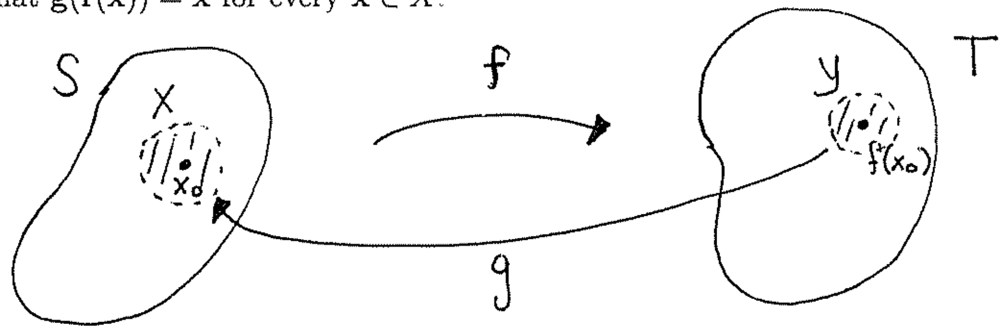
Inverse Function Theorem

Continuum Mechanics: Fall 2007

The Inverse Function Theorem: Let $\mathbf{f} : S \to \mathscr{E}^n$, $S \subseteq \mathscr{E}^n$, (where $\mathscr{E}^n$ is n-dimensional Euclidean space) be a smooth function. We will write $\mathbf{f} = (f_1, ..., f_n)$, where each function $f_j : S \to \mathbb{R}$. Also, let $T = f(S)$. Assume that the Jacobian $J[\mathbf{f}](\mathbf{x_o}) \neq 0$ at a point $\mathbf{x_o} \in S$. Then, there is a unique function $\mathbf{g} : Y \to X$, for some open sets $\mathbf{x_o} \in X \subseteq S$, $f(\mathbf{x_o}) \in Y \subseteq T$, such that $\mathbf{f}$ is one-to-one on X and $\mathbf{f}(X) = Y$, $\mathbf{g}$ is smooth and has the property that $\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}$ for every $\mathbf{x} \in X$.



We often think of the map $\mathbf{f}$ as defining a change of coordinates (from the $(x_1, ..., x_n)$ variables to the $(f_1, ..., f_n)$ variables). Then, a non-zero Jacobian at a point $\mathbf{x_o}$ implies that this change of coordinates is invertible (at least in a neighborhood of $\mathbf{x_o}$), and that the inverse map is smooth. In other words, if you know a point in terms of the f_j variables, you should be able to describe the point in terms of the x_j variables.

For example, consider the map from $\mathbf{x} = (x, y) \in \mathscr{E}^2 \setminus \{(0,0)\}$ to $(r, \theta) \in \mathscr{E}^2$ given by the functions $r = f_1(x, y) = \sqrt{x^2 + y^2}$ and $\theta = f_2(x, y) = \arctan\left(\frac{y}{x}\right)$. We can compute the Jacobian matrix of this mapping

$$\begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ -\dfrac{y}{(x^2+y^2)} & \dfrac{x}{(x^2+y^2)} \end{pmatrix}$$

The determinant of this matrix is $J[\mathbf{f}](\mathbf{x}) = (\sqrt{x^2+y^2})^{-1}$. If $\mathbf{x_o}$ is any non-zero point, there must exist a smooth inverse map (defined in some neighborhood of $\left(\sqrt{x_o^2 + y_o^2}, \arctan\left(\frac{y_o}{x_o}\right)\right)$) that describes points (x, y) in terms of the (r, θ) variables.

Another example is $\mathbf{f}(x, y) = (e^x \cos(y), e^x \sin(y))$. The Jacobian of this function is $J[\mathbf{f}](x, y) = e^{2x}$. Therefore, for every point (x_o, y_o), there is a neighborhood of the point $(e^{x_o} \cos y_o, e^{x_o} \sin y_o)$ on which $\mathbf{f}$ is invertible. Notice that even though $\mathbf{f}$ is one-to-one on some neighborhood of every single point, $\mathbf{f}$ is not one-to-one on the entire set $\mathscr{E}^2$, and so is not globally invertible!

For the proof, see, for example, *Mathematical Analysis: A Modern Approach to Advanced Calculus* by Tom M. Apostol or *Principles of Mathematical Analysis* by Walter Rudin.