8 Laplace's equation: properties

We have already encountered Laplace's equation in the context of stationary heat conduction and wave phenomena. Recall that in two spatial dimensions, the heat equation is $u_t - k(u_{xx} + u_{yy}) = 0$, which describes the temperatures of a two dimensional plate. Similarly, the vibrations of a two dimensional membrane are described by the wave equation in two spatial dimensions, $u_{tt} - c^2(u_{xx} + u_{yy}) = 0$. If one considers stationary heat and wave states, i.e. not changing with time, then $u_t = u_{tt} = 0$, and both the heat and wave equations reduce to the stationary equation

$$u_{xx} + u_{yy} = 0.$$

This is the two dimensional Laplace equation. Analogously, in three dimension one has the equation

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

Using the notation $\Delta = \nabla \cdot \nabla$, we can rewrite Laplace's equation in any dimension as

$$\Delta u = 0. \tag{1}$$

The operator Δ is called *Laplace's operator*, or *Laplacian*. To distinguish the Laplacian in different dimensions, we will use the subscript notation Δ_n, where n stands for the dimensions. The solutions of the Laplace equation (1) are called *harmonic functions*. The inhomogeneous Laplace's equation

$$\Delta_n u = f(x_1, x_2, \ldots, x_n),$$

is called Poisson's equation.

Besides describing stationary heat and wave phenomena, Laplace's and Poisson's equations come up in the study of electrostatics, incompressible fluid flow, analytic functions theory, Brownian motion, etc.

Notice that in one dimension Laplace's equation is the ODE $u_{xx} = 0$, so the only harmonic functions in one dimension are the linear functions $u(x) = A + Bx$.

An obvious distinction between Laplace's equation and the heat and wave equations is that the processes described by Laplace's equation do not involve dynamics or evolution of the initial data in time. Hence the natural problem to study for Laplace's equation is the boundary value problem in some given domain D. We will consider the equation

$$\Delta u = f \quad \text{in } D, \tag{2}$$

with either of the following conditions on the boundary ∂D of the domain D.

$$\text{(D)}: \; u\big|_{\partial D} = h; \qquad \text{(N)}: \; \frac{\partial u}{\partial \mathbf{n}}\bigg|_{\partial D} = h; \qquad \text{(R)}: \; \frac{\partial u}{\partial \mathbf{n}} + au\bigg|_{\partial D} = h,$$

where $\mathbf{n}$ is the outer normal vector to this boundary.

When considering heat and wave boundary value problems, we saw that the boundary in one dimension consists of the endpoints of the interval (a, b), in which the equation is being solved. In two dimensions the boundary will be a curve, while in three dimensions it is a surface, and we expect that the geometry of the boundary will play a role in solving Laplace's equation.

We will restrict our study of Laplace's equation to two and three dimensions, and will occasionally use the vector notation $\mathbf{x} = (x, y)$, or $\mathbf{x} = (x, y, z)$ to denote a point in either two dimensional, or three dimensional space. Lets us start by first discussing the properties of Laplace's equation.

8.1 Maximum principle

It turns out that harmonic functions obey a maximum principle, which is similar to the maximum principle for the heat equation. In what follows, an open bounded connected set D is a set that does not contain any of its boundary points (open), entirely lies inside some ball centered at the origin (bounded), and consists of one piece, i.e. any two points of the set can be connected by a curve that entirely lies inside the set (connected).

Maximum Principle. Let D be a connected bounded open set (in either two or three dimensional space), and u be a harmonic function in D, which is also continuous on the closure of the set, $\overline{D} = D \cup \partial D$. Then the maximum and minimum values of u are attained on the boundary of D. Moreover, the maximum and minimum values cannot be attained inside D, unless $u \equiv$ constant.

Mathematically, this means that if u is a non-constant harmonic function in an open bounded connected set D, then

$$\max_D\{u\} < \max_{\overline{D}}\{u\} = \max_{\partial D}\{u\}, \quad \text{and} \quad \min_{\partial D}\{u\} = \min_{\overline{D}}\{u\} < \min_D\{u\}.$$

If we think of harmonic functions as equilibrium states in the heat conduction, then the maximum principle makes perfect sense, since if there was a maximum temperature at an interior point of the domain D, there would have been a heat flux from this point to the points with lower temperature, which would consequently decrease the temperature of this point, making the state unsteady.

The idea of the proof of the maximum principle is similar to the one used in proving the maximum principle for the heat equation. We start by noting that if, say in two dimensions, (x_0, y_0) is a maximum point, then both $u_{xx}(x_0, y_0) \leq 0$ and $u_{yy}(x_0, y_0) \leq 0$ by the second derivative test. But then $\Delta u(x_0, y_0) = u_{xx}(x_0, y_0) + u_{yy}(x_0, y_0) \leq 0$. This would be a contradiction, if the inequality was strict, however second order derivatives may be zero at extrema points.

To eliminate this scenario, we modify the function u, and consider the new function $v(\mathbf{x}) = u(\mathbf{x}) + \epsilon|\mathbf{x}|^2$, where $\epsilon > 0$ is a small constant. But then we have

$$\Delta_2 v = \Delta_2 u + \epsilon \Delta_2(x^2 + y^2) = 0 + 4\epsilon > 0 \quad \text{in } D,$$

and similarly in three dimensions. But $\Delta_2 v = v_{xx} + v_{yy} \leq 0$ at an interior maximum point, therefore $v(\mathbf{x})$ has no interior maximum points in D. Since $v(\mathbf{x})$ is continuous, it must attain its maximum value somewhere in the closed set $\overline{D}$, so the maximum must be attained at some point $\mathbf{x}_0 \in \partial D$. Then for every point $\mathbf{x}$ in D, we have

$$u(\mathbf{x}) \leq v(\mathbf{x}) \leq v(\mathbf{x}_0) = u(\mathbf{x}_0) + \epsilon|\mathbf{x}_0|^2 \leq \max_{\partial D}\{u\} + \epsilon l^2,$$

where l is the largest distance from the origin to the boundary of the (bounded) set D. Since ϵ was arbitrary, we can make it go to zero, yielding

$$u(\mathbf{x}) \leq \max_{\partial D}\{u\}, \quad \text{for every } \mathbf{x} \in D.$$

So the maximum of u must be attained on the boundary. The proof for the minimum is similar. The stronger statement that the maximum cannot be attained inside D will be proved later via the mean value property of harmonic functions.

8.2 Uniqueness of the Dirichlet problem

As was the case for the heat equation, the maximum principle directly implies uniqueness of the Dirichlet problem for Poisson's equation. Indeed, suppose that the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } D, \\ u = h & \text{on } \partial D, \end{cases}$$

where D is open bounded and connected, has two solutions u_1, u_2. Then their difference, $w = u_1 - u_2$, is harmonic, and has zero Dirichlet data on the boundary ∂D. But by the maximum/minimum principle we have for any point $\mathbf{x} \in D$,

$$0 = \min_{\partial D}\{w\} \leq w(\mathbf{x}) \leq \max_{\partial D}\{w\} = 0,$$

so $w(x) = u_1(x) - u_2(x) \equiv 0$.

We will give an alternative proof of the uniqueness using the energy method in a subsequent lecture, which will also show that solutions to the Neumann problem are unique up to a constant.

8.3 Invariance

Laplace's equation is invariant under rigid motions, which are the translations, and rotations. A translation is a transformation $\mathbf{x} \to \mathbf{x}'$, which is given by $\mathbf{x}' = \mathbf{x} + \mathbf{a}$ for some vector $\mathbf{a}$. In two dimensions this vector equation is equivalent to

$$x' = x + a, \qquad y' = y + b,$$

and it is easy to see that $u_{xx} + u_{yy} = u_{x'x'} + u_{y'y'} = 0$. So if a function is harmonic in the variables (x, y), it must also be harmonic in the variables (x', y'). This is the invariance under translations. Clearly this holds in higher dimensions as well.

For the invariance under rotations, we need to show that Laplace's equation remains the same in the variables

$$\begin{aligned} x' &= x\cos\alpha + y\sin\alpha \\ y' &= -x\sin\alpha + y\cos\alpha, \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

where α is the angle of rotation.

Using the chain rule, one can compute u_x, u_y, and then u_{xx}, u_{yy} in terms of the partial derivatives of u with respect to (x', y') variables, and show that

$$u_{xx} + u_{yy} = (u_{x'x'} + u_{y'y'})(\cos^2\alpha + \sin^2\alpha) = u_{x'x'} + u_{y'y'}.$$

Thus, Laplace's operator is invariant under rotations in two dimensions.

One can prove the invariance under rotations in any dimension $n = 2, 3, \ldots$ using the matrix notation as follows. In any dimension n a rotation is given by

$$\mathbf{x}' = B\mathbf{x}, \quad \text{or} \quad x'_k = \sum_{i=1}^{n} b_{ki}x_i,$$

where $B = \{b_{ij}\}$ is an orthogonal matrix, that is

$$BB^t = B^tB = I, \quad \text{or} \quad \sum_{i=1}^{n} b_{ki}b_{li} = \delta_k^l,$$

where $\delta_k^l = 1$, if $k = l$, and $\delta_k^l = 0$, if $k \neq l$ is the Kronecker symbol. Using the chain rule, we can compute

$$\frac{\partial}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial x'_k}{\partial x_i}\frac{\partial}{\partial x'_k} = \sum_{k=1}^{n} b_{ki}\frac{\partial}{\partial x'_k}.$$

To compute the second order derivatives, we multiply the first order derivative operator by itself.

$$\frac{\partial^2}{\partial x_i^2} = \left(\sum_{k=1}^{n} b_{ki}\frac{\partial}{\partial x'_k}\right) \cdot \left(\sum_{l=1}^{n} b_{li}\frac{\partial}{\partial x'_l}\right) = \sum_{k,l=1}^{n} b_{ki}b_{li}\frac{\partial^2}{\partial x'_k \partial x'_l}.$$

But then

$$\Delta_{\mathbf{x}} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} = \sum_{i=1}^{n} \sum_{k,l=1}^{n} b_{ki} b_{li} \frac{\partial^2}{\partial x'_k \partial x'_l} = \sum_{k,l=1}^{n} \left(\sum_{i=1}^{n} b_{ki} b_{li} \right) \frac{\partial^2}{\partial x'_k \partial x'_l} = \sum_{k,l=1}^{n} \delta_k^l \frac{\partial^2}{\partial x'_k \partial x'_l} = \Delta_{\mathbf{x}'}.$$

So Laplace's operator is indeed invariant under rotations.

The rotation invariance also implies that Laplace's equation allows rotationally invariant solutions, that is, solutions that depend only on the radial variable $r = |\mathbf{x}|$. We will call such solutions radial.

8.4 Radial solutions of Laplace's equation

In order to find radial solutions to Laplace's equation, we make a change to polar variables in two dimensions, and to spherical variables in three dimensions. Notice that in this case the radial solution simply means that $u(r,\theta) = u(r)$, or $u(r,\theta,\phi) = u(r)$, that is, the function depends on only one variable, and, as a consequence, the PDE will reduce to an ODE.

We first make a change to polar variables in two dimensions, for which the transformation formulas are

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta, \end{cases} \qquad \text{with Jacobian matrix} \quad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}.$$

Using $r = \sqrt{x^2+y^2}$, one can compute the partial derivatives

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\theta, \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta.$$

Also, differentiating both sides of $x = r\cos\theta$ with respect to x, we get

$$1 = \frac{\partial r}{\partial x}\cos\theta - r\sin\theta\frac{\partial \theta}{\partial x} = \cos^2\theta - r\sin\theta\frac{\partial \theta}{\partial x} \qquad \Rightarrow \qquad \frac{\partial \theta}{\partial x} = -\frac{1-\cos^2\theta}{r\sin\theta} = -\frac{\sin\theta}{r}.$$

$\frac{\partial \theta}{\partial y}$ can be computed similarly. So the Jacobian matrix of the inverse transformation is

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta/r \\ \sin\theta & \cos\theta/r \end{pmatrix}.$$

By the chain rule, we will have

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \qquad \text{and} \qquad \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}.$$

Using these, one can compute

$$\Delta_2 = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

And the Laplace equation can be written in polar variables as

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

For a radial solution $u(r,\theta) = u(r)$, the last term in the above equation will vanish, yielding the equation

$$u_{rr} + \frac{1}{r}u_r = 0, \quad \text{or} \quad ru_{rr} + u_r = 0,$$

which is an ODE, as expected. The last equation can be written as

$$(ru_r)_r = 0,$$

where we used the integrating factor $\exp(\int \frac{1}{r}\,dr)$. Integrating the last equation gives

$$ru_r = c_1, \quad \text{or} \quad u_r = c_1 \frac{1}{r}.$$

Integrating once more gives the solution

$$u(r) = c_1 \log r + c_2.$$

Disregarding the constant solution c_2, we see that the function $\log r = \log|\mathbf{x}|$ is harmonic in two dimensions.

To find radial solutions in three dimensions, we need to make a change to spherical variables, which is given by the transformations

$$\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2} = \sqrt{s^2 + z^2} \\
s &= \sqrt{x^2 + y^2} \\
x &= s\cos\phi \qquad z = r\cos\theta \\
y &= s\sin\phi \qquad s = r\sin\theta.
\end{aligned}$$

Thus, the transformation to spherical variables can be thought of as the pair of successive transformations

$$(x, y, z) \to (s, \phi, z) \to (r, \theta, \phi).$$

Using the above computation in two dimensions, we have that

$$\begin{aligned}
u_{zz} + u_{ss} &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}, \quad \text{and} \\
u_{xx} + u_{yy} &= u_{ss} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}.
\end{aligned}$$

Adding these two identities, and canceling the term u_{ss} on both sides, we get

$$\Delta_3 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{s}u_s + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{s^2}u_{\phi\phi}. \tag{3}$$

We can also compute

$$u_s = \frac{\partial u}{\partial s} = u_r\frac{\partial r}{\partial s} + u_\theta\frac{\partial \theta}{\partial s} + u_\phi\frac{\partial \phi}{\partial s} = u_r\frac{s}{r} + u_\theta\frac{\cos\theta}{r}.$$

Then replacing u_s in (3) by the above expression, and substituting $s = r\sin\theta$ for all occurrences of s, we obtain Laplace's equation in the spherical variables in three dimensions

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}\left[u_{\theta\theta} + \cot\theta u_\theta + \frac{1}{\sin^2\theta}u_{\phi\phi}\right] = 0.$$

For a radial solution $u(r, \theta, \phi) = u(r)$, the entire square brackets term will vanish, so Laplace's equation will reduce to the ODE

$$u_{rr} + \frac{2}{r}u_r = 0.$$

Multiplying this equation by r^2, we can write it as

$$(r^2 u_r)_r = 0,$$

where we used the integrating factor $\exp(\int \frac{2}{r}\,dr)$. Integrating this equation gives

$$r^2 u_r = c_1, \quad \text{or} \quad u_r = c_1 \frac{1}{r^2}.$$

Integrating yet again, we obtain the solution

$$u(r) = -c_1 \frac{1}{r} + c_2.$$

So the function $1/r = 1/|\mathbf{x}|$ is harmonic in three dimensions.

Notice that both $1/r$ and $\log r$ functions are not defined at the origin $r = 0$, but they will be harmonic on any domain which does not contain the origin. We will see in subsequent lectures that these functions in the context of Laplace's equation play a role similar to that of the heat kernel in the context of the heat equation.

8.5 Conclusion

In this lecture we studied the maximum principle for Laplace's equation, which trivially implies the uniqueness of solutions to the Dirichlet problem for Poisson's equation. We also saw that Laplace's equation is invariant under translations and rotations. The last fact accounted for existence of radial solutions, which are solutions that are invariant under rotations, and hence depend only on the radial variable r. Making a change to polar variables in two dimensions, and spherical variables in three dimensions, we were able to find radial harmonic functions by solving the ODEs satisfied by these functions. We will see in a later lecture that these radial harmonic functions play a crucial role in finding the solution to the Dirichlet problem for Laplace's and Poisson's equations.