Tensor Algebra II

Continuum Mechanics: Fall 2007

Reading: Gurtin, Section 2

Definition. A number ω is an **eigenvalue** of a tensor $\mathbf{S}$ if there exists a unit vector $\mathbf{e}$ such that $\mathbf{Se} = \omega\mathbf{e}$.

Definition. The **characteristic space** of $\mathbf{S}$ corresponding to ω is $\{\mathbf{v} : \mathbf{Sv} = \omega\mathbf{v}\}$.

Spectral Theorem. Let $\mathbf{S} \in \mathrm{Sym}$. Then, there exists an orthonormal basis consisting of eigenvectors of $\mathbf{S}$. If $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ is such a basis with corresponding eigenvalues $\omega_1, \omega_2, \omega_3$, then

$$\mathbf{S} = \sum_i \omega_i e_i \otimes e_i.$$

Conversely, if $\mathbf{S}$ has the form above for three orthonormal vectors $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$, then these vectors are eigenvectors with eigenvalues $\omega_1, \omega_2, \omega_3$.

Notes. This is saying that there is always a basis in which a symmetric matrix is diagonalizable. The proof relies on knowing that eigenvectors corresponding to different eigenvalues of symmetric matrices must be orthogonal (the hard part is showing that there is at least one eigenvalue!). With this, we can list all the possibilities for the characteristic spaces of a symmetric tensor (they are related to the multiplicity of the tensor's eigenvalues).

Square-root Theorem. Let $\mathbf{C} \in \mathrm{Psym}$. Then there exists a unique tensor $\mathbf{U} \in \mathrm{Psym}$ such that $\mathbf{U}^2 = \mathbf{C}$.

Note. The proof of existence gives us a procedure for computing $\mathbf{U}$.

Polar Decomposition Theorem. Let $\mathbf{F} \in \mathrm{Lin}^+$. Then there exist positive defininite, symmetric tensors $\mathbf{U}, \mathbf{V} \in \mathrm{Psym}$ and a rotation $\mathbf{R} \in \mathrm{Orth}^+$ such that

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}.$$

Each of these decompositions (the *right* and *left* polar decompositions) is unique. The tensors $\mathbf{U}$ and $\mathbf{V}$ are given by the formulas $\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}$, $\mathbf{V} = \sqrt{\mathbf{F}\mathbf{F}^T}$.

Definition. The **principal invariants** of a tensor $\mathbf{S}$ are given by $\iota_1(\mathbf{S}) = \mathrm{tr}\mathbf{S}$, $\iota_2(\mathbf{S}) = \frac{1}{2}[(\mathrm{tr}S)^2 - \mathrm{tr}S^2]$, $\iota_3(\mathbf{S}) = \det\mathbf{S}$

Notes. There are n invariants for $n \times n$ matrices; the first one is always the trace and the last one is always the determinant. The characteristic polymonial always has coefficients equal to these invariants. In our case (three dimensions):

$$\det(\mathbf{S} - \omega\mathbf{I}) = \omega^3 + \iota_1(\mathbf{S})\omega^2 + \iota_2(\mathbf{S})\omega + \iota_3(\mathbf{S})$$

Cayley-Hamilton Theorem. Every tensor $\mathbf{S}$ satisfies its own characteristic equation:

$$\mathbf{S}^3 - \iota_1(\mathbf{S})\mathbf{S}^2 + \iota_2(\mathbf{S})\mathbf{S} - \iota_3(\mathbf{S})\mathbf{I} = \mathbf{0}.$$

Exercises. Section 2: 1, 3, 4, 5, 6

- Find the square root of the following matrix:

$$\mathbf{A} = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 4 & 0 \\ -1 & 0 & 4 \end{pmatrix}$$