

Math 118: Topology in Metric Spaces

You may recall having seen definitions of *open* and *closed* sets, as well as other topological concepts such as *limit* and *isolated points* and *density*, for sets of real numbers (or even sets of points in Euclidean space). A lot of these concepts help us define in analysis what we mean by points being “close” to one another so that we can discuss convergence and continuity. We are going to re-examine all of these concepts for subsets of more general spaces: One on which we have a nicely defined *distance*.

DEFINITION A **metric space** (X, d) is a set X , whose elements we call *points*, along with a function $d : X \times X \rightarrow \mathbb{R}$ that satisfies

(a) _____

(b) _____

(c) _____

For any two points $p, q \in X$, $d(p, q)$ is called the *distance* from p to q .

Notes: We clearly want non-negative distances, as well as symmetry. If we think of points in the Euclidean space \mathbb{R}^2 , or the complex plane, property (c) is the usual triangle inequality. Also note that for any $Y \subseteq X$, (Y, d) is a metric space.

EXAMPLES

You may have noticed that in a lot of these examples, the distance is of a special form. Consider a normed vector space (a vector space, as you learned in linear algebra, allows for the addition of the elements, as well as multiplication by a scalar. A norm is a real-valued function on the vector space that satisfies the triangle inequality, is 0 only for the $\mathbf{0}$ vector, and multiplying a vector by a scalar α just increases its norm by the factor $|\alpha|$.) The norm always allows us to define a distance: for two elements \mathbf{v}, \mathbf{w} in the space, $d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$.

In Math 117, you may have seen many topological definitions for subsets of \mathbb{R} . These definitions can be extended for subsets of a metric space, with the distance function replacing the absolute value to measure the closeness of points. For instance, one of the basic definitions was an “ ϵ -neighborhood,” which allowed us to talk about all the points in the space close to some given point.

DEFINITION Given a metric space X with a distance function d , the **neighborhood** of radius $\epsilon > 0$ of a point $p \in X$ is

$$N_\epsilon(p) = \underline{\hspace{10cm}}.$$

Using the distance, we can also define when a subset of a metric space is *bounded*.

DEFINITION Given a metric space (X, d) , a set $E \subseteq X$ is **bounded** if $\underline{\hspace{10cm}}$

$\underline{\hspace{10cm}}.$

DEFINITIONS Let (X, d) be a metric space and let $E \subseteq X$.

(i) A point $p \in X$ is a **limit point** of the set E if for every $r > 0$, $\underline{\hspace{10cm}}$

$\underline{\hspace{10cm}}.$

(ii) Any point $p \in E$ that is not a $\underline{\hspace{10cm}}$ is called an **isolated point** of E .

(iii) A point $p \in E$ is an **interior point** of E if there exists a neighborhood N of p such that $\underline{\hspace{10cm}}$.

(iii) E is **open** if $\underline{\hspace{10cm}}$.

(iv) E is **closed** if $\underline{\hspace{10cm}}$.

(Notice these definitions really only depend on knowing the neighborhoods. In topology, you will learn that you can work with spaces that don't even have the distance between two points defined! You simply need to define a topology: that is, state which of the subsets of the space are open neighborhoods.)

FACTS

- Neighborhoods are open sets.
- If p is a limit point of a set E , then every neighborhood of p contains infinitely many points of E .
- G is open if and only if G^c is closed. (And F is closed iff F^c is open.)
- There are sets that are neither open nor closed. (There are also sets that are both open and closed.)
- There are several easy-to-show theorems about collections of open or closed sets.

DEFINITION Let (X, d) be a metric space, and $E \subseteq X$. Let E' denote the set of _____
_____. Then, the **closure** of E is the set $\bar{E} =$ _____.

We will see that the closure of E is the smallest closed set containing E . Similarly, the interior of E (E°) is the largest open set inside of E .