A Quick Review of Vector Calculus: Differentiation

Continuum Mechanics: Fall 2007

We consider functions that depend on more than one variable. For example, we might want to use the equation for the surface of the upper half-sphere $f(x, y) = \sqrt{1 - x^2 - y^2}$. We can graph a function of two variables on the standard coordinate axes (with basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$).

Partial derivatives are denoted by ∂ instead of d; for example, the partial derivative of a function with respect to x may be written as $\frac{\partial f}{\partial x}$ or simply $\partial_x f$. The definition of the partial derivative is given by

$$\partial_x f(x, y) := \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h},$$

if the limit exists. Notice the similarity of this to the definition of the derivative in one dimension. In essence, to take a partial derivative the other variables are held "fixed," so that a partial derivative with respect to x measures the rate of change in the x-direction only.

Question: Find the equation of the line pictured below.

[Picture of the surface $z = \sqrt{1 - x^2 - y^2}$ and the tangent line to the curve $(\frac{1}{2}, y, \sqrt{\frac{3}{4} - y^2})$ at the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$.]

Recall the definitions of **gradiant** and **directional derivatives**.

We can take repeated derivatives of functions; "mixed" partial derivatives of nice enough functions are equal. (Q: How nice?)

The second derivatives of a function may be organized in the **Hessian matrix**: (letting $f_x = \partial_x f$, $f_{xy} = \partial_y \partial_x f$, etc),

$$H[f] = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}$$

Notice that this matrix is always symmetric (for nice functions f). (Recall the second derivative test for min/max problems; Lagrange multipliers.)

You might also want to review spherical coordinates; cylindrical coordinates.

Other important theorems (though probably not very relevant to this class) include Taylor's theorem and the implicit function theorem.