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Topological reconfiguration in expanding Hele–Shaw flow*

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Abstract. The possibility of a finite-time topological reconnection in the expanding Hele–Shaw flow of immiscible fluids is numerically investigated. The initial conditions correspond to those of a zero-surface tension exact solution found by Howison that develops cusp singularities by interface overlapping and thus constitute a natural candidate for potential topological singularities in the presence of small surface tension. Using a spectrally accurate boundary integral method it is found that in the case of an air bubble surface tension regularizes the cusped singularities and the solution clearly exist for all times forming the well known fingering and tip-splitting patterns. On the other hand, the presence of a viscous fluid in the interior of the bubble creates side-fingering and a complex evolution signalling finite-time topological reconfigurations of the fluid interface. With high resolution the collapsing exponent is obtained and it is found that the minimum distance between adjacent parts of the interface decreases linearly with time.

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Contents

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1. Introduction

A Hele–Shaw cell is a device used to investigate the two-dimensional flow of viscous fluids in a narrow gap between two parallel plates. Hele–Shaw flow provides a relatively simple model to study pattern formation which is crucial to the understanding of technologically important processes such as dendritic crystal growth and directional solidification. This type of flow can also be linked to oil displacement in a porous medium.

It is well known that surface tension plays a crucial role in pattern formation and can act as a length-scale selection mechanism. However, can topological singularities (in the form of fluid entrainment) occur in finite time or does the solution exist for all times in the presence of small surface tension? Constantin and Pugh [1] have rigorously shown that such singularity is not possible for an initial interface close to a circle and in the absence of pumping. Almgren [2] demonstrated convincingly that this result is, in general, not true and that for a Hele–Shaw flow solely driven by surface tension a topological singularity can develop in finite time. In [2] the initial interface is a smooth 'dumbbell' bounding a viscous fluid from surrounding air with no pumping of fluid. Goldstein *et al* [3] have also provided strong evidence of topological reconnection in Hele–Shaw flows driven by gravity, in the unstably stratified configuration. In [3], the flow consists of a thin fluid layer against a wall with gravity acting perpendicular to it. But for immiscible flows that expand due to the presence of a source the question is still open. This work is intended to address this question and give some insight via highly accurate numerics.

We concentrate on initial conditions that correspond to a zero-surface tension exact solution of an air bubble found by Howison [4]. This zero-surface tension exact solution develops cusp singularities by interface overlapping and thus is a good candidate for potential topological singularities in the presence of small surface tension. However, due to the ill-posedness of the zero-surface tension problem [5, 6], the numerical investigation of the small surface tension flow is quite difficult [7, 8]. In addition, the possibility of singularity formation demands extremely high resolution. Here we employ a spectrally accurate (infinite order) boundary integral method with fourth-order time integration [9] that uses the small-scale decomposition technique of Hou *et al* [10] to remove the high-order stability constraint induced by surface tension. Our highly accurate numerics reveal that in the case of an air bubble, surface tension regularizes the cusped-flow and the solution appears to exist for all times. The solutions develops the well known patterns of fingering and tip-splitting but shows no signs of an eventual topological reconnection. However, when the bubble is filled with a viscous fluid (unstable two-phase flow) the interior fluid leads to the formation of complex side fingering and extremely thin jets with clear indications that finite-time topological reconfigurations will occur. The collapsing exponent is obtained and it is found that the minimum distance between adjacent parts of the interface decreases linearly in time.

The rest of this paper is organized as follows. In section 2, we present the equations that govern the motion of the interface in a Hele–Shaw. In section 3 we describe briefly the numerical method used. In section 4 we present the numerical results and give some final remarks in section 5.

2. Governing equations

In a Hele–Shaw cell two viscous fluids are confined between two closely spaced parallel plates. We assume that the fluids are incompressible and immiscible and that they have different but constant viscosities. The gap-averaged velocity $\boldsymbol{u}_j j = 1, 2$ of each fluid is given by Darcy's law,

$$\boldsymbol{u}_j = -\frac{b^2}{12\mu_j}\nabla p_j, \tag{1}$$

where the subscripts 1 and 2 represent the interior and exterior fluids, b is the cell gap, μ_j is the viscosity and p_j is the pressure. By the incompressibility condition $\nabla \cdot \boldsymbol{u}_j = 0$ it satisfies Laplace's equation $\nabla^2 p_j = 0$. The interface motion is subject to the following conditions:

$$[\boldsymbol{u} \cdot \hat{\boldsymbol{n}}]|_\Gamma = 0, \tag{2}$$

$$[p]|_\Gamma = \tau\kappa, \tag{3}$$

where $[\cdot]$ stands for the jump across the interface taken as the difference of the interior minus the exterior quantity and Γ denotes the interface. Here, $\hat{\boldsymbol{n}}$ is the exterior unit normal to Γ, τ is the surface tension and κ is the interface mean curvature.

Let the interface Γ be represented, at any instant t, by $(x(\alpha, t), y(\alpha, t))$ where $\alpha \in [0, 2\pi]$ defines a counter-clockwise parametrization of Γ and x and y are periodic functions of α. Introducing the complex interface position $z = x + \mathrm{i}y$, the equations of motion can be recast into a vortex sheet (boundary integral) formulation. We nondimensionalize the governing equations by taking as the unit of length the initial radius of the bubble and letting the source $Q = 1$. Since we have the freedom of choosing the tangential velocity of the interface, the evolution equations can be written as [11]

$$\bar{z}_t = \frac{1}{z(\alpha, t)} + \frac{1}{2\pi \mathrm{i}}\int_0^{2\pi} \frac{\gamma(\alpha', t)}{z(\alpha, t) - z(\alpha', t)}\, \mathrm{d}\alpha' + A(\alpha, t)\frac{\bar{z}_\alpha(\alpha, t)}{|z_\alpha(\alpha, t)|}, \tag{4}$$

$$\gamma = 2A_\mu \,\mathrm{Re}\left(\frac{z_\alpha(\alpha, t)}{z(\alpha, t)} + \frac{z_\alpha(\alpha, t)}{2\pi \mathrm{i}}\int_0^{2\pi} \frac{\gamma(\alpha', t)}{z(\alpha, t) - z(\alpha', t)}\, \mathrm{d}\alpha'\right) + S\kappa_\alpha, \tag{5}$$

where the bar denotes the complex conjugate, γ is the vortex sheet strength, $A_\mu = (\mu_1 - \mu_2)/(\mu_1 + \mu_2)$ is the Atwood viscosity ratio, S is a dimensionless surface tension parameter and $A(\alpha, t)$ is arbitrary tangential velocity. The subscripts t and α denote partial differentiation with respect to those variables.

We concentrate here on initial conditions that corresponds to a zero-surface tension solution found by Howison [4]. This solution develops four symmetric 5/2-cusp singularities as the flow tends to overlap when expanding. After passing the singularity time the solution continues to exist for all times. Exact solutions of air bubbles in Hele–Shaw flows without surface tension are usually obtained via a conformal map from the unit disc (in the complex ζ-plane) into the fluid domain transforming the boundary of the disc into the fluid interface. In particular, the interface position for Howison's exact solution is given by the map

$$x + \mathrm{i}y = \frac{a(t)}{\zeta} + \log\frac{(c(t) - \zeta)}{(c(t) + \zeta)} + \mathrm{i}\log\frac{(\mathrm{i}c(t) + \zeta)}{(\mathrm{i}c(t) - \zeta)}, \tag{6}$$

where the air–fluid interface corresponds to $\zeta = \mathrm{e}^{\mathrm{i}\alpha}$ taking the principal branch of the logarithm. $c(0) = c_0 > 1$, $a(0) = a_0 > 0$ and for $t > 0$ these coefficients satisfy the equations

$$a(4c - a) = \frac{Qt}{\pi} + a_0(4c_0 - a_0) \tag{7}$$

$$g(a, c) = ac - \ln[(c^2 + 1)/(c^2 - 1)] - 2\arctan c^2 = k. \tag{8}$$

This zero-surface tension solution develops the 5/2 cusps when $a = c = 3^{1/4}$ but then continues to exist after this time (see figure 1).

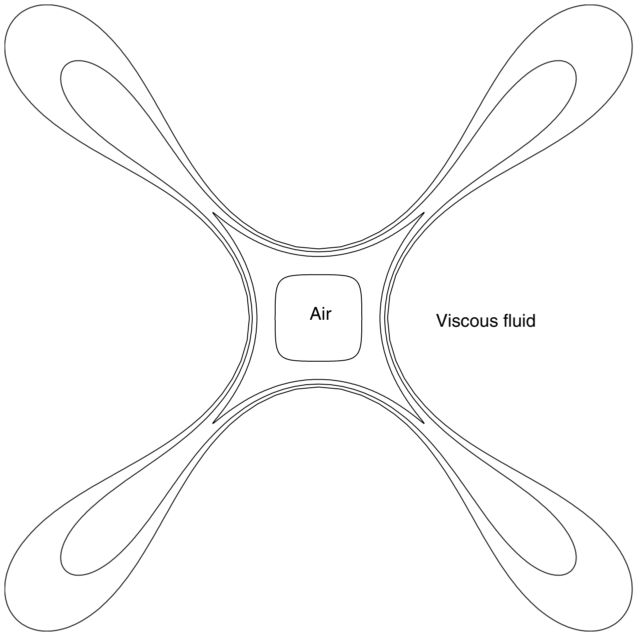

Figure 1. Howison's zero-surface tension exact solution with $c_0 = 3$, a_0 selected so that $k = \sqrt{3} - \log(2 + \sqrt{3}) - 2\pi/3$. The solution is plotted at different times, $t = 0, 1.3, 4.9$ and 10.

3. Numerical method

Surface tension introduces the term κ_α which causes high-order stiffness for explicit time integration methods and makes implicit methods difficult to implement. We use the small-scale decomposition method of Hou *et al* [10] which efficiently removes this stiffness. The method is based on the reformulation of the equations of motion in terms of the tangent angle θ to the interface and the arc-length metric $\sigma = \sqrt{x_\alpha^2 + y_\alpha^2}$ which are variables more naturally related to the curvature. It also identifies the small-scale terms that contribute to the surface-tension induced stiffness. The evolution equations in the new variables are given by,

$$\sigma_t = T_\alpha - \theta_\alpha U \tag{9}$$

$$\theta_t = \frac{1}{\sigma}(U_\alpha + T\theta_\alpha) \tag{10}$$

where T and U are the tangential and normal velocities respectively. The stiffness is hidden at the small spatial scales of U_α in the θ-equation. The leading order behaviour of U at small scales is given by [10],

$$U(\alpha, t) \sim \frac{S}{2\sigma} H\left[\left(\frac{\theta_\alpha}{\sigma}\right)_\alpha\right](\alpha, t), \tag{11}$$

where H is the Hilbert transform and S is dimensionless surface tension parameter. By choosing the tangential velocity as

$$T(\alpha, t) = \int_0^\alpha \theta_{\alpha'} U \, \mathrm{d}\alpha' - \frac{\alpha}{2\pi} \int_0^{2\pi} \alpha\theta_{\alpha'} U \, \mathrm{d}\alpha', \tag{12}$$

σ is kept constant and equal to its mean at all times. Setting $\sigma(t) = L(t)/2\pi$, where $L(t)$ is the total length of the curve, the equations of motion simplify to

$$L_t = -\int_0^{2\pi} \theta_{\alpha'} U \, \mathrm{d}\alpha', \tag{13}$$

$$\theta_t = \frac{S}{2}\left(\frac{2\pi}{L}\right)^3 H[\theta]_{\alpha\alpha\alpha} + P, \tag{14}$$

where P represents lower-order terms at small spatial scales. To remove the stiffness it is sufficient to discretize implicitly the leading order in equation (14) and treat the lower-order term P explicitly. We use the SBDF fourth-order explicit/implicit multi-step method presented in [12]. The principal value integral is approximated with the spectrally accurate alternate-point trapezoidal rule [13] and spatial derivatives are computed pseudo-spectrally. Symmetry is taken into account to speed up the computations.

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4. Numerical results

We numerically investigate the evolution of an expanding 'bubble' that is surrounded by a more viscous fluid and whose initial condition coincides with that of Howison's exact zero-surface tension solution (6). Our computations are performed with standard double precision with the Fourier filter level set to 10^{-12}. To mitigate the effects of round-off noise further we also use a selective filtering based on the four-fold symmetry of the solution (all the modes of $\theta - \alpha$ and of γ that are not multiples of 4 are set to zero). High resolution is of paramount importance in this problem. In all our calculations we maintain a spatial resolution such that the minimum distance between any two adjacent parts of the interface is greater that six mesh points. We begin all the computations with $N = 2048$ and we double N as soon as the magnitude of the last mode of $\theta(\alpha, t) - \alpha$ is greater than the filtering level or the minimum distance criterion is not satisfied. The time step is reduced by half every time that the number of grid points is doubled. At the final stage of motion, the number of interface particles increases to $N = 2^{15} = 32\,768$.

We now examine how surface tension modifies the cusped zero surface tension flow depicted in figure 1. We are particularly interested in the small surface tension values. We will see that the fluid interface develops very different behaviours depending on whether the bubble is filled with air or with a viscous fluid (whose viscosity is smaller than that of the surrounding fluid) and we study both cases separately.

4.1. Development of the interface for $A_\mu = -1$ (air bubble)

We select the nondimensional surface tension parameter $S = 0.001$. As figure 2 shows the presence of even very small surface tension regularizes the cusp singularities. Strong surface tension effects are seen well before the zero-surface tension singular time. Although physically counter-intuitive, this can be understood from the daughter singularity theory of Tanveer [14] and Siegel *et al* [15]. A complex (daughter) singularity approaches the physical domain well before the zero-surface tension singularity does. The solution exists past the zero-surface tension singular time developing the well known continuing pattern of fingering and tip-splitting competition but there is no indication that a topological singularity will form. Our numerics strongly support the all-time existence of the solution for this type of flow in the presence of fixed small but non-zero surface tension.

It is natural to ask how the presence of another fluid other than air would modify the flow dynamics and perhaps lead to topological singularity as the interior viscous fluid pushes the flow outwards in the expansion process. We now investigate this possibility.

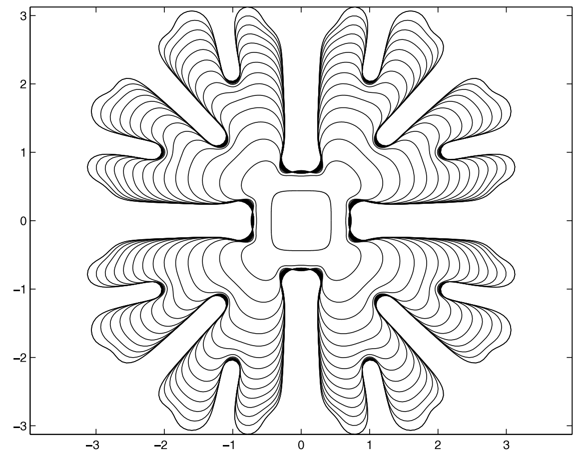

Figure 2. Interface development for $A_\mu = -1$ and $S = 0.001$. The interface is shown at times $t = 0 - 3.5$ with 0.25 time intervals.

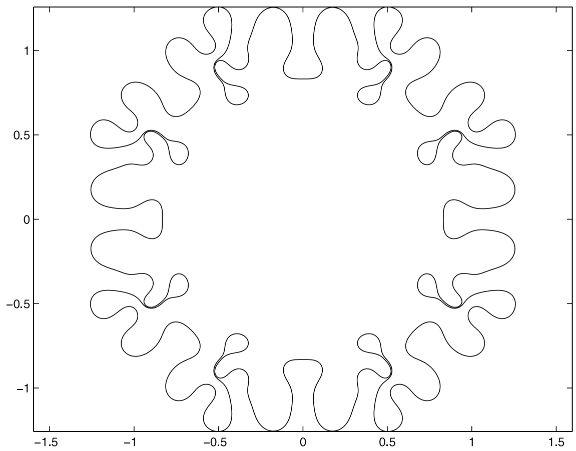

Figure 3. Interface at $t = 0.635$ for $A_\mu = -0.6$ and $S = 0.000\,25$. Initial conditions: equation (6) divided by four.

4.2. Development of the interface for $A_\mu = -0.6$

The presence of an interior viscous fluid completely changes the flow behaviour and leads to a surprising interface development. To better resolve the eventually very large expanding ‘bubble’ we scale down Howison data by a factor of four and also decrease the surface tension by a similar factor and take $S = 0.000\,25$. Similar behaviour is obtained for $S = 0.001$ and unscaled initial conditions.

The interface position at $t = 0.635$ for $A_\mu = -0.6$ and $S = 0.000\,25$ is presented in figure 3 where near topological singularities can be observed. The interface develops a convoluted pattern with ‘sideways’ merging of a finger into the base of a longer one giving rise to extremely narrow jets that are on the verge of collapsing. The time behaviour of the jet width can shed some light on the possible scenario. This is presented in figure 4. After a long nonlinear behaviour, the minimum distance appears to settle to an almost linear rate toward the late stage of the motion. This behaviour gives a strong indication that the interface would collapse in finite time. It is important to note that the resolution at all times is such that the jet width $d_{\min}(t)$

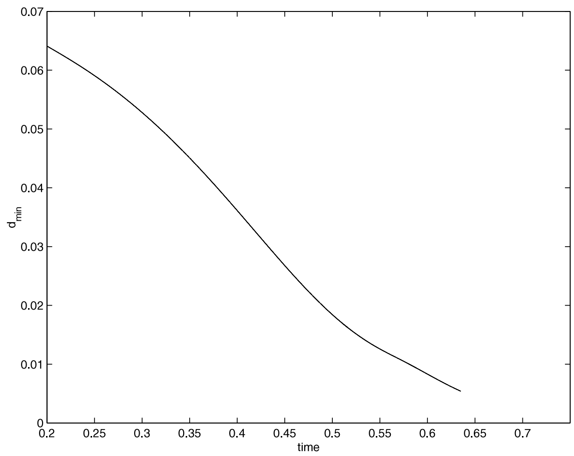

Figure 4. The time behaviour of the minimum distance between adjacent interface segments enclosing the thin jet.

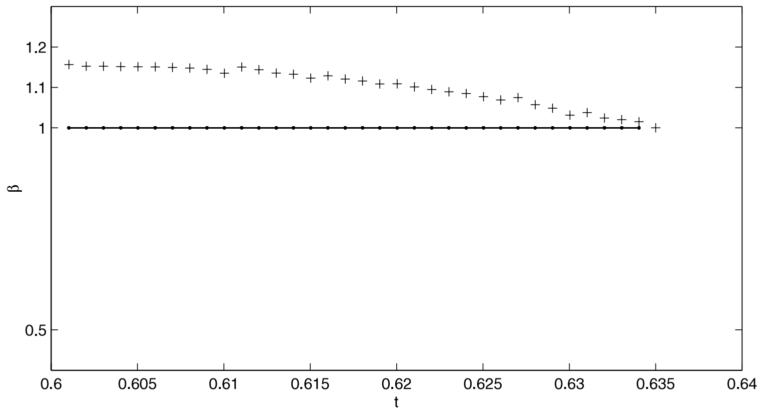

Figure 5. The collapsing exponent as a function of time.

is greater than six times the spacing between consecutive interfacial markers. It has been well documented [16, 10] that the alternate point trapezoidal rule (used to approximate the boundary integral) requires such resolution for accuracy. We fit the collapsing width with the Ansatz

$$d_{\min}(t) \propto (t^* - t)^\beta. \tag{15}$$

Using the linear least-squares method for the last ten data points we find $t^* = 0.7033$. We then perform exponential least-squares fits over a sliding set of 100 data points to determine the exponent β. The results of the fit are shown in figure 5. The collapsing exponent is fairly close to one and decreases toward this value as t approaches t^*. It is interesting to note that such approximate linear behaviour was also reported by Goldstein *et al* [3] but for a very different driving mechanism, namely gravity.

5. Conclusion

We have presented a numerical investigation of the effects of surface tension on the potential formation of topological singularities in the long-time evolution of expanding Hele–Shaw flows. With highly accurate numerics we provide evidence that strongly suggest a finite-time topological reconfiguration for a particular kind of two-phase immiscible Hele–Shaw flow in the presence of small surface tension. The topological reconfiguration occurs through 'sideways' merging of a finger into the base of a longer one. This side-fingering phenomenon has been well documented for miscible flows (see, for example, [17, 18]). Side fingering develops due to the presence of favourable pressure gradients in the flow. When these pressure gradients dominate capillary tension a topological singularity may develop in finite time. For an air bubble no such gradients can exist (constant pressure). This explains the striking difference between the behaviour of the one-phase and two-phase flows and makes the formation of topological singularities unlikely for expanding air bubbles.

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References

[1] Constantin P and Pugh M 1993 Global solutions for small data to the Hele-Shaw problem *Nonlinearity* **6** 393–415
[2] Almgren R 1996 Singularity formation in Hele-Shaw bubbles *Phys. Fluids* **8** 344–52
[3] Goldstein R E, Pesci A and Shelley M J 1998 Instabilities and singularities in Hele–Shaw flow *Phys. Fluids* **10** 2701–23
[4] Howison S D 1986 Cusp development in a Hele–Shaw flow with a free surface *SIAM J. Appl. Math.* **46** 20–6
[5] DiBenedetto E and Friedman A 1984 The ill-posed Hele–Shaw model and the Stefan problem for supercooled water *Trans. Am. Math. Soc.* **282** 183–204
[6] Howison S D, Ockendon J R and Lacey A A 1985 Singularity development in moving-boundary problems *Q. J. Mech. Appl. Math.* **38** 343–60
[7] DeGregoria A J and Schwartz L W 1986 A boundary-integral method for two-phase displacement in Hele–Shaw cells *J. Fluid Mech.* **164** 383–400
[8] Meiburg E and Homsy G M 1988 Nonlinear unstable viscous fingers in Hele–Shaw flows: 2. Numerical simulation *Phys. Fluids* **31** 429–39
[9] Ceniceros H D, Hou T Y and Si H 1999 Numerical study of Hele–Shaw flow with suction *Phys. Fluids* **11** 2471–86
[10] Hou T Y, Lowengrub J and Shelley M 1994 Removing the stiffness from interfacial flow with surface tension *J. Comput. Phys.* **114** 312–38
[11] Tryggvason G and Aref H 1983 Numerical experiments in Hele–Shaw flows with a sharp interface *J. Fluid Mech.* **136** 1–30
[12] Ascher U M, Ruuth S J and Wetton B 1995 Implicit–explicit methods for time-dependent pde's *SIAM J. Numer. Anal.* **32** 797–823
[13] Shelley M J 1992 A study of singularity formation in vortex sheet motion by a spectrally accurate vortex method *J. Fluid. Mech.* **244** 493–526
[14] Tanveer S 1993 Evolution of Hele–Shaw interface for small surface tension *Phil. Trans. R. Soc.* A **343** 155–204
[15] Siegel M, Tanveer S and Dai W 1996 Singular effects of surface tension in evolving Hele–Shaw flows *J. Fluid Mech.* **323** 201–36
[16] Baker G R and Shelley M J 1986 Boundary integral techniques for multi-connected domains *J. Comput. Phys.* **64** 112–32
[17] Zimmerman W B and Homsy G M 1992 Viscous fingering in miscible displacements: unification of effects of viscosity contrast, anisotropic dispersion, and velocity dependence of dispersion on nonlinear finger propagation *Phys. Fluids* A **4** 2348–59
[18] Petitjeans P, Chen C-Y, Meiburg E and Maxworthy T 1999 Miscible quarter five-spot displacements in Hele–Shaw cell and the role of flow-induced dispersion *Phys. Fluids* **11** 1705–16