Math CS-120: Homework 3

1. Show that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{k^2+n^2}=\frac{\pi}{4}\quad\text{and}\quad\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k^2}}=\ln(1+\sqrt{2})$$

2. Find two Riemann integrable functions $\phi : [0,1]\to\mathbb{R}$ and $f:[0,1]\to[0,1]$ such that $\phi\circ f$ is not Riemann integrable on $[0,1]$.

3. In each example, f is defined on $[0,1]$. Find f'.

(a) $f(x)=\displaystyle\int_x^1 \cos\left(\frac{1}{1+t}\right)dt$

(b) $f(x)=\displaystyle\int_{x^2}^{2x}\sin(t^2)\,dt$

(c) $f(x)=\displaystyle\int_x^{\sqrt{x}}\frac{1}{1+t^3}\,dt$

4. Define

$$f(x)=\left(\int_0^x e^{-t^2}\,dt\right)^2,\qquad g(x)=\int_0^1\frac{e^{-x^2(t^2+1)}}{t^2+1}\,dt.$$

Show that $g'(x)+f'(x)=0$ for every x and deduce that $g(x)+f(x)=\frac{\pi}{4}$. Use this to prove that

$$\lim_{x\to\infty}\int_0^x e^{-t^2}\,dt=\frac{1}{2}\sqrt{\pi}.$$

5. Prove the more general corollary to the Fundamental Theorem of Calculus: If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), and f' is Riemann integrable on $[a,b]$, then

$$\int_a^b f'(x)\,dx=f(b)-f(a)$$

6. One way to define the natural logarithm (for $x>0$) is as the integral

$$L(x)=\int_1^x\frac{1}{t}\,dt$$

(a) Prove that L is continuous, differentiable, and increasing at each point $x>0$.
(b) Prove that $L(ab)=L(a)+L(b)$ for all $a,b>0$.
(c) Prove that $L(x^n)=nL(x)$.
(d) Prove that $L(\frac{a}{b})=L(a)-L(b)$.
(e) Show that for every real number y, there exists an $x>0$ with $L(x)=y$.

7. Prove that if f is positive and continuous on $[0,1]$

$$\lim_{n\to\infty}\left[\int_0^1 f(x)^n\,dx\right]^{\frac{1}{n}}=M$$

where M denotes the maximum value of f on $[0,1]$.