

Math 246B - Partial Differential Equations

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Contents

Chapter 1: Sobolev spaces	2
1.1 H^s spaces via the Fourier transform	2
1.2 Weak derivatives	3
1.3 Sobolev spaces $W^{k,p}$	3
1.4 Smooth approximations of Sobolev functions	4
1.5 Extensions and traces of Sobolev functions	5
1.6 Sobolev embeddings and compactness results	6
1.7 Difference quotients	7
Chapter 2: Solvability of elliptic PDEs	9
2.1 Weak formulation	9
2.2 Existence of weak solutions of the Dirichlet problem	10
2.3 General linear elliptic PDEs	12
2.4 Lax-Milgram theorem, solvability of general elliptic PDEs	14
2.5 Fredholm operators on Hilbert spaces	16
2.6 The Fredholm alternative for elliptic equations	18
2.7 The spectrum of a self-adjoint elliptic operator	19
Chapter 3: Elliptic regularity theory	21
3.1 Interior regularity	21
3.2 Boundary regularity	25
Chapter 4: Variational methods	26
4.1 The Derivative of a functional	26
4.2 Solvability for the Dirichlet Laplacian	27
4.3 Constrained optimization and application to eigenvalues	29

1. Sobolev spaces

In this chapter we define the Sobolev spaces H^s and $W^{k,p}$ and give their main properties that will be used in subsequent chapters without proof. The proofs of these properties can be found in Evans's "PDE".

1.1 H^s spaces via the Fourier transform

Below all the derivatives are understood to be in the distributional sense.

Definition 1.1. Let k be a non-negative integer. The Sobolev space $H^k(\mathbb{R}^n)$ is defined as

$$H^k(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \partial^\alpha f \in L^2 \text{ for all } |\alpha| \leq k\}.$$

Theorem 1.2. $f \in H^k(\mathbb{R}^n)$ if and only if $(1 + |\xi|^2)^{\frac{k}{2}} \hat{f} \in L^2(\mathbb{R}^n)$, and the following norms are equivalent

$$f \mapsto \left[\sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}^2 \right]^{\frac{1}{2}} \quad \text{and} \quad f \mapsto \left[\int_{\mathbb{R}^n} (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 d\xi \right]^{\frac{1}{2}} = \|(1 + |\xi|^2)^{\frac{k}{2}} \hat{f}\|_{L^2(\mathbb{R}^n)}.$$

Using the equivalent definition provided by the above theorem, one can extend the notion of $H^k(\mathbb{R}^n)$ Sobolev spaces to non-integer exponents. Here we use the notation $\mathcal{S}'$ for the space of tempered distributions on $\mathbb{R}^n$ (the dual space to the Schwartz space $\mathcal{S}$).

Definition 1.3. Let $s \in \mathbb{R}$, we define the Sobolev space $H^s(\mathbb{R}^n)$ as follows

$$H^s(\mathbb{R}^n) = \{f \in \mathcal{S}' : \hat{f} \text{ is a function, and } \|f\|_{H^s(\mathbb{R}^n)} = \left[\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi \right]^{\frac{1}{2}} < \infty\}.$$

The previous theorem along with the Fourier inversion theorem imply that for non-negative integers s this definition agrees with the previous one. Also, observe that $H^0 = L^2$.

The following theorem illustrates the expected differentiation properties of Sobolev functions

Theorem 1.4. Let $k \in \mathbb{N}$, $s \in \mathbb{R}$, and $f \in \mathcal{S}'$. Then $f \in H^s(\mathbb{R}^n)$, if and only if $\partial^\alpha f \in H^{s-k}(\mathbb{R}^n)$ for all multiindices $|\alpha| \leq k$, and the following norms are equivalent

$$\|f\|_{H^s(\mathbb{R}^n)} \sim \left[\sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{H^{s-k}(\mathbb{R}^n)}^2 \right]^{\frac{1}{2}}.$$

Remark 1.5. The above theorem trivially implies that the differentiation operator ∂^α is a bounded map from H^s to H^{s-k} for each multiindex α with $|\alpha| \leq k$.

As we will see later, one can define general Sobolev spaces in which weak derivatives up to non-negative integer order k are bounded in any L^p space for $1 \leq p \leq \infty$, which will be denoted by $W^{k,p}$. The reason for using the letter H for the L^2 based Sobolev spaces is to signify that H^s spaces are Hilbert spaces. Indeed, one can show that

Theorem 1.6. *The space H^s is a Hilbert space with the inner product*

$$\langle f, g \rangle_{H^s} = \int_{\mathbb{R}^n} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} (1 + |\xi|^2)^s d\xi.$$

Moreover, the Fourier transform $\mathcal{F} : H^s \rightarrow L^2(\mathbb{R}^n, \mu)$ is a unitary isomorphism, from H^s to L^2 equipped with the measure $d\mu = (1 + |\xi|^2)^{\frac{s}{2}} d\xi$.

1.2 Weak derivatives

In what follows Ω will always denote an open subset of $\mathbb{R}^n$. Recall that the space of test functions on Ω is $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$.

Definition 1.7. Let $u, v \in L_{loc}^1(\Omega)$, and α be a multiindex. We call v the α^{th} weak derivative of u , for which we use the usual notation $v = D^\alpha u$, if

$$\int_{\Omega} u(x) \partial^\alpha \phi(x) dx = (-1)^\alpha \int_{\Omega} v(x) \phi(x) dx, \quad \forall \phi \in \mathcal{D}.$$

Remark 1.8. The weak derivative, if it exists, obviously coincides with the distributional derivative. The only difference is that the weak derivative must be an L_{loc}^1 function, while the distributional derivative is in general only a distribution.

Remark 1.9. If the weak derivative exists, it must be unique almost everywhere.

Example 1.10. Let $n = 1$, and $\Omega = (0, 2)$. Consider the functions

$$u(x) = \begin{cases} x & 0 < x \leq 1, \\ 1 & 1 < x < 2, \end{cases} \quad \text{and} \quad v(x) = \begin{cases} 1 & 0 < x \leq 1, \\ 0 & 1 < x < 2. \end{cases}$$

We claim that $v = u'$ in the weak sense.

Example 1.11. Let $n = 1$, and $\Omega = (0, 2)$. Then the function

$$u(x) = \begin{cases} x & 0 < x \leq 1, \\ 2 & 1 < x < 2, \end{cases}$$

doesn't have a weak derivative.

As expected, weak derivatives behave like ordinary derivatives in many ways, and we will state the actual properties later on in the context of general Sobolev spaces.

1.3 Sobolev spaces $W^{k,p}$

Definition 1.12. Let $1 \leq p \leq \infty$, and $k \in \mathbb{Z}_+$. We define the space $W^{k,p}(\Omega)$, and the associated norm as follows.

$$W^{k,p}(\Omega) = \{u \in L_{loc}^1(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for all multiindices } |\alpha| \leq k\},$$

$$\|u\|_{W^{k,p}(\Omega)} = \left[\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}^p \right]^{\frac{1}{p}}, \quad \text{for } 1 \leq p < \infty,$$

$$\|u\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

The defined norms provide a natural metric structure on the space $W^{k,p}$.

Definition 1.13. Let $\{u_m\}_{m=1}^\infty, u \in W^{k,p}(\Omega)$. We say that

- (i) u_m converges to u in $W^{k,p}(\Omega)$, provided $\lim_{m \rightarrow \infty} \|u_m - u\|_{W^{k,p}(\Omega)} = 0$
- (ii) u_m converges to u in $W_{loc}^{k,p}(\Omega)$, if $u_m \rightarrow u$ in $W^{k,p}(\Lambda)$ for all open subsets $\Lambda \Subset \Omega$.

We also make the following useful definition.

Definition 1.14. The space $W_0^{k,p}(\Omega)$ is the closure of the space $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$. That is $u \in W_0^{k,p}(\Omega)$, if and only if $u \in W^{k,p}(\Omega)$, and there exists a sequence $\{u_m\} \in C_c^\infty(\Omega)$, such that $u_m \rightarrow u$ in $W^{k,p}(\Omega)$.

Heuristically, the space $W_0^{k,p}(\Omega)$ consists of those functions in $W^{k,p}(\Omega)$ that “vanish” on the boundary $\partial\Omega$ with all their derivatives up to order $k-1$. This will be made precise when we study the trace operator for Sobolev spaces.

Example 1.15. Let $\Omega = B(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}$, the open unit ball in $\mathbb{R}^n$. Consider the function

$$u(x) = |x|^{-a} \quad \text{for } x \in \Omega, x \neq 0,$$

for some positive real number a . We claim that $u \in W^{1,p}(\Omega)$, iff $\alpha < \frac{n-p}{p}$.

Example 1.16. Let $n=1, \Omega = (0,1)$. Consider the function $u(x) = \sin \frac{1}{x}$, which is smooth in Ω . Then $u, u' \in L_{loc}^1(\Omega)$, but $u \notin W^{1,p}(\Omega)$ for any p .

We collect the properties of weak derivatives in the following theorem.

Theorem 1.17. Let $u, v \in W^{k,p}(\Omega)$, and $|\alpha| \leq k$. Then

- (i) $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$, and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u$, $\forall \alpha, \beta$ such that $|\alpha| + |\beta| \leq k$
- (ii) $\forall \lambda, \mu \in \mathbb{R}, \lambda u + \mu v \in W^{k,p}(\Omega)$, and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$
- (iii) $u \in W^{k,p}(\Lambda)$ for all open subsets $\Lambda \subset \Omega$
- (iv) If $\psi \in C_c^\infty(\Omega)$, then $\psi u \in W^{k,p}(\Omega)$, and the Leibniz rule holds,

$$D^\alpha(\psi u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \psi D^{\alpha-\beta} u,$$

where

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

The second statement in the above theorem implies that $W^{k,p}(\Omega)$ is a vector space. We also have that.

Theorem 1.18. $W^{k,p}(\Omega)$ is a Banach space for all $k \in \mathbb{Z}_+$ and $1 \leq p \leq \infty$.

1.4 Smooth approximations of Sobolev functions

We first state the local approximation property of Sobolev functions, for which we make use of the subset

$$\Omega_\epsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \epsilon\}.$$

Theorem 1.19. Let $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Let $u^\epsilon = \eta_\epsilon * u$ be the standard mollification of u in Ω_ϵ . Then

- (i) $u^\epsilon \in C^\infty(\Omega_\epsilon)$ for all $\epsilon > 0$

(ii) $u^\epsilon \rightarrow u$ in $W_{loc}^{k,p}(\Omega)$ as $\epsilon \rightarrow 0$.

Definition 1.20. We call v the *strong L^p derivative of order α* , if for all compact subsets $K \subset \Omega$ there exists a sequence $\{\phi_j\} \subset C^{|\alpha|}(\Omega)$, such that

$$\int_K |\phi_j - u|^p dx \rightarrow 0, \quad \text{and} \quad \int_K |D^\alpha \phi_j - v|^p dx \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

It's not hard to see that Theorem 1.19 implies that the strong L^p derivative coincides with the weak derivative.

Theorem 1.21 (Global approximation by smooth functions). *Let Ω be bounded, $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Then there exists a sequence $\{u_m\} \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$, such that $u_m \rightarrow u$ in $W^{k,p}(\Omega)$.*

Remark 1.22. Notice that the above theorem makes no assumption about the regularity of the boundary $\partial\Omega$. The trade-off is that we cannot in general expect the approximating sequence to consist of functions smooth up to the boundary.

Remark 1.23. If $\Omega = \mathbb{R}^n$, then one can always choose an approximating sequence that consists of compactly supported smooth functions. This can be achieved by considering $\tilde{u}_{lm} = \phi_l u_m$, where ϕ_l is a smooth bump function supported in the ball of radius l , and selecting a subsequence of $\tilde{u}_{lm}$ via a diagonal argument. As a corollary of the above theorem and this observation, we see that $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$.

Let us introduce the following notation:

$$\begin{aligned} \widehat{W}^{k,p}(\Omega) &= \text{closure of } C^k(\Omega) \cap W^{k,p}(\Omega) \text{ in } W^{k,p}(\Omega), \\ \widetilde{W}^{k,p}(\Omega) &= \text{closure of } C^\infty(\Omega) \cap W^{k,p}(\Omega) \text{ in } W^{k,p}(\Omega). \end{aligned}$$

Obviously $\widetilde{W}^{k,p} \subset \widehat{W}^{k,p}$, but Theorem 1.21 implies that for bounded Ω , $\widetilde{W}^{k,p}(\Omega) = W^{k,p}(\Omega)$, and hence,

$$\widetilde{W}^{k,p}(\Omega) = \widehat{W}^{k,p}(\Omega) = W^{k,p}(\Omega).$$

For domains with smooth boundary, one can approximate Sobolev functions by smooth functions up to the boundary.

Theorem 1.24. *Assume Ω is bounded, and $\partial\Omega$ is C^1 . Let $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Then there exists a sequence $\{u_m\} \in C^\infty(\bar{\Omega})$, such that $u_m \rightarrow u$ in $W^{k,p}(\Omega)$.*

1.5 Extensions and traces of Sobolev functions

The question of extending a Sobolev function in some proper open subset $\Omega \subset \mathbb{R}^n$ to the entire space may be delicate, since the extended function must have integrable weak derivatives across the boundary $\partial\Omega$. Fortunately, smooth approximation up to the boundary allows one to construct such extensions, at least when $\partial\Omega$ is C^1 .

Theorem 1.25. *Let $1 \leq p \leq \infty$, and assume $\Omega \subset \mathbb{R}^n$ is bounded with C^1 boundary $\partial\Omega$. Select an open and bounded subset $\Lambda \subset \mathbb{R}^n$, such that $\Omega \Subset \Lambda$. Then there exists a bounded linear operator*

$$E : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^n),$$

such that for all $u \in W^{k,p}(\Omega)$

- (i) $Eu = u$ a.e. in Ω
- (ii) Eu has support within Λ
- (iii) $\|Eu\|_{W^{k,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{k,p}(\Omega)}$, with $C = C(p, \Omega, \Lambda)$ independent of u

Remark 1.26. One can chose a bounded domain $\Omega' \subset \mathbb{R}^n$, such that $\Lambda \Subset \Omega'$, in which case it is clear that $Eu \in W_0^{k,p}(\Omega')$.

Since Sobolev spaces are defined as spaces of integrable functions with integrable weak derivatives, in general it doesn't make sense to talk about the value of a Sobolev function at a point, or on a set of zero measure. In particular, values of a Sobolev function in Ω are not well-defined on the boundary $\partial\Omega$. However, approximation by smooth functions up to the boundary provides a way of "restricting" Sobolev functions to the boundary.

Theorem 1.27 (Trace theorem). *Assume $\Omega \in \mathbb{R}^n$ is bounded and $\partial\Omega$ is C^1 . Then there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega),$$

such that

- (i) $Tu = u|_{\partial\Omega}$, if $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$
- (ii) $\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$, with $C = C(p, \Omega)$ independent of u

Tu is then called the trace of u on $\partial\Omega$.

It turns out that Sobolev functions with zero trace are exactly those that can be approximated by smooth compactly supported functions in Ω .

Theorem 1.28. *Assume Ω is bounded and $\partial\Omega$ is C^1 . Let $u \in W^{1,p}(\Omega)$. Then $u \in W_0^{1,p}(\Omega)$, iff $Tu = 0$ on $\partial\Omega$.*

1.6 Sobolev embeddings and compactness results

Below we state the general Sobolev embedding theorem, along with the compactness results for such embeddings. We remark that typically embedding results about $W_0^{k,p}(\Omega)$ do not require smoothness of the boundary $\partial\Omega$ of the domain Ω . However, the results for $W^{k,p}(\Omega)$ are obtained from the respective results for $W^{k,p}(\mathbb{R}^n)$ via the extension theorem. But the existence of the extension operator $E : W^{k,p}(\mathbb{R}^n) \rightarrow W^{k,p}(\mathbb{R}^n)$ only holds if $\partial\Omega$ satisfies an appropriate regularity condition, e.g. C^1 condition.

In the following theorem the inclusion $X \subset Y$ denotes a continuous embedding, while the compact inclusion $X \Subset Y$ denotes a compact continuous embedding.

Theorem 1.29. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded with C^1 boundary, $k, m \in \mathbb{N}$ with $k \geq m$, and $1 \leq p < \infty$.*

- (1) *If $kp < n$, then*

$$\begin{aligned} W^{k,p}(\Omega) &\Subset L^q(\Omega) && \text{for } 1 \leq q < np/(n - kp), \\ W^{k,p}(\Omega) &\subset L^q(\Omega) && \text{for } q = np/(n - kp). \end{aligned}$$

More generally, if $(k - m)p < n$, then

$$\begin{aligned} W^{k,p}(\Omega) &\Subset W^{m,q}(\Omega) && \text{for } 1 \leq q < np/(n - (k - m)p), \\ W^{k,p}(\Omega) &\subset W^{m,q}(\Omega) && \text{for } q = np/(n - (k - m)p). \end{aligned}$$

- (2) *If $kp = n$, then*

$$W^{k,p}(\Omega) \Subset L^q(\Omega) \quad \text{for } 1 \leq q < \infty.$$

- (3) *If $kp > n$, then*

$$W^{k,p}(\Omega) \Subset C^{0,\gamma}(\bar{\Omega})$$

for $0 < \gamma < k - n/p$ if $k - n/p < 1$, for $0 < \gamma < 1$ if $k - n/p = 1$, and for $\gamma = 1$ if $k - n/p > 1$;
and

$$W^{k,p}(\Omega) \subset C^{0,\gamma}(\bar{\Omega})$$

for $\gamma = k - n/p$ if $k - n/p < 1$.

More generally, if $(k - m)p > n$, then

$$W^{k,p}(\Omega) \subseteq C^{m,\gamma}(\bar{\Omega})$$

for $0 < \gamma < k - m - n/p$ if $k - m - n/p < 1$, for $0 < \gamma < 1$ if $k - m - n/p = 1$, and for $\gamma = 1$ if $k - m - n/p > 1$; and

$$W^{k,p}(\Omega) \subset C^{m,\gamma}(\bar{\Omega})$$

for $\gamma = k - m - n/p$ if $k - m - n/p < 1$.

Remark 1.30. As we remarked above, these results hold for arbitrary open bounded sets $\Omega \subset \mathbb{R}^n$, if $W^{k,p}(\Omega)$ is replaced by $W_0^{k,p}(\Omega)$.

We say that an open set $\Omega \in \mathbb{R}^n$ is bounded in some direction, if there exists $e \in \mathbb{R}^n$, and $a, b \in \mathbb{R}$ such that $a \leq e \cdot x \leq b$ for all $x \in \Omega$, where $e \cdot x$ is the usual Euclidean dot product. On such domains we have the following result.

Theorem 1.31. *Let Ω be open and bounded in some direction, $1 \leq p < \infty$. Then there exists a positive constant $C = C(n, \Omega, p) > 0$, such that*

$$\|u\|_{L^p(\Omega)} \leq C \|Du\|_{L^p(\Omega)}, \quad \text{for all } u \in W_0^{1,p}(\Omega). \quad (1.1)$$

Inequality (1.1) is called the Poincaré inequality. Its significance is that the L^p norm of a function is bounded by the L^p norm of the weak derivative, which differs from the Sobolev embeddings, where the bound is in terms of the full norm of $W^{k,p}$. The requirement that $u \in W_0^{1,p}(\Omega)$ instead of $u \in W^{k,p}(\Omega)$ is crucial, since for constant functions Poincaré's inequality is trivially false. However, the compactness of the embedding $W_0^{1,p} \subseteq L^p(\Omega)$ for bounded Ω , along with the extension theorem can be used to show an analogous result for $W^{1,p}(\Omega)$ functions.

Theorem 1.32. *Let $\Omega \in \mathbb{R}^n$ be open, bounded, connected with C^1 boundary, $1 \leq p < \infty$. There exists a positive constant $C = C(n, \Omega, p) > 0$, such that*

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C \|Du\|_{L^p(\Omega)}, \quad \text{for all } u \in W^{1,p}(\Omega),$$

where $u_\Omega = \frac{1}{|\Omega|} \int_\Omega u \, dx$ is the average of u on Ω .

1.7 Difference quotients

A useful way of establishing weak differentiability of functions is via a uniform bound on difference quotients. To illustrate this, let us first define the difference quotients.

Definition 1.33. Let $u : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function on $\mathbb{R}^n$, and $h \in \mathbb{R} \setminus \{0\}$. The i^{th} difference quotient of u of size h is the function

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h},$$

where e_i is the unit vector in the i^{th} direction. The difference quotient vector is

$$D^h u = (D_1^h u, D_2^h u, \dots, D_n^h u).$$

Notice that the difference quotient can be defined for any function u . The next result, however, shows that it behaves as a derivative.

Proposition 1.34. *The difference quotient has the following properties:*

(1) (commutativity with weak derivatives) If $u, \partial_i u \in L^1_{loc}(\mathbb{R}^n)$, then

$$\partial_i D_j^h u = D_j^h \partial_i u.$$

(2) (integration by parts) If $u \in L^p(\mathbb{R}^n)$ and $v \in L^{p'}(\mathbb{R}^n)$, where $1 \leq p < \infty$, and $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\int (D_i^h u) v \, dx = - \int u (D_i^{-h} v) \, dx.$$

(3) (product rule)

$$D_i^h(uv) = u_i^h(D_i^h v) + (D_i^h u)v = u(D_i^h v) + (D_i^h u)v_i^h,$$

$$\text{where } u_i^h(x) = u(x + he_i).$$

The following theorem is the main result connecting weak differentiability with the uniform boundedness of the difference quotients.

Theorem 1.35. Let $\Omega \subset \mathbb{R}^n$ be open, bounded, and $\Omega' \Subset \Omega$. Denote $d = \text{dist}(\Omega', \partial\Omega) > 0$.

(1) If $Du \in L^p(\Omega)$, where $1 \leq p < \infty$, and $0 < |h| < d/2$, then

$$\|D^h u\|_{L^p(\Omega')} \leq \|Du\|_{L^p(\Omega)}.$$

(2) If $u \in L^p(\Omega)$, where $1 \leq p < \infty$, and there exists a constant C , such that

$$\|D^h u\|_{L^p(\Omega')} \leq C, \quad \text{for all } 0 < |h| < d/2,$$

then $u \in W^{1,p}(\Omega')$, and

$$\|Du\|_{L^p(\Omega')} \leq C.$$

2. Solvability of elliptic PDEs

2.1 Weak formulation

Let us first consider the Dirichlet problem for the Laplacian with homogeneous boundary conditions in a bounded open domain $\Omega \subset \mathbb{R}^n$ with C^1 boundary,

$$-\Delta u = f \quad \text{in } \Omega, \quad (2.1)$$

$$u = 0 \quad \text{on } \partial\Omega. \quad (2.2)$$

Assuming that $u, f : \bar{\Omega} \rightarrow \mathbb{R}$ are smooth functions, and multiplying the equation by a test function $\phi \in C_c^\infty(\Omega)$, we obtain

$$-\int_{\Omega} \Delta u \phi \, dx = \int_{\Omega} f \phi \, dx.$$

Integrating by parts, and discarding the boundary terms due to the compact inclusion of the support of ϕ in Ω , we arrive at

$$\int_{\Omega} Du \cdot D\phi \, dx = \int_{\Omega} f \phi \, dx \quad \text{for all } \phi \in C_c^\infty(\Omega). \quad (2.3)$$

Conversely, if f and Ω are smooth, then any smooth u satisfying (2.3) is necessarily a solution of (2.1). However, notice that (2.3) makes sense under much weaker assumptions on Ω, u and f . There is a flexibility in how to weaken the assumptions on u , so for the added structure of Hilbert spaces, we will consider the case of the derivatives of u in the weak sense being in $L^2(\Omega)$. Then, if Du is in L^2 , which will be the case if we assume that $u \in H^1(\Omega)$, then the right hand side of (2.3) is well defined by the Cauchy-Schwartz inequality for all $\phi \in C_c^\infty(\Omega)$, and by extension, for all $\phi \in H_0^1(\Omega)$, which is the closure of $C_c^\infty(\Omega)$ under the H^1 norm. The right hand side of (2.3) will be well defined for all $\phi \in H_0^1$, if $f \in L^2$, or more generally, for all $f \in H^{-1} = (H_0^1(\Omega))^*$, in which case we understand the right hand side of (2.3) as a dual pairing of f and ϕ .

Notice that if we understand the solution u in the weaker sense that it only belongs to $H^1(\Omega)$, then the Dirichlet condition (2.2) is not suited for such functions, since they are defined up to almost everywhere, and $\partial\Omega$ has n -dimensional Lebesgue measure zero. We thus weaken the boundary condition to hold in the trace sense, i.e., by requiring that the solution $u \in H^1(\Omega)$ have trace zero on the boundary $\partial\Omega$. But these are exactly $H_0^1(\Omega)$ functions.

All of the above motivates the following definition.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be open, $f \in H^{-1}(\Omega)$. A function $u : \Omega \rightarrow \mathbb{R}$ is called a weak solution of (2.1)-(2.2), if

(i) $u \in H_0^1(\Omega)$, and

(ii)

$$\int_{\Omega} Du \cdot D\phi \, dx = \langle f, \phi \rangle \quad \text{for all } \phi \in H_0^1(\Omega). \quad (2.4)$$

The right hand side of (2.4) is the dual pairing, and the weak solution is understood to be a ‘function’ in L^2 sense, i.e. it’s an equivalence class with respect to the almost everywhere pointwise equality.

Remark 2.2. The boundary conditions (2.2) were assumed to be homogeneous for simplicity, however the general case of non-homogeneous boundary conditions can be reduced to this case as follows. Assume that $g : \partial\Omega \rightarrow \mathbb{R}$ is in the range of the trace operator $T : H^1(\Omega) \rightarrow L^2(\partial\Omega)$, say $g = Tw$, then the weak formulation for the Dirichlet problem

$$\begin{aligned} -\Delta u &= f & \text{in } \Omega, \\ u &= g & \text{on } \partial\Omega, \end{aligned}$$

is obtained by replacing the first condition in Definition 2.1 by the condition $(u - w) \in H_0^1(\Omega)$. Otherwise the definition is the same.

Remark 2.3. The roots of the weak formulation (2.4) lie in the variational approach to Dirichlet's problem, in which one looks for a solution to the Dirichlet problem as the minimizer to the energy functional

$$J(u) = \frac{1}{2} \int_{\Omega} |Du|^2 dx - \langle f, u \rangle.$$

By properly defining the (Fréchet) derivative of this functional, and considering the minimization problem of $J : H_0^1(\Omega) \rightarrow \mathbb{R}$, one can show that the minimizer must be a weak solution of (2.1) in the sense of the Definition 2.1.

2.2 Existence of weak solutions of the Dirichlet problem

Using the weak formulation given by definition 2.1, the existence of weak solutions becomes an immediate consequence of the Riesz representation theorem for a suitably defined Hilbert inner product over $H_0^1(\Omega)$, which induces an equivalent norm to the standard $\|\cdot\|_{H_0^1(\Omega)}$ norm.

Theorem 2.4. *Let $\Omega \in \mathbb{R}^n$ be open, bounded in some direction, and $f \in H^{-1}(\Omega)$. Then there exists a unique weak solution $u \in H_0^1(\Omega)$ of the Dirichlet problem (2.1)-(2.2) in the sense of Definition 2.1.*

Proof. We define a binary operation on $H_0^1(\Omega)$ as follows

$$(u, v)_0 = \int_{\Omega} Du \cdot Dv dx. \quad (2.5)$$

It's easy to see that this binary operation is an inner product over $H_0^1(\Omega)$, provided Ω is bounded in some direction, and that the induced norm, $\|\cdot\|_0$, defined by $\|u\|_0 = (u, u)_0$, is equivalent to the standard norm $\|\cdot\|_{H_0^1(\Omega)}$ by Poincaré's inequality. This then implies that the space $H_0^1(\Omega)$ equipped with the $(\cdot)_0$ inner product is a Hilbert space, and $f \in H^{-1}(\Omega)$ is a bounded linear functional on $(H_0^1(\Omega), (\cdot)_0)$. But then by Riesz representation theorem, there exists a unique function $u \in H_0^1(\Omega)$, such that

$$(u, \phi)_0 = \langle f, \phi \rangle \quad \text{for all } \phi \in H_0^1(\Omega), \quad (2.6)$$

which is equivalent to u being a weak solution. $\square$

This approach to weak solvability of the Dirichlet problem for Laplace's equation can be generalized to other elliptic operators. We consider several examples of such (symmetric) operators, and will consider non-symmetric elliptic operators in the next section.

Example 2.5. Consider the Dirichlet problem for the operator $L = -\Delta + I$,

$$\begin{aligned} -\Delta u + u &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned}$$

We call $u \in H_0^1(\Omega)$ a weak solution of this problem, if

$$\int_{\Omega} (Du \cdot D\phi + u\phi) dx = \langle f, \phi \rangle \quad \text{for all } \phi \in H_0^1(\Omega).$$

In analogy to (2.6) This is equivalent to the condition that

$$(u, \phi)_1 = \langle f, \phi \rangle \quad \text{for all } \phi \in H_0^1(\Omega), \quad (2.7)$$

where $(\cdot, \cdot)_1$ is the standard inner product on $H_0^1(\Omega)$. Hence, the Riesz representation theorem will again imply the existence of a unique weak solution.

Remark 2.6. In this example $\Omega \subset \mathbb{R}^n$ is a general open set, and doesn't have to be bounded in some direction, since we used the standard inner product, and thus do not rely on Poincaré's inequality to prove equivalence of induced norms. Moreover, (2.7) implies that $\|u\|_{H_0^1} = \|f\|_{H^{-1}}$, and hence, the operator $L = -\Delta + I$ is an isometry of $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$.

Example 2.7. We can slightly generalize the previous example, by considering the operator $L = -\Delta + \mu I$, where $\mu > 0$ is a real number. Given an open domain $\Omega \in \mathbb{R}^n$, and $f \in H^{-1}(\Omega)$, a function $u \in H_0^1(\Omega)$ is a weak solution of the Dirichlet problem

$$\begin{aligned} -\Delta u + \mu u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

if $(u, \phi)_\mu = \langle f, \phi \rangle$ for all $\phi \in H_0^1(\Omega)$, where

$$(u, v)_\mu = \int_{\Omega} (Du \cdot Dv + \mu uv) dx.$$

It's easy to see that the norm $\|\cdot\|_\mu$ induced by this inner product is again equivalent to the standard norm, precisely because $\mu > 0$. Hence, Riesz representation theorem again will imply the existence of a unique weak solution.

Example 2.8. To generalize the result of the previous example to the case of $\mu < 0$, we have to guarantee that the resulting binary operation still defines an inner product, with the associated norm being equivalent to the standard norm. If Poincaré's inequality holds for the domain Ω with some constant C , i.e. $\|u\|_{L^2(\Omega)}^2 \leq C \|Du\|_{L^2(\Omega)}^2$, then we will have

$$\int_{\Omega} \mu u^2 dx \geq -C|\mu| \int_{\Omega} |Du|^2 dx,$$

and hence, also

$$(u, u)_\mu = \int_{\Omega} (|Du|^2 + \mu u^2) dx \geq (1 - C|\mu|) \int_{\Omega} |Du|^2 dx \geq \frac{(1 - C|\mu|)}{(1 + C)} \|u\|_{H_0^1(\Omega)}^2.$$

Thus, if $-1/C < \mu < 0$, then $\|u\|_\mu$ defines a norm on $H_0^1(\Omega)$ equivalent to the standard norm (the other inequality is trivial). The existence of unique solution will then again follow from Riesz representation theorem.

Remark 2.9. For bounded domains the Dirichlet Laplacian has an infinite sequence of real eigenvalues $\{\lambda_n : n \in \mathbb{N}\}$, and it can be shown that the best constant (smallest constant, giving the sharpest inequality) in the Poincaré inequality is exactly the principal eigenvalue λ_1 . Then the above method won't work for $\mu < -1/\lambda_1$. Notice that when $\mu = -\lambda_n$, not only the solution may not exist for an arbitrary $f \in H^{-1}(\Omega)$, but even if a weak solution exists, it will not be unique, since adding an eigenfunction to a solution will still be a solution. Hence, we do not expect existence of a unique weak solution when $\mu < -1/C$, where C is the best constant in the Poincaré inequality.

Example 2.10. As the last example before embarking on the study of solvability for general elliptic operators, let us consider the operator

$$Lu = - \sum_{i,j=1}^n \partial_i (a_{ij} \partial_j u), \quad (2.8)$$

where the coefficients are assumed to be bounded, symmetric ($a_{ij} = a_{ji}$), and satisfy the uniform ellipticity condition. That is, for some $\theta > 0$,

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \quad \text{for all } x \in \Omega, \text{ and all } \xi \in \mathbb{R}^n.$$

The function $u \in H_0^1(\Omega)$ will be a weak solution of the Dirichlet problem for this operator,

$$\begin{aligned} Lu &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned}$$

if

$$a(u, \phi) = \langle f, \phi \rangle \quad \text{for all } \phi \in H_0^1(\Omega),$$

where $a : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ is the symmetric bilinear form associated with the operator, and is given by

$$a(u, v) = \sum_{i,j=1}^n \int_{\Omega} a_{ij} \partial_j u \partial_i v \, dx.$$

Now, if Ω is bounded in some direction, then boundedness of a_{ij} , uniform ellipticity, and the Poincaré inequality will imply that the symmetric bilinear form a defines an inner product on $H_0^1(\Omega)$, with the induced norm being equivalent to the standard norm of $H_0^1(\Omega)$. This will again imply that $f \in H^{-1}$ is a bounded linear functional on the Hilbert space $(H_0^1(\Omega), a)$, and hence the Riesz representation theorem will once again imply the existence of a unique weak solution of the Dirichlet problem for this operator.

Remark 2.11. The bilinear form a of course arises from integration by parts of the left hand side of the equation after multiplying by the function v . Thus, having the derivative in front of the entire term $a_{ij} \partial_j u$ is crucial, since we are not assuming that the coefficients a_{ij} are weakly differentiable. In such cases we will say that the elliptic operator is in the *divergence form*.

2.3 General linear elliptic PDEs

As in the previous section, we are interested in solving the PDE

$$Lu = f \quad \text{in } \Omega,$$

subject to homogeneous Dirichlet boundary conditions on $\partial\Omega$. Here we generalize the linear operator L , and consider an operator of the form

$$Lu = - \sum_{i,j=1}^n \partial_i (a_{ij} \partial_j u) + \sum_{i=1}^n \partial_i (b_i u) + cu. \quad (2.9)$$

Notice that the leading order terms, as well as the first order terms are in the divergence form, which will be useful when studying the weak formulation of the problem. If $a_{ij}, b_i \in C^1(\Omega)$, then an operator in non-divergence form can be always written in the divergence form, by possibly modifying the coefficient of first and zeroth order terms. However, we will assume only boundedness of the coefficients, thus, for the weak formulation the divergence form (of the highest order terms) is necessary.

Definition 2.12. The operator L given by (2.9) is called elliptic at the point $x_0 \in \Omega$, if the matrix $(a_{ij}(x_0))$ is positive definite. And the operator will be elliptic in all of Ω , if it is elliptic at every point.

We will assume the stronger notion of ellipticity, that of uniform ellipticity, given by the next definition.

Definition 2.13. The operator L given by (2.9) is called uniformly elliptic in Ω , if there exists a constant $\theta > 0$, such that

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 \quad (2.10)$$

for x almost everywhere in Ω and every $\xi \in \mathbb{R}^n$.

Remark 2.14. Uniform ellipticity means that the eigenvalues of the matrix $(a_{ij}(x))$ are bounded from below by θ uniformly in x almost everywhere in Ω . We will use the uniform ellipticity with the vector $\xi = Du$, which will in turn allow us to control the integral of $|Du|^2$ in terms of the integral of $\sum_{i,j=1}^n a_{ij} \partial_i u \partial_j u$.

Example 2.15. The Laplacian, $L = -\Delta$ is uniformly elliptic, since the matrix of coefficients of the leading order terms is the unit matrix, and thus the uniform ellipticity condition (2.10) holds with $\theta = 1$.

Let $\mu \in \mathbb{R}$, and consider the Dirichlet problem for the operator $L + \mu I$,

$$\begin{aligned} Lu + \mu u &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \quad (2.11)$$

In the sequel we will always make the following assumptions on the operator L given by (2.9):

- (i) (boundedness) the coefficient functions $a_{ij}, b_i, c : \Omega \rightarrow \mathbb{R}$ satisfy

$$a_{ij}, b_i, c \in L^\infty(\Omega) \quad (2.12)$$

- (ii) (symmetry in the leading terms) the coefficients of the leading terms are symmetric: $a_{ij} = a_{ji}$

- (iii) (uniform ellipticity) the operator is uniformly elliptic, i.e. (2.10) holds.

To obtain the weak formulation for the problem (2.11), we proceed as before: multiply the equation by a test function $\phi \in C_c^\infty(\Omega)$, integrate over Ω , and integrate by parts, assuming all the functions as well as the domain are smooth. This leads to the condition that $u \in H_0^1(\Omega)$ is a weak solution of (2.11), if

$$\int_{\Omega} \left[\sum_{i,j=1}^n a_{ij} \partial_i u \partial_j \phi - \sum_{i=1}^n b_i u \partial_i \phi + c u \phi \right] dx + \mu \int_{\Omega} u \phi dx = \langle f, \phi \rangle \quad (2.13)$$

for all $\phi \in H_0^1(\Omega)$.

We define the bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ associated with the operator L as

$$a(u, v) = \int_{\Omega} \left[\sum_{i,j=1}^n a_{ij} \partial_i u \partial_j v - \sum_{i=1}^n b_i u \partial_i v + c u v \right] dx. \quad (2.14)$$

This form is well-defined on $H_0^1(\Omega)$, and is bounded as we will see later. Notice, however, that it is not symmetric, unless $b_i = 0$. Using this bilinear form, we can write the weak formulation (2.13) in a more concise form.

Definition 2.16. Let $\Omega \in \mathbb{R}^n$ be open, $f \in H^{-1}(\Omega)$, and L is given by (2.9), whose coefficients are bounded, symmetric in the leading terms, and satisfy uniform ellipticity. Then $u : \Omega \rightarrow \mathbb{R}$ is a weak solution of (2.11), if:

- (i) $u \in H_0^1(\Omega)$, and

$$(ii) \quad a(u, \phi) + \mu(u, \phi)_{L^2} = \langle f, \phi \rangle \quad \text{for all } \phi \in H_0^1(\Omega), \quad (2.15)$$

where $(\cdot, \cdot)_{L^2}$ is the standard inner product of $L^2(\Omega)$.

Since the form a given by (2.14) is not symmetric unless $b_i = 0$, we have

$$a(v, u) = a^*(u, v),$$

where

$$a^*(u, v) = \int_{\Omega} \left[\sum_{i,j=1}^n a_{ij} \partial_i u \partial_j v - \sum_{i=1}^n b_i (\partial_i u) v + cuv \right] dx. \quad (2.16)$$

This is the bilinear form associated with the formal adjoint L^* of L ,

$$L^* u = - \sum_{i,j=1}^n \partial_i (a_{ij} \partial_j u) - \sum_{i=1}^n b_i \partial_i u + cu. \quad (2.17)$$

Using the weak formulation (2.15) via the bilinear form associated with the uniformly elliptic operator L , we would like to prove the existence of a unique weak solution by a method similar to the analogous proof for the Dirichlet Laplacian. In this case, however, the bilinear form a is not symmetric, and cannot be used to define an inner product. Fortunately, a similar result to the Riesz representation theorem holds for non-symmetric bilinear forms as well, which is due to Lax and Milgram.

2.4 Lax-Milgram theorem, solvability of general elliptic PDEs

We will state the Lax-Milgram theorem in the general setting of an abstract Hilbert space $\mathcal{H}$, and will subsequently apply it to the bilinear form associated with the uniformly elliptic operator in the Hilbert space $H_0^1(\Omega)$.

Theorem 2.17. (*Lax-Milgram*) *Let $\mathcal{H}$ be a Hilbert space with the inner product $(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$, and let $b : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ be a bilinear form on $\mathcal{H}$. Further assume that there exist constants $C_1, C_2 > 0$, such that*

- (i) $C_1 \|u\|_{\mathcal{H}}^2 \leq b(u, u)$ for all $u \in \mathcal{H}$
- (ii) $|b(u, v)| \leq C_2 \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}$ for all $u, v \in \mathcal{H}$.

Then for every bounded linear functional $f : \mathcal{H} \rightarrow \mathbb{R}$, there exists a unique element $u \in \mathcal{H}$, such that

$$\langle f, v \rangle = b(u, v) \quad \text{for all } v \in \mathcal{H}.$$

Remark 2.18. By using $v = u$ in the second condition in the above theorem, we can understand the two conditions as the two inequalities of the equivalence $b(u, u) \sim \|u\|_{\mathcal{H}}^2$. The first condition is the positive definiteness of the bilinear form, while the second condition is the boundedness. Thus the Lax-Milgram theorem states that every bounded functional on a Hilbert space can be represented by the functional $b(u, \cdot)$, provided the bilinear form b is bounded and positive definite. Since the inner product is bounded and positive definite, we can see that the Lax-Milgram theorem generalized the Riesz representation theorem, and in general no symmetry is assumed for the bilinear form b .

Proof. Notice that for every fixed $u \in \mathcal{H}$, the mapping $v \mapsto b(u, v)$ is a bounded linear functional on $\mathcal{H}$. By the Riesz representation theorem there exists a unique element $w \in \mathcal{H}$, such that

$$b(u, v) = (w, v) \quad \text{for all } v \in \mathcal{H}.$$

Denote the operator mapping u to w by B , i.e. $w = Bu$, and $b(u, v) = (Bu, v)$ for all $v \in \mathcal{H}$.

Using the hypothesis of the theorem, one can show that the operator B is linear, one to one, and that the range of B , $\text{ran}(B)$, is closed in $\mathcal{H}$. These would imply that $\text{ran}(B) = \mathcal{H}$. But then every element of $\mathcal{H}$ has a preimage under B , and from the Riesz representation theorem for f , we have

$$\langle f, v \rangle = (w, v) = (Bu, v) = b(u, v) \quad \text{for all } v \in \mathcal{H},$$

where u is the preimage of the element $w \in \mathcal{H}$ under the operator B .

Uniqueness follows from linearity of b , and condition (i). □

To use the Lax-Milgram theorem to prove the existence of a unique weak solution of (2.11), we need to show that the associated bilinear form satisfies the hypothesis of the theorem. This will depend on the following energy estimates.

Theorem 2.19. *Let a be the bilinear form on $H_0^1(\Omega)$ given by (2.14), and the coefficients are bounded, symmetric in the higher order terms, and satisfy the uniform ellipticity condition (2.10). Then there exist constants $C_1, C_2 > 0$ and $\gamma \in \mathbb{R}$, such that for all $u, v \in H_0^1(\Omega)$, the following estimates hold:*

$$C_1 \|u\|_{H_0^1(\Omega)}^2 \leq a(u, u) + \gamma \|u\|_{L^2(\Omega)}^2 \quad (2.18)$$

$$|a(u, v)| \leq C_2 \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}. \quad (2.19)$$

Remark 2.20. The constant γ in inequality (2.18) can be taken to be $\gamma = \theta - c_0$, if $b_i = 0$, and $\gamma = \frac{1}{2\theta} \sum_{i=1}^n \|b_i\|_{L^\infty}^2 + \frac{\theta}{2} - c_0$, if $b_i \neq 0$. Here θ is the constant in the uniform ellipticity condition, and $c_0 = \text{ess inf}_\Omega c$.

Proof. The second inequality, (2.19), is the boundedness of the bilinear form a , and follows directly from the boundedness of the coefficients.

The estimate (2.18) is a consequence of the uniform ellipticity. Indeed, by the uniform ellipticity,

$$\begin{aligned} \theta \|Du\|_{L^2}^2 &= \theta \int_\Omega |Du|^2 dx \leq \sum_{i,j=1}^n a_{ij} \partial_i u \partial_j u dx \\ &\leq a(u, u) + \sum_{i=1}^n \int_\Omega b_i u \partial_i u dx - \int_\Omega c u^2 dx \\ &\leq a(u, u) + \sum_{i=1}^n \|b_i\|_{L^\infty} \|u\|_{L^2} \|\partial_i u\|_{L^2} - c_0 \|u\|_{L^2}^2. \end{aligned}$$

Inequality (2.18) would follow, if one uses Cauchy's inequality with ϵ for the middle term on the right, and hides the $\|\partial_i u\|_{L^2}^2$ term with an ϵ coefficient on the left. $\square$

Remark 2.21. The estimate (2.18) is called Garding's inequality, and it is the crucial *a priori* estimate, that establishes the bound for the H_0^1 norm of the solution in terms of the bilinear form of the elliptic operator.

Using Theorem 2.19, we can now apply the Lax-Milgram theorem to problem (2.11).

Theorem 2.22. *Let $\Omega \in \mathbb{R}^n$ be open, $f \in \mathcal{H}^{-1}(\Omega)$, and L be the differential operator (2.9). Suppose the coefficients are bounded, symmetric in the highest order terms, and satisfy the uniform ellipticity condition, and let $\gamma \in \mathbb{R}$ be the constant for which Theorem 2.19 holds. Then for every $\mu \geq \gamma$ there exists a unique weak solution $u \in H_0^1(\Omega)$ of the Dirichlet problem (2.11).*

Proof. For $\mu \in \mathbb{R}$, we define the bilinear form $a_\mu : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ by

$$a_\mu(u, v) = a(u, v) + \mu(u, v)_{L^2}, \quad (2.20)$$

where a is the bilinear form associated with the operator L and is given by (2.14).

It is easy to see that a_μ is bounded. It also satisfies condition (i) of the Lax-Milgram theorem by Garding's inequality (2.18), provided $\mu \geq \gamma$. Hence, we can apply the Lax-Milgram theorem to show that for every $f \in H^{-1}(\Omega)$, there exists a unique function $u \in H_0^1(\Omega)$, such that

$$\langle f, v \rangle = a_\mu(u, v) \quad \text{for all } v \in H_0^1,$$

which is equivalent to u being a weak solution. $\square$

Remark 2.23. The above proof of existence of a unique weak solution applies to L^* given by (2.17) as well, with a replaced by a^* from (2.16) in the proof, even though the first order term is not in the divergence form.

2.5 Fredholm operators on Hilbert spaces

The solvability of the problem (2.11) for μ large enough implies that the operator $K = (L + \mu I)^{-1} : \mathcal{H}^{-1}(\Omega) \rightarrow H_0^1(\Omega)$ is well defined and, as we will later see, bounded. If we restrict K to $L^2(\Omega)$, and think of it as a map into $L^2(\Omega)$, then, provided Ω is bounded, the operator K will be compact, since $\text{ran}(K) \subset H_0^1(\Omega)$, which is compactly embedded into $L^2(\Omega)$ for bounded Ω by Relich's theorem. The operator $(L - \lambda I)^{-1}$ is called the resolvent of L , thus the above property states that L has a compact resolvent. As we will see in the next section, this fact leads to characterization of solvability of the equation $Lu - \lambda u = f$ for arbitrary $\lambda \in \mathbb{R}$, $f \in L^2(\Omega)$. In this section we give the formal definitions of compact and Fredholm operators on a Hilbert space, and state some of the properties of such operators without proof.

Let $\mathcal{H}$ be a Hilbert space equipped with the inner product $(\cdot, \cdot)$, and the associated norm $\|\cdot\|$. The space of bounded linear operators $T : \mathcal{H} \rightarrow \mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$. This space is a Banach space with respect to the operator norm

$$\|T\| = \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \in \mathcal{H}, x \neq 0 \right\}.$$

The adjoint of $T \in \mathcal{L}(\mathcal{H})$ is the linear operator $T^* \in \mathcal{L}(\mathcal{H})$, such that

$$(Tx, y) = (x, T^*y) \quad \text{for all } x, y \in \mathcal{H}.$$

An operator is self-adjoint, if $T = T^*$. The kernel and range of $T \in \mathcal{L}(\mathcal{H})$ are the subspaces

$$\ker(T) = \{x \in \mathcal{H} : Tx = 0\}, \quad \text{ran}(T) = \{y \in \mathcal{H} : y = Tx \text{ for some } x \in \mathcal{H}\}.$$

Definition 2.24. A linear operator $T \in \mathcal{L}(\mathcal{H})$ is called compact, if it maps bounded sets to pre-compact sets.

This is equivalent to the following: for every bounded sequence $\{x_n\} \subset \mathcal{H}$, there exists a converging subsequence of the sequence $\{Tx_n\}$.

Example 2.25. Any bounded linear map of finite rank, i.e. a linear operator whose range is finite dimensional is compact. As a consequence, every linear operator on a finite-dimensional Hilbert space is compact.

For compact self-adjoint operators the following spectral theorem holds.

Theorem 2.26. *Let $T \in \mathcal{L}(\mathcal{H})$ be a compact self-adjoint operator. T has at most countably many distinct real eigenvalues. If there are infinitely many eigenvalues $\{\lambda_n \in \mathbb{R}, n \in \mathbb{N}\}$, then necessarily $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. The eigenspace corresponding to each nonzero eigenvalue is finite dimensional, and the eigenvectors associated with distinct eigenvalues are orthogonal. Moreover, $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors of T , including those, if any, for eigenvalue zero.*

Remark 2.27. The fact that the eigenvalues are real and the corresponding eigenspaces are mutually orthogonal follows from self-adjointness of the operator. The compactness of the operator, on the other hand, implies that the spectrum cannot have an accumulation point other than zero, since otherwise we can always choose a bounded set consisting of unit pairwise orthogonal eigenvectors corresponding to the eigenvalues converging to the nonzero accumulation point, which will not have a precompact image. Hence, there can be at most countably many eigenvalues, and they must converge to zero.

This spectral theorem will be used to characterize the spectrum of a uniformly elliptic self-adjoint operator on a bounded domain via the spectrum of its compact resolvent.

We next turn to Fredholm operators.

Definition 2.28. A linear operator $T \in \mathcal{L}(\mathcal{H})$ is called a Fredholm operator, if

- (i) $\ker(T)$ has finite dimension

(ii) $\text{ran}(T)$ is closed, and has finite codimension.

The projection theorem for Hilbert spaces, coupled with property (ii) in the definition implies that $\mathcal{H} = \text{ran}(T) \oplus \text{ran}(T)^\perp$, and $\dim \text{ran}(T)^\perp = \text{codim } \text{ran}(T) < \infty$.

Definition 2.29. If $T \in \mathcal{L}(\mathcal{H})$ is Fredholm, then the index of T is the integer

$$\text{ind}(T) = \dim \ker(T) - \dim \text{ran}(T)^\perp.$$

Example 2.30. Every linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ on a finite dimensional Hilbert space is Fredholm with zero index, since any finite dimensional linear subspace is closed. The index will be zero due to the formula

$$\dim \mathcal{H} = \dim \ker(T) + \dim \text{ran}(T).$$

Example 2.31. The identity map I on any Hilbert space is Fredholm. Moreover, $\dim \ker(I) = \text{codim } \text{ran}(I) = 0$, and hence, $\text{ind}(I) = 0$.

Using the definition of the Fredholm operator, it's not hard to see the following property.

Theorem 2.32. If $T \in \mathcal{L}(\mathcal{H})$ is Fredholm, then so is T^* , and

$$\dim \ker(T^*) = \text{codim } \text{ran}(T), \quad \text{codim } \text{ran}(T^*) = \dim \ker(T), \quad \text{ind}(T^*) = -\text{ind}(T).$$

One can show that the set of compact operators is open as a subset of $\mathcal{L}(\mathcal{H})$ in the topology of the operator norm. Moreover, the set of Fredholm operators is closed under addition of compact operators.

Theorem 2.33. Suppose $T \in \mathcal{L}(\mathcal{H})$ is Fredholm, and $K \in \mathcal{L}(\mathcal{H})$ is compact, then:

- (i) There exists an $\epsilon > 0$, such that for any $H \in \mathcal{H}$ with $\|H\| < \epsilon$, $T + H$ is Fredholm. Moreover, for every such operator, $\text{ind}(T + H) = \text{ind}(T)$.
- (ii) $T + K$ is Fredholm and $\text{ind}(T + K) = \text{ind}(T)$.

Remark 2.34. The first statement of the theorem implies that not only the set of Fredholm operators is open in the operator norm topology, but that it is the union of connected components characterized by the index.

For Fredholm operators with zero index the following result holds, known as the Fredholm alternative, which characterizes the solvability of the linear equation corresponding to a Fredholm operator.

Theorem 2.35. Let $T \in \mathcal{L}(\mathcal{H})$ be a Fredholm operator with $\text{ind}(T) = 0$, then one of the following alternatives holds:

- (1) $\ker(T^*) = \ker(T) = 0$; $\text{ran}(T) = \text{ran}(T^*) = \mathcal{H}$.
- (2) $\ker(T^*) \neq 0$; $\dim \ker(T) = \dim \ker(T^*) < \infty$; $\text{ran}(T) = \ker(T^*)^\perp$, $\text{ran}(T^*) = \ker(T)^\perp$.

Remark 2.36. The Fredholm alternative for the Fredholm operator T with zero index can be interpreted as the solvability of the linear equation $Tx = y$. Indeed, the two alternatives are equivalent to the following:

- (1) $T^*z = 0$ has the only solution $z = 0$; and $Tx = y$ has a unique solution $x \in \mathcal{H}$ for every $y \in \mathcal{H} = \text{ran}(T)$.
- (2) $T^*z = 0$ has a nonzero solution, in which case the dimension of the solution space is equal to the dimension of the solution space of the equation $Tx = 0$; the equation $Tx = y$ is solvable, iff $(y, z) = 0$ for every z solving $T^*z = 0$.

Remark 2.37. The Fredholm alternative is a consequence of the fact that, if $T \in \mathcal{L}(\mathcal{H})$, then

$$\mathcal{H} = \overline{\text{ran}(T)} \oplus \ker(T^*), \quad \text{and } \overline{\text{ran}(T)} = \ker(T^*)^\perp.$$

In the case of a Fredholm operator, there are finitely many solvability conditions expressed by the orthogonality in the second alternative.

2.6 The Fredholm alternative for elliptic equations

As we mentioned in the beginning of the previous section, Theorem 2.22 for the weak solvability implies that the operator $L + \mu I$ for $\mu \geq \gamma$ is invertible, and we may define the inverse operator $K = (L + \mu I)^{-1}$, which maps $H^{-1}(\Omega)$ onto $H_0^1(\Omega)$. That is, $Kf = u$, iff $a_\mu(u, v) = \langle f, v \rangle$ for all $v \in H_0^1(\Omega)$, where a_μ is the bilinear form (2.20).

Clearly K is linear, and it is bounded due to the Garding inequality (2.18). Let us now assume that Ω is bounded. If we restrict K to $L^2(\Omega)$, and use the compact embedding of $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ for bounded domains, then the map

$$K : L^2(\Omega) \rightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega)$$

maps bounded sets in $L^2(\Omega)$ to bounded sets in $H_0^1(\Omega)$, which are precompact in $L^2(\Omega)$. Hence, as a map from $L^2(\Omega)$ to $L^2(\Omega)$, K is compact.

If $f \in L^2(\Omega)$, then for the dual pairing of $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ we have $\langle f, v \rangle = (f, v)_{L^2}$. Hence,

$$Kf = u \quad \text{iff} \quad a_\mu(u, v) = (f, v)_{L^2} \quad \text{for all } v \in H_0^1(\Omega). \quad (2.21)$$

Using the bilinear form $a_\mu^*(u, v) = a^*(u, v) + (u, v)_{L^2}$, where a^* is the bilinear form associated with L^* , the formal dual of L , given by (2.16), we can define the operator K^* in a similar way:

$$K^*g = v \quad \text{iff} \quad a_\mu^*(v, u) = (g, u)_{L^2} \quad \text{for all } u \in H_0^1(\Omega). \quad (2.22)$$

That is,

$$K^* = (L^* + \mu I)^{-1}|_{L^2(\Omega)}.$$

It is not hard to see that K^* is the adjoint of the operator K .

Theorem 2.38. *The operator K defined by (2.21) is a linear bounded operator $K : L^2(\Omega) \rightarrow L^2(\Omega)$. Its adjoint is the operator K^* given by (2.22). If Ω is bounded, then K is a compact operator.*

Proof. Boundedness follows directly from Garding's inequality (2.18). To show that K^* is the adjoint of K , take $f, g \in L^2(\Omega)$, for which $Kf = u$, $K^*g = v$. Then using (2.21) and (2.22), we have

$$(Kf, g)_{L^2} = (u, g)_{L^2} = (g, u)_{L^2} = a_\mu^*(v, u) = a_\mu(u, v) = (f, v)_{L^2} = (f, K^*g).$$

Compactness follows from Relich's theorem, as explained above. $\square$

Observe that, if K is compact on $L^2(\Omega)$, then so is the operator σK for every $\sigma \in \mathbb{R}$. But then the operator $(I + \sigma K)$ will be Fredholm by Theorem 2.33, and $\text{ind}(I + \sigma K) = \text{ind}(I) = 0$. Hence, the Fredholm alternative, Theorem 2.35 holds for this operator. We then have a Fredholm alternative for the elliptic operator $(L - \lambda I)$ as well, as encapsulated in the following theorem.

Theorem 2.39. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and L is a uniformly elliptic operator (2.9), for which Theorem 2.22 holds. Let L^* be the formal adjoint of L , given by (2.17), and $\lambda \in \mathbb{R}$. Then one of the following alternatives holds:*

- (1) *The only weak solution of the equation $L^*v - \lambda v = 0$ is $v = 0$. For every $f \in L^2(\Omega)$ there exists a unique weak solution $u \in H_0^1(\Omega)$ of the equation $Lu - \lambda u = f$. In particular, the only solution of $Lu - \lambda u = 0$ is $u = 0$.*
- (2) *The equation $L^*v - \lambda v = 0$ has a nonzero weak solution v . The solution space of the equations $Lu - \lambda u = 0$ and $L^*v - \lambda v = 0$ are finite dimensional and have the same dimension. For $f \in L^2(\Omega)$ the equation $Lu - \lambda u = f$ has a weak solution $u \in H_0^1(\Omega)$, iff $(f, v)_{L^2} = 0$ for every weak solution $v \in H_0^1(\Omega)$ of $L^*v - \lambda v = 0$, and if the solution u exists, it is not unique.*

Remark 2.40. Notice that for smooth u, v , for which we can perform integration by parts, $(Lu, v) = (u, L^*v)$, and we can see that if $Lu = f$, then necessarily

$$(f, v) = (Lu, v) = (u, L^*v) = 0$$

for all v solving $L^*v = 0$. The Fredholm alternative implies that this condition is also sufficient for the solvability of the elliptic equations, which is a consequence of the index of the operator I being equal to zero.

Proof. Since $K = (L + \mu I)^{-1}$ is compact, and hence $(I + \sigma K)$ is Fredholm on $L^2(\Omega)$, the Fredholm alternative, Theorem 2.35, holds for the equation

$$u + \sigma Ku = g \quad u, g \in L^2(\Omega), \quad (2.23)$$

for any $\sigma \in \mathbb{R}$. We consider the two alternatives separately.

(1) Suppose the only solution of $v + \sigma K^*v = 0$ is $v = 0$. Then, applying $(L^* + \mu I)$ to this equation, we see that the only solution of $L^*v + (\mu + \sigma)v = 0$ is $v = 0$. The Fredholm alternative then implies that for every $g \in L^2(\Omega)$ there is a unique solution of (2.23). Now take an arbitrary function $f \in L^2(\Omega)$, and let $g = Kf$, then the unique solution of (2.23) for this g will be in the range of K . We may then apply $(L + \mu I)$ to (2.23) to conclude that there is a weak solution $u \in \text{ran}(K) \subset H_0^1(\Omega)$ of the equation

$$Lu + (\mu + \sigma)u = f. \quad (2.24)$$

This solution must be unique, since otherwise (2.23) would have multiple solutions. Taking $\sigma = -(\lambda + \mu)$ leads to the first alternative in the theorem.

(2) Suppose $v + \sigma K^*v = 0$ has a finite dimensional subspace of solutions $v \in L^2(\Omega)$. Then $v = -\sigma K^*v \in \text{ran}(K^*)$, and applying $(L^* + \mu I)$ to this equation leads to

$$L^*v + (\mu + \sigma)v = 0.$$

By the Fredholm alternative, equation $u + \sigma Ku = 0$ has a finite dimensional solution space of the same dimension, and by applying $(L + \mu I)$ to this, so does the equation

$$Lu + (\mu + \sigma)u = 0.$$

Also, for any $f \in L^2(\Omega)$, (2.23) is solvable for $g = Kf$, if and only if $(g, v) = 0$ for all solutions v of $v + \sigma K^*v = 0$. But

$$(v, g)_{L^2} = (v, Kf)_{L^2} = (K^*v, f)_{L^2} = -\frac{1}{\sigma}(v, f)_{L^2},$$

and hence $Lu - \lambda u = f$ will have a weak solution iff (2.23) does for $\sigma = -(\lambda + \mu)$ and $g = Kf$, which by Fredholm alternative will happen iff $(v, g) = 0$, or equivalently $(f, v) = 0$ for every solution v of $v + \sigma K^*v = 0$. But the solutions of the last equation are exactly the solutions of $L^*v - \lambda v = 0$. $\square$

2.7 The spectrum of a self-adjoint elliptic operator

Suppose L is a symmetric uniformly elliptic operator in some domain $\Omega \subset \mathbb{R}^n$ of the form

$$Lu = - \sum_{i,j=1}^n \partial_i(a_{ij}\partial_j u) + cu, \quad (2.25)$$

where $a_{ij} = a_{ji}$, and $a_{ij}, c \in L^\infty(\Omega)$. The associated symmetric bilinear form will be

$$a(u, v) = \int_{\Omega} \left(\sum_{i,j=1}^n a_{ij} \partial_i u \partial_j v + cuv \right) dx.$$

If the domain Ω is bounded, then the resolvent $K = (L + \mu I)^{-1}$ is a compact self-adjoint operator on $L^2(\Omega)$ for μ large enough. Hence, Theorem 2.26 holds for this K . Since, as we saw in the last section, L has the same eigenfunctions as K , we have a corresponding spectral theorem for the elliptic operator L .

Theorem 2.41. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, then the operator L given by (2.25) has an increasing sequence of real eigenvalues of finite multiplicity*

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n \leq \dots,$$

such that $\lambda_n \rightarrow \infty$. Moreover, there is an orthonormal basis $\{\phi_n : n \in \mathbb{N}\}$ of $L^2(\Omega)$ consisting of eigenfunctions $\phi_n \in H_0^1(\Omega)$, which are weak solutions of

$$L\phi_n = \lambda_n \phi_n.$$

Proof. First, notice that if $K\phi = 0$ for some $\phi \in L^2(\Omega)$, then applying $(L + \mu I)$ to this equation will yield $\phi = 0$, so K doesn't have zero as one of its eigenvalues. This in particular will imply that K must necessarily have infinitely many eigenvalues, since otherwise K could have only finitely many linearly independent eigenfunctions, which could not span $L^2(\Omega)$.

Now if $K\phi = \kappa\phi$, for $\phi \in L^2(\Omega)$, then $\phi \in \text{ran}(K) \subset H_0^1(\Omega)$, and applying $(L + \mu I)$ to this equation gives

$$L\phi = \left(\frac{1}{\kappa} - \mu\right)\phi.$$

So ϕ is an eigenfunction of L corresponding to the eigenvalue $\lambda = 1/\kappa - \mu$. This means that $a(\phi, \phi) = \lambda \|\phi\|_{L^2}^2$, hence by Garding's inequality (2.18),

$$C_1 \|\phi\|_{H_0^1} \leq a(\phi, \phi) + \gamma \|\phi\|_{L^2}^2 = (\lambda + \gamma) \|\phi\|_{L^2}^2.$$

for some $\gamma \in \mathbb{R}$. It follows that $\lambda > -\gamma$, and so the eigenvalues of L are bounded from below. The limiting property follows from the spectral theorem for the compact operator K . $\square$

Remark 2.42. The boundedness of the domain Ω is crucial, since otherwise the operator K may not be compact, and the spectrum of L then may not be discrete. As an example, consider the Laplacian $L = -\Delta$ on $\mathbb{R}^n$, which has the purely continuous spectrum $[0, \infty)$.

3. Elliptic regularity theory

In this chapter we show that the solution to elliptic PDEs are smooth, provided so are the forcing term and the coefficients of the linear operator. It is convenient to start with the interior regularity of solutions.

3.1 Interior regularity

As a motivation to the regularity estimates, let us first consider the case of the Laplacian. Suppose $u \in C_c^\infty(\mathbb{R}^n)$. Integrating by parts twice, we get

$$\int (\Delta u)^2 dx = \int \left(\sum_{i=1}^n (\partial_i^2 u) \right) \left(\sum_{j=1}^n (\partial_j^2 u) \right) dx = \sum_{i,j=1}^n \int (\partial_{ij}^2 u)(\partial_{ij}^2 u) dx = \int |D^2 u|^2 dx.$$

Thus, if $\Delta u = f$, then we just computed that

$$\|D^2 u\|_{L^2} = \|f\|_{L^2}.$$

That is, we can control the L^2 -norm of all second order derivatives of u by the L^2 norm of the Laplacian of u . This identity suggests that if $f \in L^2$, and $u \in H^1$ is a weak solution of the Poisson's equation $\Delta u = f$, then $u \in H^2$. However, the above computation may not work for weak solutions that belong to H^1 , since the use of second and higher weak derivatives is not justified in the integration by parts.

Let us now consider the uniformly elliptic operator L given by

$$Lu = - \sum_{i,j=1}^n \partial_j (a_{ij} \partial_i u), \quad (3.1)$$

and the respective PDE

$$Lu = f \quad \text{in } \Omega, \quad (3.2)$$

where $\Omega \subset \mathbb{R}^n$ is open and $f \in L^2(\Omega)$. It is straightforward, and will be apparent from the proof how to extend the regularity theory to operators that contain lower-order terms.

We define a weak solution as the function $u \in H^1(\Omega)$ that satisfies the identity

$$a(u, v) = (f, v) \quad \text{for all } v \in H_0^1(\Omega), \quad (3.3)$$

where the bilinear form a associated with the elliptic operator (3.1) is given by

$$a(u, v) = \sum_{i,j=1}^n \int_{\Omega} a_{ij} \partial_i u \partial_j v dx. \quad (3.4)$$

Notice that we do not impose any boundary condition, so the interior regularity theorem will apply to any weak solution of (3.2), no matter what the boundary conditions are.

Before stating and proving the elliptic regularity theorem, let us first try to emulate the above integration by parts method used in the case of the Laplacian for the elliptic operator (3.1). For the

purpose of obtaining a local estimate for D^2u on a subdomain $\Omega' \Subset \Omega$, we take a cut-off function $\eta \in C_c^\infty(\Omega)$, such that $0 \leq \eta \leq 1$, and $\eta = 1$ on Ω' . As a test function we take

$$v = -\partial_k(\eta^2 \partial_k u). \quad (3.5)$$

Multiplying (3.2) by v , and integrating over Ω gives $(Lu, v) = (f, v)_{L^2}$. Then integration by parts gives

$$\begin{aligned} (Lu, v) &= \sum_{i,j=1}^n \int_{\Omega} \partial_j(a_{ij} \partial_i u) \partial_k(\eta^2 \partial_k u) dx \\ &= \sum_{i,j=1}^n \int_{\Omega} \partial_k(a_{ij} \partial_i u) \partial_j(\eta^2 \partial_k u) dx \\ &= \sum_{i,j=1}^n \int_{\Omega} \eta^2 a_{ij} (\partial_i \partial_k u) (\partial_j \partial_k u) dx + F, \end{aligned}$$

where F contains all the remaining terms from the product rule, i.e.

$$F = \sum_{i,j=1}^n \int_{\Omega} \{ \eta^2 (\partial_k a_{ij}) (\partial_i u) (\partial_j \partial_k u) + 2\eta \partial_j \eta [a_{ij} (\partial_i \partial_k u) (\partial_k u) + (\partial_k a_{ij}) (\partial_i u) (\partial_k u)] \} dx$$

Notice that F is linear in the second order derivatives in u , which, as we will see, is crucial to obtaining the a priori estimate for D^2u . Using the definition of η , and the uniform ellipticity with the vector $\xi = \eta D \partial_k u$, we see that

$$\theta \int_{\Omega'} |D \partial_k u|^2 dx = \theta \int_{\Omega'} |\eta D \partial_k u|^2 dx \leq \sum_{i,j=1}^n \int_{\Omega} \eta^2 a_{ij} (\partial_i \partial_k u) (\partial_j \partial_k u) dx = (f, v)_{L^2} - F.$$

Using the definition of v , we can bound the $(f, v)_{L^2}$ term on the right as follows.

$$\begin{aligned} (f, v)_{L^2} &= \int_{\Omega} f [\partial_k(\eta^2 \partial_k u)] dx = \int_{\Omega} f [\eta \partial_k \eta (\partial_k u) + \eta^2 \partial_k^2 u] dx \\ &\leq \|f\|_{L^2(\Omega)} \|\partial_k u\|_{L^2(\Omega')} + \|f\|_{L^2(\Omega)} \|\partial_k^2 u\|_{L^2(\Omega')} \\ &\leq C \left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2 + \frac{1}{\epsilon} \|f\|_{L^2(\Omega)}^2 + \epsilon \|D \partial_k u\|_{L^2(\Omega')}^2 \right), \end{aligned}$$

where we used Cauchy's inequality with ϵ for the term with second order derivatives of u . Since second order derivatives of u enter only linearly into the F term, we can bound it similarly to the above.

$$F \leq C \left(\|u\|_{H^1(\Omega)}^2 + \frac{1}{\epsilon} \|Du\|_{L^2(\Omega)}^2 + \epsilon \|D \partial_k u\|_{L^2(\Omega')}^2 \right).$$

Combining these estimates, and absorbing all the second order derivative terms of u on the left hand side (they enter the right hand side with a factor of ϵ , which can be made small), we obtain the estimate

$$\|D \partial_k u\|_{L^2(\Omega')}^2 \leq C \left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2 \right). \quad (3.6)$$

Remark 3.1. The H^1 norm on the right hand side of 3.6 can be bounded by the L^2 norm of f and the L^2 norm of u essentially in the same way as above, by taking as a test function $v = u$. This will lead to an estimate of the second order derivatives of u in terms of the L^2 norms of Lu and u .

Remark 3.2. Notice that in the derivation of (3.6) we assumed that u is twice differentiable (weakly) from the beginning. However, if this is not known a priori, as is the case for a weak solution $u \in H^1$, one can not use second order derivatives, and instead must work with difference quotients. Obtaining an estimate on the difference quotients of $\partial_k u$ uniformly in the size of the difference quotient, h , will imply that u is twice weakly differentiable and is in H_{loc}^2 . This is the gist of the next result.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be open, and assume that L is given by (3.1) with the coefficients $a_{ij} \in C^1(\Omega)$, and $f \in L^2(\Omega)$. If $u \in H^1(\Omega)$ is a weak solution of (3.2), then $u \in H^2(\Omega')$ for every $\Omega' \Subset \Omega$. Moreover,*

$$\|u\|_{H^2(\Omega')} \leq C \left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \right), \quad (3.7)$$

where the constant $C = C(n, a_{ij}, \Omega', \Omega)$ is independent of u and f .

Proof. We use a similar argument to the one that lead to estimate (3.6) in the smooth case. Let $\eta \in C_c^\infty(\Omega)$ be a smooth cut-off function, such that $0 \leq \eta \leq 1$, and $\eta = 1$ on Ω' . We use the following test function in (3.3),

$$v = -D_k^{-h} (\eta^2 D_k^h u) \in H_0^1(\Omega).$$

Integrating by parts, we obtain

$$\begin{aligned} a(u, v) &= - \sum_{i,j=1}^n \int_{\Omega} a_{ij} (\partial_i u) D_k^{-h} \partial_j (\eta^2 D_k^h u) \, dx = \sum_{i,j=1}^n \int_{\Omega} D_k^h (a_{ij} \partial_i u) \partial_j (\eta^2 D_k^h u) \, dx \\ &= \sum_{i,j=1}^n \int_{\Omega} \eta^2 a_{ij}^h (D_k^h \partial_i u) (D_k^h \partial_j u) \, dx + F, \end{aligned}$$

where $a_{ij}^h(x) = a_{ij}(x + h e_k)$, and F contains all the remaining terms coming from the product rule,

$$F = \sum_{i,j=1}^n \int_{\Omega} \left\{ \eta^2 (D_k^h a_{ij}) (\partial_i u) (D_k^h \partial_j u) + 2\eta \partial_j \eta [a_{ij}^h (D_k^h \partial_i u) (D_k^h u) + (D_k^h a_{ij}) (\partial_i u) (D_k^h u)] \right\} dx.$$

Using the uniform ellipticity of L with the vector $\xi = \eta D_k^h Du$, we get

$$\theta \int_{\Omega} \eta^2 \|D_k^h Du\|^2 \, dx \leq \sum_{i,j=1}^n \int_{\Omega} \eta^2 a_{ij} (D_k^h \partial_i u) (D_k^h \partial_j u) \, dx.$$

From the weak formulation (3.3) and the above, we have

$$\theta \int_{\Omega} \eta^2 \|D_k^h Du\|^2 \, dx \leq - \int_{\Omega} f D_k^{-h} (\eta^2 D_k^h u) \, dx - F. \quad (3.8)$$

We estimate the right hand side of this inequality using Cauchy-Schwartz and Cauchy's inequality as was done in obtaining estimate (3.6).

$$\begin{aligned} \left| \int_{\Omega} f D_k^{-h} (\eta^2 D_k^h u) \, dx \right| &\leq \|f\|_{L^2(\Omega)} \|D_k^{-h} (\eta^2 D_k^h u)\|_{L^2(\Omega)} \\ &\leq \|f\|_{L^2(\Omega)} \|\partial_k (\eta^2 D_k^h u)\|_{L^2(\Omega)} \\ &\leq \|f\|_{L^2(\Omega)} \left(\|\eta^2 D_k^h \partial_k u\|_{L^2(\Omega)} + \|2\eta (\partial_k \eta) D_k^h u\|_{L^2(\Omega)} \right) \\ &\leq \|f\|_{L^2(\Omega)} \left(\|\eta D_k^h \partial_k u\|_{L^2(\Omega)} + C \|Du\|_{L^2(\Omega)} \right), \end{aligned}$$

where we used the fact that η is compactly supported in Ω , and hence the L^2 norm of the difference quotient is bounded by the norm of the weak derivative for sufficiently small h .

We can similarly bound the F term in (3.8),

$$|F| \leq C \left(\|Du\|_{L^2(\Omega)} \|\eta D_k^h Du\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)}^2 \right).$$

Now, using these bounds in (3.8) gives,

$$\begin{aligned} \theta \|\eta D_k^h Du\|_{L^2(\Omega)}^2 &= \theta \int_{\Omega} \eta^2 \|D_k^h Du\|^2 \, dx \leq C \left(\|f\|_{L^2(\Omega)} \|\eta D_k^h Du\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \|Du\|_{L^2(\Omega)} \right. \\ &\quad \left. + \|Du\|_{L^2(\Omega)} \|\eta D_k^h Du\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)}^2 \right). \end{aligned}$$

Applying Cauchy's inequality with ϵ to the terms containing $\eta D_k^h Du$, and with constant 1 to the rest of the terms, we obtain the bound

$$\begin{aligned} \theta \|\eta D_k^h Du\|_{L^2(\Omega)}^2 &\leq C \left(\frac{1}{\epsilon} \|f\|_{L^2(\Omega)}^2 + \epsilon \|\eta D_k^h Du\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2 + \|Du\|_{L^2(\Omega)}^2 \right. \\ &\quad \left. + \frac{1}{\epsilon} \|Du\|_{L^2(\Omega)}^2 + \epsilon \|\eta D_k^h Du\|_{L^2(\Omega)}^2 + \|Du\|_{L^2(\Omega)}^2 \right). \end{aligned}$$

Finally, absorbing the $\|\eta D_k^h Du\|_{L^2(\Omega)}^2$ terms on the right into the left hand side by choosing ϵ small enough, and using the fact that $\eta = 1$ on Ω' , we arrive at the estimate

$$\|D_k^h Du\|_{L^2(\Omega')}^2 \leq C \left(\|f\|_{L^2(\Omega)}^2 + \|Du\|_{L^2(\Omega)}^2 \right), \quad (3.9)$$

where the constant $C = C(\Omega, \Omega', a_{ij})$ is independent of h, u, f . Notice that this estimate holds with Ω replaced by Ω'' in the norms on the right hand side, where $\Omega' \Subset \Omega'' \Subset \Omega$.

We can estimate $\|Du\|_{L^2(\Omega'')}^2$ in terms of $\left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \right)$ by taking $v = \zeta u \in H_0^1(\Omega)$ in (3.3), where $\zeta \in C_c^\infty(\Omega)$ is a smooth cut-off function, such that $0 \leq \zeta \leq 1$ and $\zeta = 1$ on Ω'' . Then uniform ellipticity of L implies

$$\begin{aligned} \theta \int_{\Omega''} |Du|^2 dx &\leq \int_{\Omega} |\zeta Du|^2 \leq \sum_{i,j=1}^n \int_{\Omega} \zeta^2 a_{ij} \partial_i u \partial_j u \\ &\leq \int_{\Omega} f u dx \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \right). \end{aligned}$$

Combining this with (3.9) (where Ω is replaced by Ω'' on the right) gives the estimate

$$\|D_k^h Du\|_{L^2(\Omega')}^2 \leq C \left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \right),$$

which is uniform in h , hence, u has second weak derivatives which belong to $L^2(\Omega')$. Moreover, estimate (3.7) holds. $\square$

Remark 3.4. If the operator L contains lower order terms, then estimate (3.7) can be proved in much the same way, with several more terms being estimated using Cauchy-Schwartz and Cauchy's inequality.

Remark 3.5. If $u \in H_{\text{loc}}^2(\Omega)$ and $f \in L^2(\Omega)$, then equation $Lu = f$, where the derivatives are understood in the weak sense, holds pointwise almost everywhere in Ω . Such solutions are called *strong solutions* to distinguish them from weak solutions that may not possess weak second order derivatives, and from classical solutions, which have continuous second order derivatives. The last theorem then implies that if L is uniformly elliptic, then any weak solution is necessarily a strong solution.

The repeated application of the interior elliptic regularity estimate (3.7) leads to higher interior regularity.

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^n$ be open, and assume that L is given by (3.1) with the coefficients $a_{ij} \in C^{k+1}(\Omega)$, and $f \in H^k(\Omega)$. If $u \in H^1(\Omega)$ is a weak solution of (3.2), then $u \in H^2(\Omega')$ for every $\Omega' \Subset \Omega$. Moreover,*

$$\|u\|_{H^{k+2}(\Omega')} \leq C \left(\|f\|_{H^k(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \right),$$

where the constant $C = C(n, k, a_{ij}, \Omega', \Omega)$ is independent of u and f .

The proof of this theorem uses induction on k and arguments similar to those in the proof of Theorem 3.3. The details are left as an exercise. Note that if the hypothesis of the theorem hold for $k > \frac{n}{2}$, then $f \in C(\Omega)$, and $u \in C^2(\Omega)$, so u is a classical solution of $Lu = f$. Furthermore, if f and a_{ij} are smooth, then so is the solution.

Corollary 3.7. *If $a_{ij}, f \in C^\infty(\Omega)$, and $u \in H^1(\Omega)$ is a weak solution of (3.2) with L given by (3.1), then $u \in C^\infty(\Omega)$.*

The proof of the corollary is left as an exercise, with the observation that smoothness is a local property, so it is enough to show that $u \in C^\infty(\Omega')$ for every open subset $\Omega' \Subset \Omega$.

We observe that Remark 3.4 applies to Theorem 3.6 also, as well as to the last Corollary.

3.2 Boundary regularity

4. Variational methods

In this chapter we revisit the question of weak solvability of elliptic PDE's via a variational approach. The calculus of variations methods have the advantage of being applicable to more general, including nonlinear problems, which differs from the purely linear methods based on the Riesz representation theorem. Although most of what we discuss can be directly generalized to general second order uniformly elliptic operators, here we will consider only the Laplacian.

4.1 The Derivative of a functional

Let X be a Banach space, and F a continuous functional on X , i.e., a continuous, but not necessarily linear map from X to $\mathbb{R}$. We will define the derivative and differentiability of such maps in analogy to the finite dimensional case $F : \mathbb{R}^n \rightarrow \mathbb{R}$. Recall that for such functions of several variables the derivative at a point $x \in \mathbb{R}^n$ in the direction of $y \in \mathbb{R}^n$, $F'(x)y = DF(x) \cdot y = D_y F(x)$ is defined to be

$$F'(x)y = \lim_{\epsilon \rightarrow 0} \frac{F(x + \epsilon y) - F(x)}{\epsilon}.$$

Differentiability requires the stronger condition of local linearity,

$$F(x + y) - F(x) = A(x)y + o(|y|), \quad \text{as } y \rightarrow 0,$$

where $A(x)$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}$. Of course in this case the directional derivative $F'(x)y$ must exactly coincide with $A(x)y$, so the directional derivative $F'(x)y$ must be linear in y . Indeed, for differentiable functions on $\mathbb{R}^n$, the directional derivative simply becomes $F'(x)y = DF(x) \cdot y$, where $DF(x)$ is the gradient of F at x . So we can think of $F'(x)$ as a linear functional on $\mathbb{R}^n$, that is, $F'(x) \in (\mathbb{R}^n)^*$, where $(\mathbb{R}^n)^*$ is the dual space of $\mathbb{R}^n$, and the derivative F' as a map $F' : \mathbb{R}^n \rightarrow (\mathbb{R}^n)^*$.

In analogy to functions on $\mathbb{R}^n$, we define the directional derivative of a functional $F : X \rightarrow \mathbb{R}$ on any Banach space X as

$$F'(x)y = \lim_{\epsilon \rightarrow 0} \frac{F(x + \epsilon y) - F(x)}{\epsilon}.$$

If the directional derivative at the point $x \in X$ exists in every direction $y \in X$, then the functional F is called Gâteaux differentiable. We further define Fréchet differentiability of F by the local linearity condition, i.e. F is (Fréchet) differentiable at x , if there exists a linear functional $A(x) : X \rightarrow \mathbb{R}$, so that

$$F(x + y) - F(x) = A(x)y + o(|y|), \quad \text{as } y \rightarrow 0.$$

Similar to the $\mathbb{R}^n$ case, we can see that Fréchet differentiability implies Gâteaux differentiability, and the directional derivative $F'(x)y$ must be linear in y . Then the derivative becomes a map from X to its dual X^* , $F' : X \rightarrow X^*$. We say that F is C^1 , if this map is continuous. From now on we will mean differentiability in the Fréchet sense.

A point $x \in X$ is called a critical point of a differentiable functional F , if $F'(x) = 0$ as a linear map, i.e.

$$F'(x)y = 0, \quad \text{for all } y \in X.$$

If F has the form of an action integral for some Lagrangian, then the last condition is equivalent to the Euler-Lagrange equation satisfied by x .

We further observe that if F attains its global minimum at x , then x must be a critical point of F . Indeed, if we define $f(t) = F(x + ty)$, where $t \in \mathbb{R}$, then $t = 0$ must be a minimum point for f . Hence, $f'(0) = F'(x)y = 0$. Since this can be done for every y , we see that x must be a critical point of F .

Definition 4.1. A functional $F : X \rightarrow \mathbb{R}$ on a Banach space X is called coercive, if there exists constants $C_1 > 0$, $C_2 \in \mathbb{R}$, such that

$$F(x) \geq C_1 \|x\|_X^2 - C_2 \quad \text{for all } x \in X. \quad (4.1)$$

Remark 4.2. Notice that coercive functionals are bounded from below, since $F(x) \geq -C_2 > -\infty$. In this case F will achieve its global minimum, if there exists $x \in X$, such that $F(x) = \inf\{F(x) : x \in X\}$. Graphically, one can visualize coercivity as a condition for the graph of $F(x)$ to lie above some parabola in $\|x\|_X$.

Remark 4.3. Coercivity can also be used to show boundedness of a sequence $\{x_n\} \subset X$, provided the sequence $\{F(x_n)\} \subset \mathbb{R}^n$ is bounded. In this sense, if one considers a “generalized norm” $\|x\|_F^2 = F(x) + C_2 \|x\|_X^2$, then (4.1), along with boundedness of F will imply the equivalence of this generalized norm to the Banach space norm $\|x\|_X$.

Our goal in this chapter is to establish weak solvability of elliptic equations by showing the existence of a global minimizer for a suitably chosen functional. This minimizer then will be a critical point for the functional, and by the choice of the functional, also a weak solution of the corresponding equation. If we denote $I = \inf\{F(x) : x \in X\}$, then there exists a sequence $\{x_j\} \subset X$, such that $F(x_j) \rightarrow I$. But then the sequence $\{F(x_j)\}$ is bounded, which implies the boundedness of the sequence $\{x_j\}$. If X is finite dimensional, then the boundedness of the sequence would imply by compactness that there is a converging subsequence $x_{j_k} \rightarrow x \in X$, which would be a minimizer for X . If, however, X is infinite dimensional, then bounded sets need not be precompact. In this case we will rely on weakly convergent subsequences, which exist by the Banach-Alaoglu theorem, provided X is reflexive, and the following property of functionals.

Definition 4.4. A continuous functional F on a Banach space X is called weakly lower semicontinuous, if

$$F(x) \leq \liminf_{j \rightarrow \infty} F(x_j), \quad \text{whenever } x_j \rightharpoonup x \text{ weakly in } X.$$

4.2 Solvability for the Dirichlet Laplacian

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $f \in L^2(\Omega)$. We consider the question of weak solvability of the Dirichlet problem for the Laplacian,

$$\begin{aligned} \Delta u &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \quad (4.2)$$

Let us define the functional

$$F(u) = \int_{\Omega} \left(\frac{1}{2} |Du|^2 + fu \right) dx, \quad (4.3)$$

which is well defined for functions $u \in X = H_0^1(\Omega)$ by Cauchy-Schwartz. We use the definition of partial derivatives to find $F'(u)v$:

$$\begin{aligned} F'(u)v &= \lim_{\epsilon \rightarrow 0} \frac{\int_{\Omega} \left(\frac{1}{2} |D(u + \epsilon v)|^2 + f(u + \epsilon v) \right) dx - \int_{\Omega} \left(\frac{1}{2} |Du|^2 + fu \right) dx}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \int_{\Omega} \left(Du \cdot Dv + \frac{\epsilon}{2} |Dv|^2 + fv \right) dx. \end{aligned}$$

Hence,

$$F'(u)v = \int_{\Omega} (Du \cdot Dv + fv) dx. \quad (4.4)$$

Notice that $F'(u)v$ is linear in v , and $F'(u)$ is bounded in $H_0^1(\Omega)$, since

$$|F'(u)v| \leq \|Du\|_{L^2(\Omega)} \|Dv\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq C \|v\|_{H_0^1(\Omega)}$$

by Cauchy Schwartz, where $C = C(u, f)$. The function F is differentiable everywhere in $H_0^1(\Omega)$, since

$$|F(u+v) - F(u) - F'(u)v| = \frac{1}{2} \int_{\Omega} |Dv|^2 dx \leq \frac{1}{2} \|Dv\|_{L^2(\Omega)}^2 = o(\|v\|_{H_0^1(\Omega)}).$$

We further check that F is C^1 on $X = H_0^1(\Omega)$. For this, observe that

$$|(F'(u) - F'(v))w| = \left| \int_{\Omega} (Du - Dv) \cdot Dw dx \right| \leq \|D(u-v)\|_{L^2(\Omega)} \|Dw\|_{L^2(\Omega)} \leq \|u-v\|_X \|w\|_X,$$

hence,

$$\|F'(u) - F'(v)\|_{X^*} = \sup_{w \neq 0} \frac{|(F'(u) - F'(v))w|}{\|w\|_X} \rightarrow 0, \quad \text{as } \|u-v\|_X \rightarrow 0.$$

So $F' : X \rightarrow X^*$ is a continuous map.

We see from (4.4) that if u is a critical point for F , then it is a weak solution of (4.2). Then to prove the existence of a weak solution for the Dirichlet problem (4.2) it is enough to find a critical point of the functional (4.3). We will do this by showing the existence of a minimizer for F , i.e. that F attains its minimum on $H_0^1(\Omega)$.

Let us start by checking that F is coercive on $H_0^1(\Omega)$. Indeed,

$$F(u) = \int_{\Omega} \left(\frac{1}{2} |Du|^2 + fu \right) dx \leq \frac{1}{2} \|Du\|_{L^2(\Omega)}^2 - \int_{\Omega} |uf| dx.$$

But

$$\int_{\Omega} |fu| dx \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq \frac{1}{2\epsilon^2} \|f\|_{L^2(\Omega)}^2 + \frac{\epsilon^2}{2} \|u\|_{L^2(\Omega)}^2 \leq \frac{1}{2\epsilon^2} \|f\|_{L^2(\Omega)}^2 + C \frac{\epsilon^2}{2} \|Du\|_{L^2(\Omega)}^2,$$

where we used Cauchy's inequality with ϵ , and Poincaré's inequality. Hence, by choosing ϵ small enough, we have

$$F(u) \geq \frac{1}{2} \|Du\|_{L^2(\Omega)}^2 - \frac{1}{2\epsilon^2} \|f\|_{L^2(\Omega)}^2 - C \frac{\epsilon^2}{2} \|u\|_{L^2(\Omega)}^2 \geq \frac{1}{4} \|Du\|_{L^2(\Omega)}^2 - C_2 \geq C_1 \|u\|_{H_0^1(\Omega)} - C_2,$$

where we again used Poincaré's inequality in the last step. So F is indeed coercive, and as a consequence, is also bounded from below.

Define $I = \inf\{F(u) : u \in H_0^1(\Omega)\}$. There exists a sequence $\{u_j\} \subset H_0^1(\Omega)$, such that $F(u_j) \rightarrow I$. But coercivity implies that the sequence $\{u_j\}$ is bounded. By the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, there exists a subsequence $\{u_{j_k}\}$ which converges in $L^2(\Omega)$, i.e. there exists $u \in L^2(\Omega)$, such that $u_{j_k} \rightarrow u$ in L^2 . By Banach-Alaoglu's theorem, we can take a further subsequence, which we will also call $\{u_{j_k}\}$, which converges weakly in $H_0^1(\Omega)$. That is, there exists $u' \in H_0^1(\Omega)$, such that $u_{j_k} \rightharpoonup u'$ weakly in $H_0^1(\Omega)$. But since the same subsequence converges in L^2 , and hence also weakly in L^2 to u , we must have $u = u'$, and in particular, $u \in H_0^1(\Omega)$. We claim that this u is a minimizer for F , indeed,

$$\begin{aligned} F(u) &= \frac{1}{2} \|Du\|_{L^2(\Omega)}^2 + \int_{\Omega} fu dx = \frac{1}{2} \|u\|_{H_0^1(\Omega)}^2 - \frac{1}{2} \|u\|_{L^2(\Omega)}^2 + \int_{\Omega} fu dx \\ &\leq \liminf_{j_k \rightarrow \infty} \left\{ \frac{1}{2} \|u_{j_k}\|_{H_0^1(\Omega)}^2 - \frac{1}{2} \|u_{j_k}\|_{L^2(\Omega)}^2 + \int_{\Omega} fu_{j_k} dx \right\} \\ &= \liminf_{j_k \rightarrow \infty} F(u_{j_k}) = I, \end{aligned}$$

where we used the fact that the norm of a Hilbert space is weakly lower semicontinuous. So u must be a critical point for F , and, hence, a weak solution of (4.2). The uniqueness of this weak solution can be proved by showing that F defined by (4.3) is strictly convex on $H_0^1(\Omega)$.

We chose the Dirichlet condition in (4.2) to be homogeneous for simplicity, but the above variational method can be used to show weak solvability in the case of inhomogeneous boundary conditions as well, which we briefly discuss next. Consider the following Dirichlet problem for the Laplacian,

$$\begin{aligned} \Delta u &= 0 & \text{in } \Omega, \\ u &= g & \text{on } \partial\Omega, \end{aligned} \quad (4.5)$$

where $g \in H^1(\Omega)$ (g is identified with its preimage under the trace map). We can express the boundary condition via the requirement that $(u - g) \in H_0^1(\Omega)$ (c.f. Remark 2.2). Then the weak solution must belong to the admissibility set

$$\mathcal{A} = \{u \in H^1(\Omega) : (u - g) \in H_0^1(\Omega)\} = \{u = g + v : v \in H_0^1(\Omega)\}.$$

Now, if we define the functional

$$F(u) = \frac{1}{2} \int_{\Omega} |Du|^2 dx, \quad (4.6)$$

then the minimizer of F on the admissibility set $\mathcal{A}$ will be a weak solution of (4.5). Indeed, defining $f(t) = F(u + tv)$ for $v \in H_0^1(\Omega)$, we see that $t = 0$ is a minimum point for f , hence, $f'(0) = F'(u)v = 0$. Since this can be done for every $v \in H_0^1(\Omega)$, we have

$$\int_{\Omega} Du \cdot Dv dx = 0, \quad \text{for all } v \in H_0^1(\Omega),$$

which means that u is a weak solution of (4.5). Thus, to show the existence of a weak solution, it is enough to find a minimizer for (4.6) over the set $\mathcal{A}$. Let us denote $I = \inf\{F(u) : u \in \mathcal{A}\} \geq 0$. We can find a sequence $\{u_j = g + v_j\} \subset \mathcal{A}$, for which $F(u_j) \rightarrow I$, then from coercivity of F the sequence $\{u_j\}$ must be bounded, and hence the sequence $\{v_j\} \subset H_0^1(\Omega)$ must be bounded as well. But then the existence of a minimizer of F on $\mathcal{A}$ can be shown exactly as before, by extracting a subsequence of $\{v_j\}$ that converges to some $v \in H_0^1(\Omega)$ both in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$. Then $u = g + v$ will be a minimizer for F . The details are left as an exercise.

4.3 Constrained optimization and application to eigenvalues

In the previous section we were concerned in minimizing a C^1 functional F associated with the Dirichlet problem for the Laplacian. Similarly, we may have C^1 functionals $F : X \rightarrow \mathbb{R}$ and $G : X \rightarrow \mathbb{R}$ on a Banach space X , and want to minimize $F(x)$ subject to the constraint $G(x) = 0$. The constraint is equivalent to the admissibility set $\mathcal{C} = \{x \in X : G(x) = 0\}$, which is a closed linear subspace of X as the null set of a continuous functional G . It turns out that one has the Lagrange multiplier condition for functionals as well, i.e. if $x_0 \in \mathcal{C}$ minimizes F subject to the constraint $G(x) = 0$, then $F'(x_0)$ must be a constant multiple of $G'(x_0)$.

Theorem 4.5. *Suppose F and G are C^1 functionals on a Banach space X . Let $x_0 \in X$ be such that $G(x_0) = 0$, and x_0 is a local extremum of F on the constraint set $\mathcal{C} = \{x : G(x) = 0\}$. Then either*

- (i) $G'(x_0)y = 0$ for all $y \in X$, or
- (ii) *there exists $\mu \in \mathbb{R}$, such that $F'(x_0)y = \mu G'(x_0)y$ for all $y \in X$.*

The proof is left as an exercise. Here we show how this theorem can be used to give a variational characterization of the eigenvalues of the Laplacian.

Let $\Omega \subset \mathbb{R}^n$ be open, bounded, with sufficiently smooth boundary. We say λ is an eigenvalue for the Laplacian $L = -\Delta$, if there exists a nontrivial solution of

$$\begin{aligned} \Delta u + \lambda u &= 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned} \quad (4.7)$$

Let us study the weak solutions $u \in H_0^1(\Omega)$ of (4.7), which satisfy

$$\int_{\Omega} Du \cdot Dv \, dx = \lambda \int_{\Omega} uv \, dx, \quad \text{for all } v \in H_0^1(\Omega). \quad (4.8)$$

From elliptic regularity theory these solutions must be smooth, and hence the pair (λ, u) will also solve (4.7) in the classical sense. However, in this section we understand solutions in the weak sense (4.8), and use the variational approach on $H_0^1(\Omega)$ to characterize the eigenvalues.

By linearity a solution (λ, u) determines a one-dimensional eigenspace of solutions, $\{\alpha u\} \subset H_0^1(\Omega)$. We can normalize these solutions by e.g. requiring that $\|u\|_{L^2(\Omega)} = 1$. This condition will then act as a constraint. Let us define the functionals

$$F(u) = \int_{\Omega} |Du|^2 \, dx \quad \text{and} \quad G(u) = \int_{\Omega} u^2 \, dx - 1.$$

These functionals are well defined on $H_0^1(\Omega)$, and are C^1 , with derivatives

$$F'(u)v = 2 \int_{\Omega} Du \cdot Dv \, dx \quad \text{and} \quad G'(u)v = 2 \int_{\Omega} uv \, dx.$$

If $u \in H_0^1(\Omega)$ minimizes F subject to the constraint $G(u) = 0$, then according to Theorem 4.5 there exists $\mu \in R$, such that (4.8) is satisfied with $\lambda = \mu$, since if $G'(u) = 0$, then $u = 0$, and u does not satisfy the constraint $\|u\|_{L^2(\Omega)} = 1$. Hence, to find a solution of the weak eigenvalue problem (4.8), it is enough to find a minimizer of F on the constraint set $\mathcal{C} = \{u \in H_0^1(\Omega) : G(u) = 0\}$. But this is a subspace of the Hilbert space $H_0^1(\Omega)$ as the nullset of a continuous functional G , and we can use exactly the same minimization procedure as in the last section to find a minimizer u with $\|u\|_{L^2(\Omega)} = 1$, such that $F(u) = I = \inf\{F(u) : u \in \mathcal{C}\}$. Furthermore, if we take $v = u$ in (4.8), we get

$$I = F(u) = \int_{\Omega} |Du|^2 \, dx = \lambda \int_{\Omega} u^2 \, dx = \lambda.$$

Thus, we can also define λ by the Rayleigh quotient

$$\lambda = \inf_{u \in H_0^1(\Omega)} \frac{\int_{\Omega} |Du|^2 \, dx}{\int_{\Omega} u^2 \, dx}.$$

From the last identity it is also clear that the best constant in Poincaré's inequality for Ω is exactly the principal eigenvalue of the Dirichlet Laplacian on Ω . Notice that clearly $\lambda \geq 0$, and if $\lambda = 0$, then $F(u) = 0$, hence $|Du| = 0$ a.e. But, since $u \in H_0^1(\Omega)$, this would mean that $u = 0$ contradicting the assumption that $u \in \mathcal{C}$. Hence, $\lambda > 0$. We will denote this solution of the weak eigenproblem (4.8) by (λ_1, u_1) .

To find the other eigenvalues, we first observe that if u_1 and u_2 are normalized eigenfunctions corresponding to distinct eigenvalues λ_1 and λ_2 respectively, then

$$(\lambda_1 - \lambda_2) \int_{\Omega} u_1 u_2 \, dx = \int_{\Omega} \lambda_1 u_1 u_2 \, dx - \int_{\Omega} \lambda_2 u_1 u_2 \, dx = \int_{\Omega} Du \cdot Dv \, dx - \int_{\Omega} Du \cdot Dv \, dx = 0,$$

which is, of course, a consequence of the self-adjointness of the Laplacian. Thus, $\int_{\Omega} u_1 u_2 \, dx = (u_1, u_2)_{L^2(\Omega)} = 0$, since $\lambda_1 \neq \lambda_2$. So we should look for u_2 in the orthogonal complement to the subspace spanned by u_1 , that is, in the space

$$X_2 = \{v \in X_1 = H_0^1(\Omega) : (v, u_1)_{L^2(\Omega)} = 0\}.$$

Since X_2 is the nullspace of the continuous function $(\cdot, u_1)_{L^2(\Omega)}$ on $H_0^1(\Omega)$, it must be closed, and thus is itself a Hilbert space with the same inner product as $H_0^1(\Omega)$. We can then obtain the second eigenvalue λ_2 by minimizing F on X_2 subject to the constraint $G(v) = 0$, which will give

$$\lambda_2 = \inf\{F(v) : v \in X_2, \text{ and } \|v\|_{L^2(\Omega)} = 1\} = \inf_{v \in X_2} \frac{\int_{\Omega} |Dv|^2 \, dx}{\int_{\Omega} v^2 \, dx},$$

and $\lambda_2 = F(v)$ is attained at some $v = u_2 \in X$. Since $X_2 \subset X_1$, it is clear that $\lambda_2 \geq \lambda_1$.

Proceeding inductively, we define $X_n = \{v \in H_0^1(\Omega) : (v, u_i)_{L^2(\Omega)} = 0 \text{ for } i = 1, 2, \dots, n-1\}$. Then

$$\lambda_n = \inf\{F(v) : v \in X_n \text{ and } \|v\|_{L^2(\Omega)} = 1\} = \inf_{v \in X_n} \frac{\int_{\Omega} |Dv|^2 dx}{\int_{\Omega} v^2 dx},$$

is an eigenvalue, which is attained at $u_n \in X_n$. This procedure will generate the sequence of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots,$$

whose associated eigenfunctions u_n are orthonormal in $L^2(\Omega)$:

$$(u_n, u_n)_{L^2(\Omega)} = \|u_n\|_{L^2(\Omega)}^2 = 1, \quad \text{and} \quad (u_n, u_m)_{L^2(\Omega)} = 0 \text{ if } n \neq m.$$

As we know from section 2.7, the sequence of eigenvalues $\lambda_n \rightarrow \infty$, a trivial consequence of which is that each λ_n occurs only finitely many times. Thus, the eigenspace associated with a given eigenvalue is finite dimensional. One can also show that the set of eigenfunctions $\{u_n\}$ forms an orthonormal basis of $L^2(\Omega)$.