) h'(t) - \frac{x}{l} j'(t), \\ \tilde{\phi}(x) &= \phi(x) - \left(1 - \frac{x}{l}\right) h(0) - \frac{x}{l} j(0). \end{aligned} \quad (11)$$

Problem (10) is equivalent to (1), so the function $v(x, t)$ can be found by formula (7), in which ϕ_n and $f_n(s)$ must be replaced by respectively $\tilde{\phi}_n$ and $\tilde{f}_n(s)$. Clearly, $\tilde{\phi}_n$ and $\tilde{f}_n(t)$ can be found in terms of the Fourier coefficients of $\phi(x)$ and $f(x, t)$ using (11). Having found the function $v(x, t)$ in the series form,

$$v(x, t) = \sum_{n=1}^{\infty} \left[\tilde{\phi}_n e^{-k(n\pi/l)^2 t} + \int_0^t e^{k(n\pi/l)^2 (s-t)} \tilde{f}_n(s) ds \right] \sin \frac{n\pi x}{l},$$

we can write the solution to problem (9) as

$$\begin{aligned} u(x, t) &= v(x, t) + U(x, t) \\ &= \left(1 - \frac{x}{l}\right) h(t) + \frac{x}{l} j(t) + \sum_{n=1}^{\infty} \left[\tilde{\phi}_n e^{-k(n\pi/l)^2 t} + \int_0^t e^{k(n\pi/l)^2 (s-t)} \tilde{f}_n(s) ds \right] \sin \frac{n\pi x}{l}. \end{aligned} \quad (12)$$

In the case of Neumann boundary conditions,

$$u_x(0, t) = h(t), \quad u_x(l, t) = j(t),$$

one should use the function

$$V(x, t) = \int U(x, t) dx = \left(x - \frac{x^2}{2l}\right) h(t) + \frac{x^2}{2l} j(t)$$

to shift the data, since $V_x(0, t) = U(0, t) = h(t)$, and $V_x(l, t) = U(l, t) = j(t)$.

For the inhomogeneous boundary value problem for the wave equation,

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x, t), & \text{for } 0 < x < l, \\ u(x, 0) = \phi(x), u_t(x, 0) = \psi(x), \\ u(0, t) = h(t), u(l, t) = j(t), \end{cases} \quad (13)$$

one can shift the data in the same way to reduce the problem to one equivalent to (8).

An alternative method to shifting the data, is to use expansion (2) to find the solution of problem (9) directly. The difference between this case and the case of problem (1) is in that the solution does not satisfy the homogeneous boundary conditions, so the series (2) can not be differentiated term by term when substituting into the equation. The way out is to separately expand $u_{xx}(x, t)$ as

$$u_{xx}(x, t) = \sum_{n=1}^{\infty} w_n(t) \sin \frac{n\pi x}{l}. \quad (14)$$

But then using Green's second identity,

$$\int_a^b (f''g - gf'') dx = (f'g - fg') \Big|_a^b,$$

we can compute

$$\begin{aligned} w_n(t) &= \frac{2}{l} \int_0^l u_{xx}(x, t) \sin \frac{n\pi x}{l} dx \\ &= -\frac{2}{l} \int_0^l \left(\frac{n\pi}{l}\right)^2 u(x, t) \sin \frac{n\pi x}{l} dx + \frac{2}{l} \left(u_x \sin \frac{n\pi x}{l} - \frac{n\pi}{l} u \cos \frac{n\pi x}{l}\right) \Big|_0^l. \end{aligned}$$

Noticing that the first boundary term vanishes, and using the boundary conditions for u from (9) in the second term, we have

$$w_n(t) = -\left(\frac{n\pi}{l}\right)^2 u_n(t) - \frac{2n\pi}{l^2} (-1)^n j(t) + \frac{2n\pi}{l^2} h(t). \quad (15)$$

Then substituting (14) and (5) into the equation, and using (15), we get

$$\sum_{n=1}^{\infty} \left[u'_n(t) - k \left(-\left(\frac{n\pi}{l}\right)^2 u_n(t) - \frac{2n\pi}{l^2} (-1)^n j(t) + \frac{2n\pi}{l^2} h(t) \right) \right] \sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{l}.$$

Using the completeness, we obtain the following ODEs for the coefficients $u_n(t)$.

$$u'_n + k \left(\frac{n\pi}{l}\right)^2 u_n = f_n(t) - \frac{2n\pi}{l^2} [(-1)^n j(t) - h(t)], \quad \text{with } u_n(0) = \phi_n.$$

One can solve these ODEs using the same integrating factor as before to get

$$u_n(t) = \phi_n e^{-k(n\pi/l)^2 t} + \int_0^t e^{k(n\pi/l)^2 (s-t)} \left(f_n(s) - \frac{2n\pi}{l^2} [(-1)^n j(s) - h(s)] \right) ds.$$

The solution to (9) will then be given by the series (2). It is left as an exercise to convince yourselves that this solution is equivalent to (12).

For the inhomogeneous wave problem (13) this method will yield the following ODEs for the coefficients

$$u_n'' + c^2 \left(\frac{n\pi}{l} \right)^2 u_n = f_n(t) - \frac{2n\pi}{l^2} [(-1)^n j(t) - h(t)],$$

with the initial conditions

$$u_n(0) = \phi_n, \quad u_n'(0) = \psi_n.$$

Solving these ODEs by variation of parameters, one can find the solution to (13) as the series (2).

7.2 Conclusion

Using the eigenfunctions corresponding to the associated homogeneous boundary value problems as building blocks, we searched for a solution to the inhomogeneous heat and wave equations in the form of a series in terms of these eigenfunctions. This lead to ODEs for the coefficients of the series, which is akin to the method of variation of parameters, where one varies the coefficients in the linear combination of solutions to the homogeneous equation to obtain a solution of the inhomogeneous one.

We studied two methods of solving problems with inhomogeneous boundary conditions, the first of which amounted to merely shifting the boundary data to reduce the problem to one with homogeneous boundary conditions. The second method was to directly look for a solution in the form of a series in terms of the eigenfunctions of the associated homogeneous problem. In this case one can no longer directly substitute the series into the equation due to the issue of boundary terms when differentiating Fourier series. However, after taking care of these boundary terms, we arrived at ODEs for the coefficients of this expansion as well, solving which will give the series solution (2).