Math CS-120: Homework 1

1. Show that if $S, T \in \mathcal{A}$ and $S \subseteq T$, then $a(S) \leq a(T)$.

2. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = x$ when x is rational and $f(x) = 0$ when x is irrational. Compute the upper and lower integrals of f. Is f Riemann integrable?

3. Given a function $\alpha : [a,b] \to \mathbb{R}$, and a partition P of $[a,b]$, let $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$. The upper and lower Riemann-Stieltjes integrals ($\overline{\int_a^b} f d\alpha$ and $\underline{\int_a^b} f d\alpha$) of a bounded function $f : [a,b] \to \mathbb{R}$ can be defined by using the sums

$$U(P, f, \alpha) = \sum_{k=1}^{n} M_k(f)\Delta\alpha_k$$

$$L(P, f, \alpha) = \sum_{k=1}^{n} m_k(f)\Delta\alpha_k$$

(The Riemann integral is the special case when $\alpha(x) = x$.) Let α be defined on $[-1,1]$ by $\alpha(x) = 0$ if $x < 0$ and $\alpha(x) = 1$ if $x \geq 0$. For any continuous $f : [-1,1] \to \mathbb{R}$, compute the upper and lower Riemann-Stieltjes integrals of f.

4. A set A is dense in $[0,1]$ if for every nonempty interval (a,b) such that $[0,1] \cap (a,b) \neq \emptyset$, it is also true that $(a,b) \cap A \neq \emptyset$. Prove that if f is Riemann integrable and $f(x) = 0$ for all $x \in A$, then $\int_a^b f = 0$.

5. If $f : [a,b] \to \mathbb{R}$ is a bounded, continuous, non-negative function such that $\int_a^b f = 0$, prove that $f(x) = 0$ for every $x \in [a,b]$.

6. Let $f : [a,b] \to \mathbb{R}$ be bounded. If P and Q are partitions of $[a,b]$, then

(i) If $P \subseteq Q$, then $L(P,f) \leq L(Q,f) \leq U(Q,f) \leq U(P,f)$.

(ii) $L(P,f) \leq U(Q,f)$

(iii) $\underline{\int_a^b} f \leq \overline{\int_a^b} f$

Hint: It is sufficient to prove (i) for the case when Q contains only one additional point.