Math CS-120: Homework 2

1. Prove Theorem 1: If $f : [a, b] \to \mathbb{R}$ then the following are equivalent:

(i) f is Riemann integrable on $[a, b]$

(ii) $\underline{\int_a^b} f = \overline{\int_a^b} f$

(Hint: Show (ii) $\Rightarrow$ (i) directly, but show (i) $\Rightarrow$ Riemann's condition.)

2. (a) Using Theorem 2, prove that $f(x) = x^3$ is Riemann integrable on $[0, 1]$. (That is, show Riemann's condition holds.)

(b) Compute the integral $\int_0^1 x^3\, dx$ by finding a number A such that $L(P, f) \leq A \leq U(P, f)$ for every partition $P \in \mathcal{P}[a, b]$.

3. Prove that if g is continuous on $[a, b]$ except at the point c ($a < c < b$), then g is Riemann integrable.

4. (a) Let f be Riemann integrable on $[a, b]$ and let $\frac{1}{f}$ be bounded. Prove that $\frac{1}{f}$ is Riemann integrable.

(b) Find an example where f is Riemann integrable on $[a, b]$ and $f(x) \neq 0$ for all $x \in [a, b]$, but $\frac{1}{f}$ is not Riemann integrable.

5. Prove that $\dfrac{1}{3\sqrt{2}} \leq \displaystyle\int_0^1 \frac{x^2}{\sqrt{1 + x^2}} \leq \frac{1}{3}$.

6. (a) Show that if f is Riemann integrable on $[a, b]$, then f^2 is Riemann integrable on $[a, b]$.

(b) Show that if f and g are Riemann integrable on $[a, b]$, then fg is Riemann integrable on $[a, b]$.