9 Harmonic functions in rectangles and cubes

In this lecture we demonstrate how boundary value problems for Laplace's equation can be solved by separation of variables in the case of rectangles in two dimensions and cubes in three dimensions.

Let us consider the two dimensional Laplace's equation in a rectangle $D = (0,a)\times(0,b)$, with boundary conditions prescribed on the four edges of the rectangle,

$$I_1 = (0,a)\times\{0\}, \quad I_2 = (0,a)\times\{b\}; \qquad I_3 = \{0\}\times(0,b), \quad I_4 = \{a\}\times(0,b).$$

We choose some particular boundary conditions, say Neumann conditions on the entire boundary, and study the problem

$$\begin{cases} u_{xx}+u_{yy}=0 \quad \text{in } D=(0,a)\times(0,b), \\ u_y(x,0)=h(x), \quad u_y(x,b)=g(x); \qquad u_x(0,y)=j(y), \quad u_x(a,y)=k(y). \end{cases} \tag{1}$$

Other boundary conditions, Dirichlet on the entire boundary, or mixed – Dirichlet on some of the edges, and Neumann on the others, can be handled in much the same way.

Notice that the solution to problem (1) can be written as a sum $u = u_1+u_2+u_3+u_4$, where each of the summands u_i solves Laplace's equation in the rectangle with vanishing boundary data on all edges but I_i, for $i = 1,2,3,4$. It is then enough to find only one of u_i's, since finding the others will be similar. Let us, for example, find u_3, i.e. we would like to solve the boundary value problem

$$\begin{cases} u_{xx}+u_{yy}=0 \quad \text{in } D=(0,a)\times(0,b), \\ u_y(x,0)=0, \quad u_y(x,b)=0; \qquad u_x(0,y)=j(y), \quad u_x(a,y)=0. \end{cases} \tag{2}$$

We first look for separated solutions $u(x,y) = X(x)Y(y)$. Plugging this separated solution into the equation gives

$$X''Y + Y''X = 0.$$

Dividing by XY, and separating the variables on different sides of the equation, we obtain

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda.$$

Clearly λ is independent of both x and y, and hence is a constant. So we have the following ODEs for the components X and Y.

$$X'' = \lambda X, \qquad \text{and} \qquad Y'' = -\lambda Y.$$

We next observe that the first two boundary conditions of (2) imply

$$\begin{aligned} u_y(x,0) &= X(x)Y'(0) = 0, \\ u_y(x,b) &= X(x)Y'(b) = 0, \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} Y'(0) &= 0, \\ Y'(b) &= 0, \end{aligned}$$

since we are looking for solutions that do not vanish everywhere, hence $X(x) \not\equiv 0$. We thus have the following eigenvalue problem for the Y component.

$$\begin{cases} Y'' = -\lambda Y, \\ Y'(0) = Y'(b) = 0. \end{cases} \tag{3}$$

This eigenvalue problem was solved before. The eigenvalues and the eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{b}\right)^2, \qquad Y_n(y) = \cos\frac{n\pi y}{b}, \quad \text{for } n = 0,1,\ldots.$$

Plugging these values of λ into the equation for X, we have

$$\text{For } \lambda_0 = 0, \quad X'' = 0 \quad \Rightarrow \quad X_0(x) = \frac{A_0}{2} + \frac{B_0}{2}x.$$

$$\text{For } \lambda_n = \left(\frac{n\pi}{b}\right)^2, \ n = 1, 2, \ldots, \quad X'' = \left(\frac{n\pi}{b}\right)^2 X \quad \Rightarrow \quad X_n(x) = A_n e^{n\pi x/b} + B_n e^{-n\pi x/b}.$$

One can alternatively use the hyperbolic sine and cosine functions to write $X_n(x) = A'_n \cosh(n\pi x/b) + B'_n \sinh(n\pi x/b)$.

Notice that the forth boundary condition of (2) implies

$$u_x(a, y) = X'(a)Y(y) = 0 \quad \Rightarrow \quad X'(a) = 0.$$

This gives the following conditions for the coefficients A_n, B_n.

$$X'_0(a) = 0 \quad \Rightarrow \quad \frac{B_0}{2} = 0,$$

and $X'_n(a) = 0$ gives

$$A_n \frac{n\pi}{b} e^{n\pi a/b} - B_n \frac{n\pi}{b} e^{-n\pi a/b} = 0 \quad \Rightarrow \quad B_n = A_n e^{2n\pi a/b}.$$

Thus, $X_n(x) = A_n e^{n\pi x/b} + A_n e^{2n\pi a/b} e^{-n\pi x/b} = A_n \left(e^{n\pi x/b} + e^{-n\pi(x-2a)/b}\right)$.

Now using $X_n(x)$ and $Y_n(y)$, we can write the series solution as

$$u(x, y) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \left(e^{n\pi x/b} + e^{-n\pi(x-2a)/b}\right) \cos\frac{n\pi y}{b} \tag{4}$$

Finally, the inhomogeneous boundary condition of (2) implies

$$u_x(0, y) = j(y) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{b} \left(1 - e^{2n\pi a/b}\right) \cos\frac{n\pi y}{b},$$

hence,

$$A_n \frac{n\pi}{b} \left(1 - e^{2n\pi a/b}\right) = j_n, \quad \text{or} \quad A_n = \frac{j_n}{(1 - e^{2n\pi a/b})n\pi/b},$$

where

$$j_n = \frac{2}{b} \int_0^b j(y) \cos\frac{n\pi y}{b}\, dy, \quad n = 1, 2, \ldots,$$

are the Fourier cosine coefficients of $j(y)$. The zeroth coefficient of $j(y)$ vanishes due to the condition that the integral of the Neumann data over the boundary must be zero, for the problem (1) to be well posed.

Substituting the coefficients A_n into the series (4) gives the solution

$$u(x, y) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \frac{j_n}{(1 - e^{2n\pi a/b})n\pi/b} \left(e^{n\pi x/b} + e^{-n\pi(x-2a)/b}\right) \cos\frac{n\pi y}{b}.$$

Notice that the solution is determined up to the constant $A_0/2$, which is expected for a Neumann problem.

The strategy of solving boundary value problems for Laplace's equation in a rectangle is always the same, which we summarize in the following scheme.

(i) Look for separated solutions

(ii) Solve the eigenvalue problem for the component with two homogeneous boundary conditions

(iii) Find the other component using the obtained eigenvalues, and the third homogeneous boundary condition

(iv) Form the series solution, and find the coefficients from the inhomogeneous boundary condition

9.1 Cubes in three dimensions

A similar strategy works for cubes in three dimensions as well. Indeed, let us consider the following Dirichlet problem.

$$\begin{cases} \Delta_3 u = u_{xx} + u_{yy} + u_{zz} = 0 \quad \text{in } D = (0,\pi) \times (0,\pi) \times (0,\pi), \\ u(\pi, y, z) = g(y, z), \\ u(0, y, z) = u(x, 0, z) = u(x, \pi, z) = u(x, y, 0) = u(x, y, \pi) = 0 \end{cases}$$

For the separated solution $u(x, y, z) = X(x)Y(y)Z(z)$, the equation will reduce to

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0.$$

The homogeneous boundary conditions imply

$$X(0) = Y(0) = Y(\pi) = Z(0) = Z(\pi) = 0.$$

So we have the following ODEs with the associated boundary conditions

$$\begin{cases} Y'' = -\lambda Y, \\ Y(0) = Y(\pi) = 0, \end{cases} \qquad \begin{cases} Z'' = -\delta Z, \\ Z(0) = Z(\pi) = 0, \end{cases} \qquad \begin{cases} X'' = (\lambda + \delta)X, \\ X(0) = 0. \end{cases}$$

The eigenvalue problems for Y and Z give

$$\lambda_m = m^2, \quad Y_m(y) = \sin my; \qquad \delta_n = n^2, \quad Z_n(z) = \sin nz.$$

Using these values of λ and δ, we can solve the X equation with the homogeneous boundary condition, which will give

$$X_{mn}(x) = A_{mn} \sinh(\sqrt{m^2 + n^2} x).$$

The series solution will then be

$$u(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sinh(\sqrt{m^2 + n^2} x) \sin my \sin nz. \tag{5}$$

For this solution the inhomogeneous boundary condition implies

$$u(\pi, y, z) = g(y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sinh(\sqrt{m^2 + n^2} \pi) \sin my \sin nz. \tag{6}$$

This is the Fourier series of the function $g(y, z)$ in two variables, and one can find its coefficients in the same way as before. Notice that the elements of the set $\{\sin my \sin nz\}_{m,n=1}^{\infty}$ are pairwise orthogonal in the sense of the dot product

$$(f, g) = \int_0^{\pi} \int_0^{\pi} f(y, z) g(y, z) \, dy dz,$$

and

$$\int_0^{\pi} \int_0^{\pi} \sin^2 my \sin^2 nz \, dy dz = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}.$$

So the coefficients formula in this case will be

$$g_{mn} = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi g(y,z) \sin my \sin nz \, dydz.$$

Then, the coefficients in the series (5) can be found from (6) to be

$$A_{mn} = \frac{g_{mn}}{\sinh(\sqrt{m^2+n^2}\pi)} = \frac{4}{\pi^2 \sinh(\sqrt{m^2+n^2}\pi)} \int_0^\pi \int_0^\pi g(y,z) \sin my \sin nz \, dydz,$$

and the series solution can be written in terms of the Fourier coefficients of $g(y,z)$ as follows.

$$u(x,y,z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} g_{mn} \frac{\sinh(\sqrt{m^2+n^2}x)}{\sinh(\sqrt{m^2+n^2}\pi)} \sin my \sin nz.$$

9.2 Conclusion

In the case of rectangular domains, we demonstrated how the separation of variables can be used to solve boundary value problems for Laplace's equation in two dimensions. This is a consequence of the fact that the rectangle itself can be separated into a product of two intervals $R = (0,a) \times (0,b)$. The main idea was to break any boundary value problem to ones with homogeneous boundary conditions on all but one of the edges of the rectangle. Then using these homogeneous boundary conditions one can solve the resulting eigenvalue problem and obtain the solution in a series form, the coefficients of which will be determined by the inhomogeneous boundary condition. The same goes for cubes in three dimensions, where one can find the solutions as a double series. We will see next time that the method of separation of variables can be applied to Laplace's equation in a disk as well, which can be thought of as a rectangle in polar coordinates.