5. Elliptic regularity theory

In this chapter we show that the solution to elliptic PDEs are smooth, provided so are the forcing term and the coefficients of the linear operator. It is convenient to start with the interior regularity of solutions.

5.1 Interior regularity

As a motivation to the regularity estimates, let us first consider the case of the Laplacian. Suppose $u \in C_c^\infty(\mathbb{R}^n)$. Integrating by parts twice, we get

$$\int (\Delta u)^2\, dx = \int \left(\sum_{i=1}^n (\partial_i^2 u)\right)\left(\sum_{j=1}^n (\partial_j^2 u)\right) dx = \sum_{i,j=1}^n \int (\partial_{ij}^2 u)(\partial_{ij}^2 u)\, dx = \int |D^2 u|^2\, dx.$$

Thus, if $\Delta u = f$, then we just computed that

$$\|D^2 u\|_{L^2} = \|f\|_{L^2}.$$

That is, we can control the L^2-norm of all second order derivatives of u by the L^2 norm of the Laplacian of u. This identity suggests that if $f \in L^2$, and $u \in H^1$ is a weak solution of the Poisson's equation $\Delta u = f$, then $u \in H^2$. However, the above computation may not work for weak solutions that belong to H^1, since the use of second and higher weak derivatives is not justified in the integration by parts.

Let us now consider the uniformly elliptic operator L given by

$$Lu = -\sum_{i,j=1}^n \partial_j(a_{ij}\partial_i u), \tag{5.1}$$

and the respective PDE

$$Lu = f \quad \text{in } \Omega, \tag{5.2}$$

where $\Omega \in \mathbb{R}^n$ is open and $f \in L^2(\Omega)$. It is straightforward, and will be apparent from the proof how to extend the regularity theory to operators that contain lower-order terms.

We define a weak solution as the function $u \in H^1(\Omega)$ that satisfies the identity

$$a(u,v) = (f,v) \qquad \text{for all } v \in H_0^1(\Omega), \tag{5.3}$$

where the bilinear form a associated with the elliptic operator (5.1) is given by

$$a(u,v) = \sum_{i,j=1}^n \int_\Omega a_{ij}\partial_i u \partial_j v\, dx. \tag{5.4}$$

Notice that we do not impose any boundary condition, so the interior regularity theorem will apply to any weak solution of (5.2), no matter what the boundary conditions are.

Before stating and proving the elliptic regularity theorem, let us first try to emulate the above integration by parts method used in the case of the Laplacian for the elliptic operator (5.1). For the purpose of obtaining a local estimate for D^2u on a subdomain $\Omega' \Subset \Omega$, we take a cut-off function $\eta \in C_c^\infty(\Omega)$, such that $0 \leqslant \eta \leqslant 1$, and $\eta = 1$ on Ω'. As a test function we take

$$v = -\partial_k(\eta^2 \partial_k u). \tag{5.5}$$

Multiplying (5.2) by v, and integrating over Ω gives $(Lu, v) = (f, v)_{L^2}$. Then integration by parts gives

$$\begin{aligned}(Lu, v) &= \sum_{i,j=1}^{n} \int_\Omega \partial_j(a_{ij}\partial_i u)\partial_k(\eta^2\partial_k u)\, dx \\ &= \sum_{i,j=1}^{n} \int_\Omega \partial_k(a_{ij}\partial_i u)\partial_j(\eta^2\partial_k u)\, dx \\ &= \sum_{i,j=1}^{n} \int_\Omega \eta^2 a_{ij}(\partial_i\partial_k u)(\partial_j\partial_k u)\, dx + F,\end{aligned}$$

where F contains all the remaining terms from the product rule, i.e.

$$F = \sum_{i,j=1}^{n} \int_\Omega \left\{\eta^2(\partial_k a_{ij})(\partial_i u)(\partial_j\partial_k u) + 2\eta\partial_j\eta\left[a_{ij}(\partial_i\partial_k u)(\partial_k u) + (\partial_k a_{ij})(\partial_i u)(\partial_k u)\right]\right\}\, dx$$

Notice that F is linear in the second order derivatives in u, which, as we will see, is crucial to obtaining the a priori estimate for D^2u. Using the definition of η, and the uniform ellipticity with the vector $\xi = \eta D\partial_k u$, we see that

$$\theta\int_{\Omega'} |D\partial_k u|^2\, dx = \theta\int_{\Omega'} |\eta D\partial_k u|^2\, dx \leqslant \sum_{i,j=1}^{n} \int_\Omega \eta^2 a_{ij}(\partial_i\partial_k u)(\partial_j\partial_k u)\, dx = (f, v)_{L^2} - F.$$

Using the definition of v, we can bound the $(f, v)_{L^2}$ term on the right as follows.

$$\begin{aligned}(f, v)_{L^2} = \int_\Omega f[\partial_k(\eta^2\partial_k u)]\, dx &= \int_\Omega f[\eta\partial_k\eta(\partial_k u) + \eta^2\partial_k^2 u]\, dx \\ &\leqslant \|f\|_{L^2(\Omega)}\|\partial_k u\|_{L^2(\Omega')} + \|f\|_{L^2(\Omega)}\|\partial_k^2 u\|_{L^2(\Omega')} \\ &\leqslant C\left(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{H^1(\Omega)} + \frac{1}{\epsilon}\|f\|^2_{L^2(\Omega)} + \epsilon\|D\partial_k u\|^2_{L^2(\Omega')}\right),\end{aligned}$$

where we used Cauchy's inequality with ϵ for the term with second order derivatives of u. Since second order derivatives of u enter only linearly into the F term, we can bound it similarly to the above.

$$F \leqslant C\left(\|u\|^2_{H^1(\Omega)} + \frac{1}{\epsilon}\|Du\|^2_{L^2(\Omega)} + \epsilon\|D\partial_k u\|^2_{L^2(\Omega')}\right).$$

Combining these estimates, and absorbing all the second order derivative terms of u on the left hand side (they enter the right hand side with a factor of ϵ, which can be made small), we obtain the estimate

$$\|D\partial_k u\|^2_{L^2(\Omega')} \leqslant C\left(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{H^1(\Omega)}\right). \tag{5.6}$$

Remark 5.1. The H^1 norm on the right hand side of 5.6 can be bounded by the L^2 norm of f and the L^2 norm of u essentially in the same way as above, by taking as a test function $v = u$. This will lead to an estimate of the second order derivatives of u in terms of the L^2 norms of Lu and u.

Remark 5.2. Notice that in the derivation of (5.6) we assumed that u is twice differentiable (weakly) from the beginning. However, if this is not know a priori, as is the case for a weak solution $u \in H^1$, one can not use second order derivatives, and instead must work with difference quotients. Obtaining an estimate on the difference quotients of $\partial_k u$ uniformly in the size of the difference quotient, h, will imply that u is twice weakly differentiable and is in H^2_{loc}. This is the gist of the next result.

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^n$ be open, and assume that L is given by (5.1) with the coefficients $a_{ij} \in C^1(\Omega)$, and $f \in L^2(\Omega)$. If $u \in H^1(\Omega)$ is a weak solution of (5.2), then $u \in H^2(\Omega')$ for every $\Omega' \Subset \Omega$. Moreover,*

$$\|u\|_{H^2(\Omega')} \leqslant C\left(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\right), \tag{5.7}$$

where the constant $C = C(n, a_{ij}, \Omega', \Omega)$ is independent of u and f.

Proof. We use a similar argument to the one that lead to estimate (5.6) in the smooth case. Let $\eta \in C_c^\infty(\Omega)$ be a smooth cut-off function, such that $0 \leqslant \eta \leqslant 1$, and $\eta = 1$ on Ω'. We use the following test function in (5.3),

$$v = -D_k^{-h}\left(\eta^2 D_k^h u\right) \in H_0^1(\Omega).$$

Integrating by parts, we obtain

$$\begin{aligned} a(u,v) = -\sum_{i,j=1}^n \int_\Omega a_{ij}(\partial_i u) D_k^{-h}\partial_j\left(\eta^2 D_k^h u\right)\,dx &= \sum_{i,j=1}^n \int_\Omega D_k^h(a_{ij}\partial_i u)\partial_j\left(\eta^2 D_k^h u\right)\,dx \\ &= \sum_{i,j=1}^n \int_\Omega \eta^2 a_{ij}^h (D_k^h\partial_i u)\left(D_k^h \partial_j u\right)\,dx + F, \end{aligned}$$

where $a_{ij}^h(x) = a_{ij}(x + he_k)$, and F contains all the remaining terms coming from the product rule,

$$F = \sum_{i,j=1}^n \int_\Omega \left\{\eta^2\left(D_k^h a_{ij}\right)(\partial_i u)\left(D_k^h\partial_j u\right) + 2\eta\partial_j\eta\left[a_{ij}^h\left(D_k^h\partial_i u\right)\left(D_k^h u\right) + \left(D_k^h a_{ij}\right)(\partial_i u)\left(D_k^h u\right)\right]\right\}dx.$$

Using the uniform ellipticity of L with the vector $\xi = \eta D_k^h Du$, we get

$$\theta\int_\Omega \eta^2\|D_k^h Du\|^2\,dx \leqslant \sum_{i,j=1}^n \int_\Omega \eta^2 a_{ij}\left(D_k^h\partial_i u\right)\left(D_k^h\partial_j u\right)\,dx.$$

From the weak formulation (5.3) and the above, we have

$$\theta\int_\Omega \eta^2\|D_k^h Du\|^2\,dx \leqslant -\int_\Omega f D_k^{-h}\left(\eta^2 D_k^h u\right)\,dx - F. \tag{5.8}$$

We estimate the right hand side of this inequality using Cauchy-Schwartz and Cauchy's inequality as was done in obtaining estimate (5.6).

$$\begin{aligned} \left|\int_\Omega f D_k^{-h}\left(\eta^2 D_k^h u\right)\,dx\right| &\leqslant \|f\|_{L^2(\Omega)}\|D_k^{-h}\left(\eta^2 D_k^h u\right)\|_{L^2(\Omega)} \\ &\leqslant \|f\|_{L^2(\Omega)}\|\partial_k\left(\eta^2 D_k^h u\right)\|_{L^2(\Omega)} \\ &\leqslant \|f\|_{L^2(\Omega)}\left(\|\eta^2 D_k^h\partial_k u\|_{L^2(\Omega)} + \|2\eta(\partial_k\eta)D_k^h u\|_{L^2(\Omega)}\right) \\ &\leqslant \|f\|_{L^2(\Omega)}\left(\|\eta D_k^h\partial_k u\|_{L^2(\Omega)} + C\|Du\|_{L^2(\Omega)}\right), \end{aligned}$$

where we used the fact that η is compactly supported in Ω, and hence the L^2 norm of the difference quotient is bounded by the norm of the weak derivative for sufficiently small h.

We can similarly bound the F term in (5.8),

$$|F| \leqslant C\left(\|Du\|_{L^2(\Omega)}\|\eta D_k^h Du\|_{L^2(\Omega)} + \|Du\|^2_{L^2(\Omega)}\right).$$

Now, using these bounds in (5.8) gives,

$$\begin{aligned} \theta\|\eta D_k^h Du\|^2_{L^2(\Omega)} = \theta\int_\Omega \eta^2\|D_k^h Du\|^2\,dx \leqslant & C\Big(\|f\|_{L^2(\Omega)}\|\eta D_k^h Du\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\|Du\|_{L^2(\Omega)} \\ & + \|Du\|_{L^2(\Omega)}\|\eta D_k^h Du\|_{L^2(\Omega)} + \|Du\|^2_{L^2(\Omega)}\Big). \end{aligned}$$

Applying Cauchy's inequality with ϵ to the terms containing $\eta D_k^h Du$, and with constant 1 to the rest of the terms, we obtain the bound

$$\theta\|\eta D_k^h Du\|^2_{L^2(\Omega)} \leqslant C\Big(\frac{1}{\epsilon}\|f\|^2_{L^2(\Omega)} + \epsilon\|\eta D_k^h Du\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} + \|Du\|^2_{L^2(\Omega)} + \frac{1}{\epsilon}\|Du\|^2_{L^2(\Omega)} + \epsilon\|\eta D_k^h Du\|^2_{L^2(\Omega)} + \|Du\|^2_{L^2(\Omega)}\Big).$$

Finally, absorbing the $\|\eta D_k^h Du\|^2_{L^2(\Omega)}$ terms on the right into the left hand side by choosing ϵ small enough, and using the fact that $\eta = 1$ on Ω', we arrive at the estimate

$$\|D_k^h Du\|^2_{L^2(\Omega')} \leqslant C\Big(\|f\|^2_{L^2(\Omega)} + \|Du\|^2_{L^2(\Omega)}\Big), \tag{5.9}$$

where the constant $C = C(\Omega, \Omega', a_{ij})$ is independent of h, u, f. Notice that this estimate holds with Ω replaced by Ω'' in the norms on the right hand side, where $\Omega' \Subset \Omega'' \Subset \Omega$.

We can estimate $\|Du\|^2_{L^2(\Omega'')}$ in terms of $\Big(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\Big)$ by taking $v = \zeta u \in H_0^1(\Omega)$ in (5.3), where $\zeta \in C_c^\infty(\Omega)$ is a smooth cut-off function, such that $0 \leqslant \zeta \leqslant 1$ and $\zeta = 1$ on Ω''. Then uniform ellipticity of L implies

$$\theta\int_{\Omega''} |Du|^2\, dx \leqslant \int_\Omega |\zeta Du|^2 \leqslant \sum_{i,j=1}^n \int_\Omega \zeta^2 a_{ij}\partial_i u \partial_j u \leqslant \int_\Omega fu\, dx \leqslant \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \leqslant C\left(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\right).$$

Combining this with (5.9) (where Ω is replaced by Ω'' on the right) gives the estimate

$$\|D_k^h Du\|^2_{L^2(\Omega')} \leqslant C\Big(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\Big),$$

which is uniform in h, hence, u has second weak derivatives which belong to $L^2(\Omega')$. Moreover, estimate (5.7) holds. ☐

Remark 5.4. If the operator L contains lower order terms, then estimate (5.7) can be proved in much the same way, with several more terms being estimated using Cauchy-Schwartz and Cauchy's inequality.

Remark 5.5. If $u \in H^2_{\mathrm{loc}}(\Omega)$ and $f \in L^2(\Omega)$, then equation $Lu = f$, where the derivatives are understood in the weak sense, holds pointwise almost everywhere in Ω. Such solutions are called *strong solutions* to distinguish them from weak solutions that may not posses weak second order derivatives, and from classical solutions, which have continuous second order derivatives. The last theorem then implies that if L is uniformly elliptic, then any weak solution is necessarily a strong solution.

The repeated application of the interior elliptic regularity estimate (5.7) leads to higher interior regularity.

Theorem 5.6. *Let $\Omega \subset \mathbb{R}^n$ be open, and assume that L is given by* (5.1) *with the coefficients $a_{ij} \in C^{k+1}(\Omega)$, and $f \in H^k(\Omega)$. If $u \in H^1(\Omega)$ is a weak solution of* (5.2)*, then $u \in H^2(\Omega')$ for every $\Omega' \Subset \Omega$. Moreover,*

$$\|u\|_{H^{k+2}(\Omega')} \leqslant C\left(\|f\|^2_{H^k(\Omega)} + \|u\|^2_{L^2(\Omega)}\right),$$

where the constant $C = C(n, k, a_{ij}, \Omega', \Omega)$ is independent of u and f.

The proof of this theorem uses induction on k and arguments similar to those in the proof of Theorem 5.3. The details are left as an exercise. Note that if the hypothesis of the theorem hold for $k > \frac{n}{2}$, then $f \in C(\Omega)$, and $u \in C^2(\Omega)$, so u is a classical solution of $Lu = f$. Furthermore, if f and a_{ij} are smooth, then so is the solution.

Corollary 5.7. *If $a_{ij}, f \in C^\infty(\Omega)$, and $u \in H^1(\Omega)$ is a weak solution of* (5.2) *with L given by* (5.1), *then $u \in C^\infty(\Omega)$.*

The proof of the corollary is left as an exercise, with the observation that smoothness is a local property, so it is enough to show that $u \in C^\infty(\Omega')$ for every open subset $\Omega' \Subset \Omega$.

We observe that Remark 5.4 applies to Theorem 5.6 also, as well as to the last Corollary.

5.2 Boundary regularity