Math 122B: Complex Variables

The Cauchy-Goursat Theorem

Cauchy-Goursat Theorem. *If a function f is analytic at all points interior to and on a* simple closed contour C *(i.e., f is analytic on some simply connected domain D containing C), then*

$$\int_C f(z)\,dz = 0.$$

Note. If we assume that f' is continuous (and therefore the partial derivatives of u and v are continuous where $f(z) = u(x, y) + iv(x, y)$), this result follows immediately from Green's theorem: Letting R be the region enclosed by the curve C,

$$\begin{aligned}\int_C f(z)\,dz = \int_C (u(x,y) + iv(x,y))\,(dx + i\,dy) &= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) \\ = \iint_R (-v_x - u_y)\,dA + i\iint_R (u_x - v_y)\,dA &= 0\end{aligned}$$

since f is analytic (use the Cauchy-Riemann equations!) However, the Cauchy-Goursat theorem says we don't need to assume that f' is continuous (only that it exists!)

Theorem. (An extension of Cauchy-Goursat)
If f is analytic in a simply connected domain D, then

$$\int_C f(z)\,dz = 0$$

for every closed contour C *lying in D.*

Notes.

• Combining this theorem with Theorem (§42), every function f that is analytic on a simply connected domain D must have an antiderivative on the domain D.

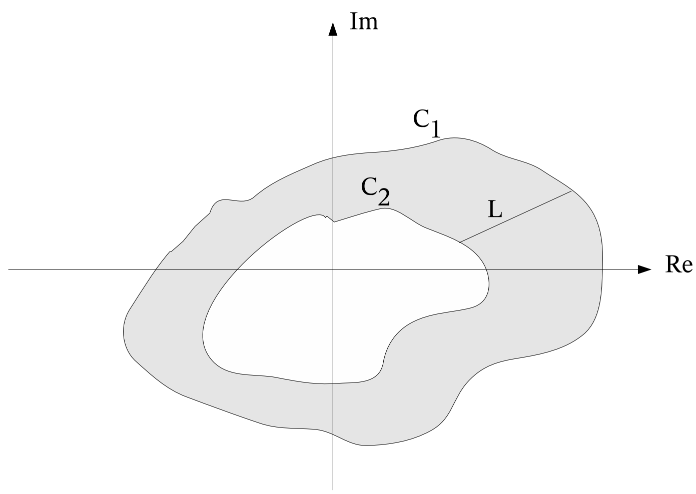
• Given two simple closed contours such that one can be continuous deformed into the other through a region where f is analytic, the contour integrals of f over these two contours have the same value! In other words, f might not be analytic in some region R, but if it is analytic outside of R, then the value of the contour integrals of f must be the same for all closed contours that enclose R – of course, this value doesn't have to be 0 since f is not analytic everywhere. (See the corollary below.)

Corollary. *Let C_1 be a positively oriented simple closed contour. Then, C_1 breaks the complex plane up into two regions: the interior of C_1 and the exterior of C_1 (by the* Jordan curve theorem*). Let C_2 be a positively oriented simple closed contour entirely inside the interior of C_1. If f is analytic in between and on C_1 and C_2, then*

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Proof. Connect the contours C_1 and C_2 with a line L (which starts at a point a on C_1 and ends at a point b on C_2). Integrate over a new contour C that both begins and ends at a: $C = (-C_2) \cup L \cup C_1 \cup (-L)$ (see the picture below – as you travel along C notice that the orientation is such that the domain in between C_1 and C_2 is always to the left!) Then, since this is a closed contour, the extension of Cauchy-Goursat implies that

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0.$$



Example. We can show that $\displaystyle\int_{C_o} \frac{1}{z}\,dz = 2\pi i$, where C_o is the positively oriented circle of radius ϵ_o centered at the origin (for any $\epsilon_o > 0$).

Therefore, for any positively oriented simple closed contour C whose interior contains the origin,

$$\int_C \frac{1}{z}\,dz = 2\pi i.$$

(Write out the details!)