

## LENS RIGIDITY WITH PARTIAL DATA IN THE PRESENCE OF A MAGNETIC FIELD

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ABSTRACT. In this paper we consider the lens rigidity problem with partial data for conformal metrics in the presence of a magnetic field on a compact manifold of dimension  $\geq 3$  with boundary. We show that one can uniquely determine the conformal factor and the magnetic field near a strictly convex (with respect to the magnetic geodesics) boundary point where the lens data is accessible. We also prove a boundary rigidity result with partial data assuming the lengths of magnetic geodesics joining boundary points near a strictly convex boundary point are known. The local lens rigidity result also leads to a global rigidity result under some strictly convex foliation condition. A discussion of a weaker version of the lens rigidity problem with partial data for general smooth curves is given at the end of the paper.

**1. Introduction and main results.** Given a Riemannian manifold  $(M, g)$ , endowed with a magnetic field  $\Omega$ , that is a closed 2-form, we consider the law of motion described by

$$(1) \quad \nabla_{\dot{\gamma}} \dot{\gamma} = Y(\dot{\gamma}),$$

where  $\nabla$  is the Levi-Civita connection of  $g$  with the Christoffel symbols  $\{\Gamma_{jk}^i\}$  and  $Y : TM \rightarrow TM$  is the *Lorentz force* associated with  $\Omega$ , i.e. the bundle map uniquely determined by

$$(2) \quad \Omega_z(v, w) = \langle Y_z(v), w \rangle_g$$

for all  $z \in M$  and  $v, w \in T_z M$ . A curve  $\gamma : \mathbb{R} \rightarrow M$ , satisfying (1) is called a *magnetic geodesic*. The equation (1) defines a flow  $\phi_t : t \rightarrow (\gamma(t), \dot{\gamma}(t))$  on  $TM$  that we call a *magnetic flow*. It is not difficult to show that the generator  $\mathbf{G}_\mu$  of the magnetic flow is

$$(3) \quad \mathbf{G}_\mu(z, v) = \mathbf{G}(z, v) + Y_i^j(z) v^i \frac{\partial}{\partial v^j},$$

where  $\mathbf{G}(z, v) = v^i \frac{\partial}{\partial x^i} - \Gamma_{jk}^i v^j v^k \frac{\partial}{\partial v^i}$  is the generator of the geodesic flow. Note that time is not reversible on the magnetic geodesics, unless  $\Omega = 0$ . When  $\Omega = 0$  we obtain the ordinary geodesic flow. We call the triple  $(M, g, \Omega)$  a *magnetic system*.

It turns out that the magnetic flow is the Hamiltonian flow of  $H(v) = \frac{1}{2}|v|_g^2$ ,  $v \in TM$  with respect to the symplectic form  $\beta = \beta_0 + \pi^* \Omega$ , where  $\beta_0$  is the canonical

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symplectic form on  $TM$  and  $\pi : TM \rightarrow M$  is the canonical projection. Magnetic flows and magnetic geodesics were first considered in [3, 2]. Since the magnetic flow preserves the level sets of the Hamiltonian function  $H$ , every magnetic geodesic has constant speed. In the current paper we fix the energy level  $H^{-1}(\frac{1}{2})$ , i.e. we only consider the unit speed magnetic geodesics. However, this is not a constraint at all, it is easy to check from (1) that one can obtain the behavior (up to time scale) of magnetic geodesics at any energy level by rescaling the magnetic field  $\Omega$ .

From now on, we assume that  $\partial M \neq \emptyset$ . Let  $z \in \partial M$ ,  $S\partial M$  be the unit sphere bundle of the boundary  $\partial M$ , we say  $M$  is *strictly magnetic convex* at  $z$  if

$$\Lambda(z, v) > \langle Y_z(v), \nu(z) \rangle_g$$

for all  $v \in S_z\partial M$ , where  $\Lambda$  is the second fundamental form of  $\partial M$ ,  $\nu(z)$  is the inward unit vector normal to  $\partial M$  at  $z$ . When  $Y = 0$ , this is consistent with the ordinary definition of convexity.

In this paper, we mainly consider the lens rigidity problem for magnetic systems. To define the lens data of a magnetic system, we introduce the manifolds

$$\partial_{\pm}SM = \{(z, v) : z \in \partial M, v \in S_zM, \pm(v, \nu(z)) \geq 0\}.$$

We define the *scattering relation*

$$L : \partial_+SM \rightarrow \partial_-SM$$

as follows: given  $(z, v) \in \partial_+SM$ ,  $L(z, v) = (z', v')$ , where  $z'$  is the exit point,  $v'$  is the exit direction, if exist, of the maximal unit speed magnetic geodesic  $\gamma_{z,v}$  issued from  $(z, v)$ . Let

$$\ell : \partial_+SM \rightarrow \mathbb{R} \cup \infty$$

be the *travel time*, which is the length of  $\gamma_{z,v}$ , possibly infinite. If  $\ell < \infty$ , we say the magnetic system  $(M, g, \Omega)$  is *non-trapping*.  $L$  and  $\ell$  together are called the *lens data* of the magnetic system. It is easy to check that given a diffeomorphism  $\psi : M \rightarrow M$  fixing the boundary, the magnetic systems  $(M, g, \Omega)$  and  $(M, \psi^*g, \psi^*\Omega)$  have the same lens data. The *lens rigidity* problem for magnetic systems asks whether the lens data  $(L, \ell)$  uniquely determines a magnetic system up to the natural obstruction above.

The lens rigidity problem is closely related to another rigidity problem, namely the boundary rigidity problem. The latter, also known as the travel time tomography, is motivated by the geophysical problem of recovering the inner structure of the earth (such as the sound speed or the index of refraction) from the travel times of seismic waves at the surface [18, 55]. Mathematically, the ordinary *boundary rigidity* problem (no presence of magnetic fields) deals with the reconstruction of a compact Riemannian manifold  $(M, g)$  with smooth boundary from its boundary distance function  $d_g$ , where the value of  $d_g : \partial M \times \partial M \rightarrow \mathbb{R}$  at any two boundary points  $x, y$ , denoted by  $d_g(x, y)$ , is defined as the infimum of the lengths of geodesics in  $M$  connecting  $x$  and  $y$ . Similarly the boundary distance function is invariant under the diffeomorphism (it is actually an isometry) mentioned above, thus one can only expect to reconstruct  $g$  up to the natural obstruction.

There are examples showing that a general compact manifold with boundary may not be boundary rigid, see also Remark 1.6, one needs to impose additional geometric conditions. One such condition is simplicity. A compact Riemannian manifold  $M$  is *simple* if the boundary  $\partial M$  is strictly convex and any two points can be joined by a unique distance minimizing geodesic. It is known that on simple manifolds, the boundary rigidity problem and the lens rigidity problem are equivalent [27]. Michel

[27] conjectured that simple manifolds are boundary rigid, Pestov and Uhlmann showed that this is true for simple surfaces [39]. In higher dimensions, Stefanov and Uhlmann proved that a generic simple metric is boundary rigid [45], they also gave stability estimates. See [7, 46, 53] for recent surveys on the ordinary boundary rigidity problem. Rigidity problems for simple magnetic systems were studied in [12, 19] (see the definition of simple magnetic systems there), [13] provided a reconstruction procedure of simple 2D magnetic systems from scattering relations. Rigidity problems for more general Hamiltonian systems were considered in [4].

On non-simple manifolds, it is natural to consider the lens rigidity problem. Croke showed that the finite quotient of a lens rigid manifold is lens rigid [8], and the torus is lens rigid too [9]. Lens rigidity also holds on real-analytic manifolds which satisfy some non-conjugacy condition [54]. Stefanov and Uhlmann have shown lens rigidity for metrics close to a generic class of non-simple metrics including the real-analytic ones [48]. Recently, Guillarmou [17] proved lens rigidity up to conformal diffeomorphism for compact surfaces with convex boundary, free of conjugate points, with hyperbolic trapped sets, including the negatively curved ones. However, generally manifolds with trapped geodesics are not necessarily lens rigid [10]. Stability estimates for lens rigidity problem was studied in [5]. Reconstruction of a real-analytic magnetic system from its scattering relation was considered in [20].

Given a strictly magnetic convex boundary point  $p$ , the *partial data (or local lens rigidity)* problem for magnetic systems is whether we can determine the metric  $g$  and magnetic field  $\Omega$  near  $p$  from the (*partial*) lens data known near  $S_p\partial M$ . In this paper, we consider the case that  $g$  is in some conformal class, i.e.  $g = c^2g_0$  for some smooth positive function  $c$ , where  $g_0$  is known, we want to recover the conformal factor  $c$  (also the magnetic field  $\Omega$ ). Rigidity for the full data problem in the same conformal class was proven in [31, 32] for simple metrics, see also [6]. Then it was extended to simple magnetic systems in [12]. Notice that there is no natural obstruction to the unique determination in the conformal case, one expects to recover the system uniquely from its lens data.

Let  $\iota : \partial M \rightarrow M$  be the canonical inclusion,  $(L, \ell)$  and  $(\tilde{L}, \tilde{\ell})$  be the lens data of  $(c^2g_0, \Omega)$  and  $(\tilde{c}^2g_0, \tilde{\Omega})$  respectively. Assume that the conformal factor and the tangential part of the magnetic field, i.e.  $\iota^*\Omega$ , are known on  $\partial M$  near some strictly magnetic convex point  $p$ , we get the following local magnetic lens rigidity result.

**Theorem 1.1.** *Let  $n = \dim M \geq 3$ , let  $c, \tilde{c} > 0$  be smooth functions,  $\Omega, \tilde{\Omega}$  be smooth closed 2-forms and let  $\partial M$  be strictly magnetic convex with respect to both  $(c^2g_0, \Omega)$  and  $(\tilde{c}^2g_0, \tilde{\Omega})$  near  $p \in \partial M$ . Assume that on  $\partial M$  near  $p$ ,  $c = \tilde{c}$  and  $\iota^*\Omega = \iota^*\tilde{\Omega}$ . If  $L = \tilde{L}$ ,  $\ell = \tilde{\ell}$  near  $S_p\partial M$ , then  $c = \tilde{c}$  and  $\Omega = \tilde{\Omega}$  in  $M$  near  $p$ .*

This is the first partial data rigidity result for smooth magnetic systems, and it generalizes local rigidity results from [50] of the geodesic flow. Previously such results are only known in the real-analytic category, see e.g. [25]. In the last section, we will discuss a weaker version of local lens rigidity problem for a general family of smooth curves.

**Remark 1.2.** Notice that our data in Theorem 1.1 contains both the travel time data  $\ell$  and the boundary restriction of  $c$ . Here  $c$  is necessary for defining the lens data  $L$  and  $\ell$ .

Similar to [50], the main idea of the proof of Theorem 1.1 is a reduction to a local uniqueness problem of the magnetic X-ray transform by applying an integral

identity from [43]. However, instead of considering the flow on the cotangent bundle as in [50], we do all the analysis on the tangent bundle (see Section 2), which turns out to be much more convenient. Our approach even simplifies the original method of [50] for the ordinary boundary and lens rigidity problems.

The geodesic X-ray transform (or *tensor tomography*) problem is concerned with the recovery of a function or tensor field from its integrals along geodesics joining boundary points, and it is the linearization of the boundary and lens rigidity problems. This problem has been extensively studied in the literature, see e.g. [30, 37, 41, 38, 44, 11, 42, 47, 49, 33, 34, 28, 29, 36] and the references therein. Recent studies of the magnetic tensor tomography problem can be found in [12, 1]. The *local* tensor tomography problem was considered in [23, 24] for real-analytic metrics, [52, 51] for smooth metrics; the case of magnetic geodesics appears in [52, Appendix] and [56].

For the ordinary partial data rigidity problems, the reduction mentioned above ends with a local weighted geodesic ray transform of vector functions. It turns out that the local linear problem we need to consider in the current paper is the invertibility of some local magnetic ray transform of the combination of vector functions (related to the conformal factor) and 1-1 tensors (related to the Lorentz force, which can be viewed as a vector of 1-forms) with matrix weights, see Section 2 and 3 for more details. Microlocal analysis of weighted X-ray transforms can be found in [15, 22, 21] for the global problems and [50, 35] for the local problems. Comparing to the papers mentioned above, the matrix weights on the functions and 1-forms are different in our case, it is not an attenuated ray transform as in [35]. Generally the kernel of the X-ray transform of functions plus 1-forms is nontrivial (unless some ‘divergence free’ condition is assumed), however, since the 1-forms in our X-ray transform satisfy an additional property, we can show that actually the kernel is trivial in our case. This is also one of the contributions of our paper.

We also study the magnetic boundary rigidity problem with partial data. The following boundary action function on  $\partial M \times \partial M$  was introduced in [12] which plays the role of boundary distance function for simple magnetic systems

$$(4) \quad \mathbb{A}(x, y) := T(x, y) - \int_{\gamma_{x,y}} \alpha,$$

where  $\gamma_{x,y}$  is the unique unit speed magnetic geodesic joining  $x, y$  with  $T(x, y)$  its length, and  $\alpha$  is the magnetic potential (1-form) satisfying  $\Omega = d\alpha$ . The existence of  $\alpha$  is due to the trivial topology of simple systems and the fact that  $\Omega$  is closed. The function  $\mathbb{A}(x, y)$  is referred to as Mañé’s action potential (of energy 1/2) in classical mechanics, which corresponds to the minimum of the integrals (or the action) of the Lagrangian  $L(x, v) = \frac{1}{2}|v|_g^2 - \alpha_x(v) + \frac{1}{2}$  on all the absolutely continuous curves  $\gamma : [0, T(x, y)] \rightarrow M$  satisfying  $\gamma(0) = x$ ,  $\gamma(T(x, y)) = y$ . Notice that the definition of  $\mathbb{A}$  depends on the choice of  $\alpha$ , given any  $f \in C^\infty(M)$ ,  $\Omega = d\alpha = d(\alpha + df)$ .

For the local problem, assume  $p \in \partial M$  is strictly magnetic convex, one investigates the determination of a magnetic system near  $p$  from the knowledge of the boundary action functional near  $(p, p)$ . Notice that given a sufficiently small neighborhood of  $p$  in  $M$ , we can always assume that it has trivial topology, thus such magnetic potentials always exist locally near  $p$ . Moreover, due to the strict convexity, points  $x, y \in \partial M$  near  $p$  can be joined by a unique unit speed magnetic geodesic, therefore we can define a boundary action functional  $\mathbb{A}$  locally near  $(p, p)$ .

Note that in the geodesic case, the local boundary distance data is equivalent to the local lens data near a strictly convex point  $p$ . However, for magnetic systems, such equivalence only holds between the boundary action functional and the scattering relation [12]. It is unclear that whether one can derive the travel time data (the length of  $\gamma_{x,y}$ ) from  $\mathbb{A}(x,y)$ , except in the case of real-analytic systems [20]. The following rigidity result related to the boundary action functional is an immediate corollary of Theorem 1.1.

**Theorem 1.3.** *Let  $n = \dim M \geq 3$ , let  $c, \tilde{c} > 0$  be smooth functions,  $\Omega, \tilde{\Omega}$  be smooth closed 2-forms and let  $\partial M$  be strictly magnetic convex with respect to both  $(c^2 g_0, \Omega)$  and  $(\tilde{c}^2 g_0, \tilde{\Omega})$  near  $p \in \partial M$ . Assume that there are 1-forms  $\alpha$  and  $\tilde{\alpha}$ , satisfying  $\Omega = d\alpha$  and  $\tilde{\Omega} = d\tilde{\alpha}$ , such that  $\mathbb{A} = \tilde{\mathbb{A}}$ ,  $T = \tilde{T}$  on  $\partial M \times \partial M$  near  $(p, p)$ . Then  $c = \tilde{c}$  and  $\Omega = \tilde{\Omega}$  in  $M$  near  $p$ .*

**Remark 1.4.** One needs to know both  $\mathbb{A}$  and  $T$  to reduce Theorem 1.3 to the settings of Theorem 1.1. If  $L(x, v) = (y, v')$ , then near  $p$  we have  $\ell(x, v) = T(x, y)$ . The advantage of considering  $T$ , instead of  $\ell$ , is that we do not need to know the restriction of the conformal factor on the boundary. On the other hand, it is not difficult to see that one can determine  $c$  on the boundary from  $T$ .

Another application of Theorem 1.1 is a global lens rigidity result under some foliation condition related to the magnetic flow as follows: Given a compact Riemannian manifold  $(M, g)$  with smooth boundary and a magnetic field  $\Omega$ , we say that  $M$  can be foliated by strictly magnetic convex hypersurfaces for the magnetic system  $(M, g, \Omega)$  if there exist a smooth function  $f : M \rightarrow \mathbb{R}$  and  $a < b$ , such that the level set  $f^{-1}(t)$  (as a hypersurface) is strictly magnetic convex with respect to  $(M, g, \Omega)$  from  $f^{-1}((-\infty, t])$  for any  $t \in (a, b]$ ,  $df$  is non-zero on these level sets and  $M \setminus f^{-1}((a, b])$  has empty interior. Note that  $\partial M$  is not necessarily a level set of  $f$ . The next theorem is an analog of [50, Theorem 1.3].

**Theorem 1.5.** *Let  $n = \dim M \geq 3$ , let  $c, \tilde{c} > 0$  be smooth functions,  $\Omega, \tilde{\Omega}$  be smooth closed 2-forms and let  $\partial M$  be strictly magnetic convex with respect to both  $(c^2 g_0, \Omega)$  and  $(\tilde{c}^2 g_0, \tilde{\Omega})$ . Assume that  $c = \tilde{c}$  and  $\iota^* \Omega = \iota^* \tilde{\Omega}$  on  $\partial M$ , and  $M$  can be foliated by strictly magnetic convex hypersurfaces for  $(M, c^2 g_0, \Omega)$ . If  $L = \tilde{L}$ ,  $\ell = \tilde{\ell}$  on  $\partial_+ SM$ , then  $c = \tilde{c}$  and  $\Omega = \tilde{\Omega}$  in  $M$ .*

The global foliation condition implies non-trapping on  $f^{-1}([t, b])$  for any  $a < t < b$ , but allows the existence of conjugate points on  $M$ . Notice that in [17] trapped sets are allowed, but the manifolds need to be free of conjugate points. In the case of absence of magnetic fields, the condition is an analog of the Herglotz and Wieckert & Zoeppritz condition  $\frac{\partial}{\partial r} \frac{r}{c(r)} > 0$ , see also [50, Section 6]. Examples of manifolds satisfying the foliation conditions include compact submanifolds of complete manifolds with positive curvature [16], compact manifolds with non-negative sectional curvature [14], and compact manifolds with no focal points [40]. Our foliation condition defined just before Theorem 1.5 is the corresponding version for magnetic systems. Given a compact Riemannian manifold  $(M, g)$  with strictly convex boundary, satisfying the foliation condition (w.r.t. the usual geodesics), one can construct examples for magnetic systems by adding a magnetic field  $\Omega$  supported away from  $\{f \leq a\}$  (here  $f$  and  $a$  are w.r.t. the geodesic case) with sufficiently small norm. Then it is easy to check by definition that the magnetic system  $(M, g, \Omega)$  satisfies the magnetic foliation condition, and the boundary is also strictly magnetic convex.

One can derive similar stability estimates for these rigidity problems as in [50], however, in the current paper we only deal with uniqueness.

**Remark 1.6.** Theorem 1.5 is regarding the lens rigidity problem. Generally under the foliation condition, the global boundary rigidity problem is not solvable. One example is the unit ball with radial symmetric metric and very large curvature at the center, then it is impossible to recover the metric near the center as no distance minimizing geodesics will reach a small neighborhood of the center.

The outline of the paper is as follows: In Section 2, we reduce the rigidity problems to a local X-ray transform problem through a pseudo-linearization. The invertibility of the X-ray transform is shown in Section 3 by using Melrose' scattering calculus. Section 4 consists the proofs of various local and global rigidity theorems. Finally, we discuss a weak version of the lens rigidity problem for general smooth curves in Section 5.

**2. Reducing to a linear problem.** First we can determine the jets of  $c$  and  $\Omega$  at any boundary point  $p$  at which  $\partial M$  is strictly magnetic convex from the scattering relation near  $S_p\partial M$ .

**Lemma 2.1.** *Let  $c, \tilde{c} > 0$  be smooth functions,  $\Omega, \tilde{\Omega}$  be smooth closed 2-forms and let  $\partial M$  be strictly magnetic convex with respect to both  $(c^2g_0, \Omega)$  and  $(\tilde{c}^2g_0, \tilde{\Omega})$  near a fixed  $p \in \partial M$ . Let  $c = \tilde{c}$ ,  $\iota^*\Omega = \iota^*\tilde{\Omega}$  on  $\partial M$  near  $p$  and  $L = \tilde{L}$  near  $S_p\partial M$ . Then in any local coordinate system  $\partial^a c = \partial^a \tilde{c}$ ,  $\partial^a \Omega = \partial^a \tilde{\Omega}$  on  $\partial M$  near  $p$  for any multiindex  $a$ .*

If  $\Omega = \Omega_{ij} dz^i \wedge dz^j$ , we define  $\partial^a \Omega := \{\partial^a \Omega_{ij}\}$ .

The proof of Lemma 2.1 was essentially given in [12, 20], for the sake of completeness we give the sketch here.

*Proof.* By [20, Lemma 2], there exist 1-forms  $\alpha$  and  $\tilde{\alpha}$  in a neighborhood  $U$  of  $p$  such that  $\Omega = d\alpha$ ,  $\tilde{\Omega} = d\tilde{\alpha}$  in  $U$  and  $\iota^*\alpha = \iota^*\tilde{\alpha}$  on  $U \cap \partial M$ . Now since  $\partial M$  is strictly magnetic convex with respect to both magnetic systems near  $p$ , applying [12, Lemma 2.6], the boundary action functional  $\mathbb{A}$  and  $\tilde{\mathbb{A}}$  (locally defined) with respect to  $(c^2g_0, \alpha)$  and  $(\tilde{c}^2g_0, \tilde{\alpha})$  respectively are equal on  $\partial M \times \partial M$  near  $(p, p)$ . Then [12, Theorem 2.2] implies that  $\partial^a c = \partial^a \tilde{c}$ ,  $\partial^a \Omega = \partial^a \tilde{\Omega}$  on  $\partial M$  near  $p$  for any multiindex  $a$ . Note that Lemma 2.6 and Theorem 2.2 of [12] are global theorem on simple systems, but the proof works locally near a strictly magnetic convex boundary point if the local data is given. Moreover, for conformal metrics, the gauge in [12, Theorem 2.2] is trivial.  $\square$

Similar to [50], the starting point of the paper is an integral identity first proved in [43] for the usual geodesic flow. Actually the proof of this identity works for general (not necessarily Hamiltonian) flows. Let  $V, \tilde{V}$  be two vector fields on some smooth manifold  $N$  (no metric assigned). Denote by  $X(s, X^{(0)})$  the solution of  $\dot{X} = V(X)$ ,  $X(0) = X^{(0)}$ , and we use the same notation for  $\tilde{V}$  with the corresponding solution denoted by  $\tilde{X}$ . We state the identity here without proof, please see [43, p87] and [50, Lemma 2.2] for the detailed proof. The differences  $X - \tilde{X}$  and  $V - \tilde{V}$  in the identity make sense under local coordinates, which works well for our local rigidity problem.

**Lemma 2.2.** *For any  $t > 0$  and initial condition  $X^{(0)}$ , if  $X(\cdot, X^{(0)})$  and  $\tilde{X}(\cdot, X^{(0)})$  exist on the interval  $[0, t]$ , then*

$$(5) \quad X(t, X^{(0)}) - \tilde{X}(t, X^{(0)}) = \int_0^t \frac{\partial \tilde{X}}{\partial X^{(0)}}(t-s, X(s, X^{(0)}))(V - \tilde{V})(X(s, X^{(0)}))ds.$$

One can check that given the metric  $g$ , the restriction of the Lorentz force  $Y$  on  $SM$  uniquely determines the magnetic field  $\Omega$ , vice versa. From now on, we use  $(g, Y)$  and  $(\tilde{g}, \tilde{Y})$  to represent the magnetic systems. Note that given any  $u \in TM$ ,  $\langle Y(u), u \rangle_g = \Omega(u, u) = 0$ .

For  $(g, Y)$ , let  $(z, v) \in SM$  (with respect to metric  $c^2 g_0$ ), then by (3)

$$(6) \quad V(z, v) = \left( \frac{dz}{dt}, \frac{dv}{dt} \right) |_{t=0} = \left( v, -\Gamma_{jk}^i(z)v^j v^k \frac{\partial}{\partial v^i} + Y_j^i(z)v^j \frac{\partial}{\partial v^i} \right).$$

On the other hand, if we let  $(z, v)$  be the initial vector for some magnetic flow  $(\tilde{\gamma}, \dot{\tilde{\gamma}})$  of  $(\tilde{g}, \tilde{Y})$  with fixed energy level, then generally  $(z, v)$  is not on the unit sphere bundle with respect to metric  $\tilde{g}$ . Since  $|v|_g = 1$ , we get that  $\tilde{\gamma}$  has constant speed  $|v|_{\tilde{g}} = \sqrt{\tilde{c}^2 |v|_g^2} = \sqrt{\tilde{c}^2 / c^2} = \tilde{c} / c$ . Then similar to  $V$ , we have

$$(7) \quad \tilde{V}(z, v) = \left( \frac{d\tilde{z}}{d\tilde{t}}, \frac{d\tilde{v}}{d\tilde{t}} \right) |_{\tilde{t}=0} = \left( v, -\tilde{\Gamma}_{jk}^i(z)v^j v^k \frac{\partial}{\partial v^i} + \tilde{Y}_j^i(z)v^j \frac{\partial}{\partial v^i} \right).$$

**Remark 2.3.** If we consider the Hamiltonian flow on the cotangent bundle with the Hamiltonian  $H(\xi) = \frac{1}{2}|\xi|_g^2 = \frac{1}{2}g^{ij}\xi_i\xi_j$ , then the corresponding generating vector  $V, dH = \beta(V, \cdot)$ , is

$$\begin{aligned} V &= g^{ij}\xi_j \frac{\partial}{\partial z^i} + \left( -\frac{1}{2} \frac{\partial g^{ij}}{\partial z^k} \xi_i \xi_j + (\Omega_{ik} - \Omega_{ki})g^{ij}\xi_j \right) \frac{\partial}{\partial \xi_k} \\ &= g^{ij}\xi_j \frac{\partial}{\partial z^i} + \left( -\frac{1}{2} \frac{\partial g^{ij}}{\partial z^k} \xi_i \xi_j + Y_k^j \xi_j \right) \frac{\partial}{\partial \xi_k}. \end{aligned}$$

However, for the cotangent bundle, the proof will be more complicated, since both the  $\partial_z$  and  $\partial_\xi$  terms of  $V$  and  $\tilde{V}$  are different, see [50]. The arguments in the current paper also simplifies the proof of [50] for the geodesic case.

We denote points on the tangent bundle  $TM$  by  $\sigma = (z, v)$ , then the flow with initial point  $\sigma$  is  $X(t, \sigma) = (Z(t, \sigma), \Xi(t, \sigma))$ . Thus the identity (5) is rewritten as

$$(8) \quad X(t, \sigma) - \tilde{X}(t, \sigma) = \int_0^t \frac{\partial \tilde{X}}{\partial \sigma}(t-s, X(s, \sigma))(V - \tilde{V})(X(s, \sigma))ds.$$

Applying the lens data, note that  $c = \tilde{c}$  on  $\partial M$  near  $p$  by Lemma 2.1, we get the following proposition:

**Proposition 2.4.** *Assume  $L(z_0, v_0) = \tilde{L}(z_0, v_0)$ ,  $\ell(z_0, v_0) = \tilde{\ell}(z_0, v_0)$  for some  $\sigma_0 = (z_0, v_0) \in \partial_+ SM$ . Then*

$$(9) \quad \int_0^{\ell(\sigma_0)} \frac{\partial \tilde{X}}{\partial \sigma}(\ell(\sigma_0) - s, X(s, \sigma_0))(V - \tilde{V})(X(s, \sigma_0))ds = 0.$$

We take the second  $n$ -dimensional components of (9) and plug in (6) and (7) to the identity. We assume that both  $X(t, \sigma)$  and  $\tilde{X}(t, \sigma)$  exist for  $t \in [0, \ell(\sigma)]$ . We get

$$(10) \quad \int_0^{\ell(\sigma)} \frac{\partial \tilde{\Xi}^l}{\partial v^i}(\ell(\sigma) - s, X(s, \sigma))((\tilde{\Gamma}_{jk}^i - \Gamma_{jk}^i)v^j v^k + (Y_j^i - \tilde{Y}_j^i)v^j)(X(s, \sigma))ds = 0.$$

for  $l = 1, 2, \dots, n$  and any  $\sigma \in \partial_+ SM$  if the two systems  $(g, Y)$  and  $(\tilde{g}, \tilde{Y})$  have the same lens data at  $\sigma$ . Actually we can integrate over  $s \in \mathbb{R}$  after extending the magnetic geodesics. To see this, by Lemma 2.1 any smooth extension of  $(g, Y)$  near  $p \in \partial M$  is also a smooth extension of  $(\tilde{g}, \tilde{Y})$ , we denote the extended manifold by  $\widetilde{M}$ , thus we can assume outside  $M$  near  $p$ ,  $(g, Y) = (\tilde{g}, \tilde{Y})$ , which means  $V = \tilde{V}$  outside  $M$  near  $p$ . So the integrand of (10) vanishes for  $s \notin [0, \ell(\sigma)]$ .

Now we do more analysis of the term  $V - \tilde{V}$ .

$$\begin{aligned} \tilde{\Gamma}_{jk}^i - \Gamma_{jk}^i &= \frac{1}{2} \tilde{g}^{il} \left( \frac{\partial \tilde{g}_{lk}}{\partial z^j} + \frac{\partial \tilde{g}_{lj}}{\partial z^k} - \frac{\partial \tilde{g}_{jk}}{\partial z^l} \right) - \frac{1}{2} g^{il} \left( \frac{\partial g_{lk}}{\partial z^j} + \frac{\partial g_{lj}}{\partial z^k} - \frac{\partial g_{jk}}{\partial z^l} \right) \\ &= \frac{1}{\tilde{c}} (\delta_k^i \frac{\partial \tilde{c}}{\partial z^j} + \delta_j^i \frac{\partial \tilde{c}}{\partial z^k} - g_0^{il} g_{0jk} \frac{\partial \tilde{c}}{\partial z^l}) - \frac{1}{c} (\delta_k^i \frac{\partial c}{\partial z^j} + \delta_j^i \frac{\partial c}{\partial z^k} - g_0^{il} g_{0jk} \frac{\partial c}{\partial z^l}) \\ &= \delta_k^i \frac{\partial \ln(\tilde{c}/c)}{\partial z^j} + \delta_j^i \frac{\partial \ln(\tilde{c}/c)}{\partial z^k} - g_0^{il} g_{0jk} \frac{\partial \ln(\tilde{c}/c)}{\partial z^l}. \end{aligned}$$

We denote  $\ln \frac{\tilde{c}}{c} = h$ , then

$$(\tilde{\Gamma}_{jk}^i - \Gamma_{jk}^i) v^j v^k = 2 \frac{\partial h}{\partial z^j} v^j v^i - g_0^{ij} \frac{1}{c^2} \frac{\partial h}{\partial z^j}.$$

Thus (10) can be rewritten as

$$(11) \quad \int \frac{\partial \tilde{\Xi}^l}{\partial v^i} (\ell(\sigma) - s, X(s, \sigma)) \left( (2v^i v^j - g_0^{ij} \frac{1}{c^2}) \frac{\partial h}{\partial z^j} + (Y_j^i - \tilde{Y}_j^i) v^j \right) (X(s, \sigma)) ds = 0.$$

Let exit times  $\tau(z, v)$  be the minimal (and the only)  $t \geq 0$  such that  $X(t, (z, v)) \in \partial_- SM$ . They are well defined near  $S_p \partial M$  if  $\partial M$  is strictly magnetic convex at  $p$ . We write  $(z, v) = X(s, \sigma)$ , then

$$\frac{\partial \tilde{\Xi}^l}{\partial v^i} (\ell(\sigma) - s, X(s, \sigma)) = \frac{\partial \tilde{\Xi}^l}{\partial v^i} (\tau(z, v), (z, v)) = \frac{\partial \tilde{\Xi}^l}{\partial v^i} (z, v).$$

As we have extended the system outside  $M$  near  $p$ ,  $\tau$  is smooth near  $S_p \partial M$  on  $\widetilde{SM}$ . Thus we get

$$(12) \quad I_{AB}[\varphi, \Phi](\gamma) = \int_{\mathbb{R}} A(\gamma(t), \dot{\gamma}(t)) \left( B(\gamma(t), \dot{\gamma}(t)) \varphi(\gamma(t)) + \Phi(\dot{\gamma}(t)) \right) dt = 0,$$

where  $A(z, v) = (\frac{\partial \tilde{\Xi}^l}{\partial v^i}(z, v))_{n \times n}$  and  $B(z, v) = (2v^i v^j - g_0^{ij} \frac{1}{c^2})_{n \times n}$  are two smooth matrix weights,  $\varphi = (\frac{\partial h}{\partial z^1}, \dots, \frac{\partial h}{\partial z^n})$  is a vector-valued function and  $\Phi = (Y_j^i - \tilde{Y}_j^i)$  is an  $(1, 1)$ -tensor (can be viewed as a vector of 1-forms). Hence  $I_{AB}$  is a matrix weighted magnetic ray transform of the combination of vector functions and 1-forms. Notice that by Lemma 2.1,  $\varphi$  and  $\Phi$  vanish outside  $M$ , the integral is actually on a finite interval.

In particular, since  $\frac{\partial \tilde{\Xi}^l}{\partial v^i}(z, v) = \delta_i^l$  on  $S_p \partial M$ , i.e.  $A(z, v) = Id_{n \times n}$  for  $(z, v) \in S_p \partial M$ , we get that the matrix  $A$  is invertible near  $S_p \partial M$  in  $\widetilde{SM}$  (with respect to  $c^2 g_0$ ). On the other hand, the weight  $B$  is also an invertible matrix. To see this, notice that  $B = 2v^T v - g^{-1}$  where  $v = (v^1, \dots, v^n)$  and  $g^{-1}$  is the dual of the metric  $g = c^2 g_0$ . So  $gB = 2gv^T v - Id$ , and the invertibility of  $B$  is equivalent to the invertibility of  $gB$ . To show that  $gB$  is invertible, let  $u$  be an arbitrary column vector such that  $gBu = 0$ , then  $vgBu = 0$ , i.e.  $2vgv^T vu - vu = 0$ . Note that the vector  $v \in T_z \widetilde{M}$  has unit length, i.e.  $vgv^T = 1$ , we get that  $2vu = vu$ , so  $vu = 0$ . Since  $gBu = 2gv^T vu - u = u$ , this implies that  $u = 0$ . Thus the matrix function  $B$  is invertible on  $\widetilde{SM}$  (with respect to  $g$ ).

Now we have reduced the lens rigidity (non-linear) problem to the following magnetic ray transform (linear) problem:

*If  $I_{AB}[\varphi, \Phi](\gamma) = 0$ ,  $[\varphi, \Phi]$  supported in  $M$ , for magnetic geodesics close to the ones tangent to  $\partial M$  at  $p$ , does that imply  $[\varphi, \Phi] = 0$  near  $p$ ?*

We will show that the answer to above question is affirmative in the next section, actually it holds for weights  $A, B$  more general than the specific ones of (11).

### 3. Injectivity of the weighted ray transform.

**3.1. A scattering  $\Psi DO$  and its kernel.** Now let  $\rho \in C^\infty(\widetilde{M})$  be a boundary defining function of  $\partial M$ , so that  $\rho \geq 0$  on  $M$ . Suppose  $\partial M$  is strictly magnetic convex at  $p \in \partial M$ , then given a magnetic geodesic  $\gamma$  on  $\widetilde{M}$  with  $\gamma(0) = p$ ,  $\dot{\gamma}(0) \in S_p \partial M$ , one has

$$\frac{d^2 \rho}{dt^2}(\gamma(t))|_{t=0} = -\Lambda(p, \dot{\gamma}(0)) + \langle Y_p(\dot{\gamma}(0)), \nu(p) \rangle < 0.$$

Similar to [52] we consider the function  $\tilde{x}(z) = -\rho(z) - \epsilon|z - p|^2$  for some small enough  $\epsilon > 0$ , which makes  $\tilde{x}$  strictly magnetic concave from  $U_c = \{\tilde{x} \geq -c\} \subset \widetilde{M}$  for some sufficiently small  $c > 0$ . Here  $|\cdot|$  in the definition of  $\tilde{x}$  is the Euclidean norm, since we are considering (local) rigidity problems near  $p$ , it is well-defined under some initially chosen local chart near  $p$ . Let  $x = \tilde{x} + c$ , the open set we will work with is

$$O_c = U_c^{int} \cap \overline{M} = \{x > 0, \rho \geq 0\}$$

with compact closure. For the sake of simplicity, we drop the subscript  $c$ , i.e.  $U_c = U$ ,  $O_c = O$ .

One can complete  $x$  to a coordinate system  $(x, y)$  on a neighborhood of  $p$ , with  $y$  coordinates on the hypersurface  $x = 0$ . For each point  $(x, y)$  we can parameterize magnetic geodesics through this point which are ‘almost tangent’ to level sets of  $x$  (these are the curves that we are interested in) by  $\lambda \partial_x + \omega \partial_y \in TU$ ,  $\omega \in \mathbb{S}^{n-2}$  and  $\lambda$  is relatively small. Given a magnetic geodesic  $\gamma$  with  $\gamma(0) = (x, y)$ , which is parameterized by  $\lambda \partial_x + \omega \partial_y$  (i.e.  $\gamma_{x,y,\lambda,\omega}(t) = (x'(t), y'(t))$ ), define  $\alpha(x, y, \lambda, \omega) := \frac{d^2}{dt^2} x'(0)$ . In particular,  $\alpha(x, y, 0, \omega) > 0$  for  $x$  small by the concavity of  $x$ . Furthermore, it was shown in [52] that there exist  $\delta_0 > 0$  small and  $C > 0$  such that for  $|\lambda| \leq C\sqrt{x}$  (and  $|\lambda| < \delta_0$ ),  $x'(t) \geq 0$  for  $t \in (-\delta_0, \delta_0)$ , the magnetic geodesics remain in  $\{x \geq 0\}$  at least for  $|t| < \delta_0$ , i.e. they are  $O$ -local magnetic geodesics (exit  $O$  from  $\partial M$ ) for sufficiently small  $c$ . Note that [52] considers geodesics, but the settings work for general curves, see the appendix of [52]. We assume that the extended manifold  $\widetilde{M}$  is complete, so  $\gamma$  is defined on the whole real line, then

$$\begin{aligned} I_{AB}[\varphi, \Phi](x, y, \lambda, \omega) &= \int_{\mathbb{R}} \left( A(B\varphi + \Phi) \right) (\gamma_{x,y,\lambda,\omega}(t), \dot{\gamma}_{x,y,\lambda,\omega}(t)) dt \\ &= \int_{\mathbb{R}} \left( A(B\varphi + \Phi) \right) (x'(t), y'(t), \lambda'(t), \omega'(t)) dt. \end{aligned}$$

We will construct some localized version of the normal operator  $I_{AB}^* I_{AB}$  and study the microlocal properties of it. The main microlocal analysis will be carried out near  $\partial U = \{x = 0\}$  in  $U$ , which is a manifold with boundary. Since the standard pseudodifferential calculus is not suitable for working near the boundary of a manifold, we will apply the scattering pseudodifferential calculus (or scattering calculus for short) introduced by Melrose [26].

To make the paper more self-contained, we give a brief introduction of the scattering calculus, a more thorough discussion can be find in [52, Section 2]. We start with the scattering pseudodifferential algebra in the Euclidean case, as the scattering calculus is typically used to study the behavior ‘at infinity’, it is convenient to compactify the Euclidean space  $\mathbb{R}^n$  by gluing a sphere at the infinity, so topologically the compactified space  $\overline{\mathbb{R}^n}$  is a ball. More precisely, we can use the following identification (inverse polar coordinates) near the boundary (i.e. the infinity) of  $\overline{\mathbb{R}^n}$

$$(13) \quad (x, \theta) \in [0, \epsilon) \times \mathbb{S}^{n-1} \rightarrow z = x^{-1}\theta \in \overline{\mathbb{R}^n}.$$

The scattering pseudodifferential algebra  $\Psi_{sc}^{m,l}(\overline{\mathbb{R}^n})$  is the generalization of the standard pseudodifferential algebra by quantizing symbols  $a \in S^{m,l}$ ,  $m, l \in \mathbb{Z}$ , which are elements in  $C^\infty(\mathbb{R}_z^n \times \mathbb{R}_\zeta^n)$  satisfying

$$(14) \quad |D_z^\alpha D_\zeta^\beta a(z, \zeta)| \leq C_{\alpha\beta} \langle z \rangle^{l-|\alpha|} \langle \zeta \rangle^{m-|\beta|}$$

for any multiindices  $\alpha, \beta$ , where  $D_z = -i\partial_z$ ,  $\langle z \rangle = \sqrt{1 + |z|^2}$ , similar for  $D_\zeta$  and  $\langle \zeta \rangle$  respectively, and  $C_{\alpha\beta}$  is some positive constant only depending on  $\alpha, \beta$ . Similar to the identification (13), one can compactify each factor of  $\mathbb{R}_z^n \times \mathbb{R}_\zeta^n$  to define the compactified space of scattering symbols on  $\overline{\mathbb{R}_z^n} \times \overline{\mathbb{R}_\zeta^n}$ . Notice that in the identification (13),  $x$  is a boundary defining function near  $\partial\overline{\mathbb{R}^n}$ , one can extend it to a global boundary defining function on  $\overline{\mathbb{R}^n}$ . We denote  $\rho_z$  a boundary defining function on  $\overline{\mathbb{R}_z^n}$  and  $\rho_\zeta$  a boundary defining function on  $\overline{\mathbb{R}_\zeta^n}$ , by the estimate (14) a scattering symbol  $a(z, \zeta)$  of order  $(m, l)$  on  $\mathbb{R}_z^n \times \mathbb{R}_\zeta^n$  is (uniquely) lifted to a *conormal function* on  $\overline{\mathbb{R}_z^n} \times \overline{\mathbb{R}_\zeta^n}$ , i.e. one satisfying  $\rho_z^l \rho_\zeta^m V_1 \cdots V_k a(z, \zeta) \in L^\infty$  for all  $k$  and vector fields  $V_j$  on  $\overline{\mathbb{R}_z^n} \times \overline{\mathbb{R}_\zeta^n}$  that are tangent to both  $\partial\overline{\mathbb{R}_z^n}$  and  $\partial\overline{\mathbb{R}_\zeta^n}$  (so the restrictions of  $V_j$ 's to  $\mathbb{R}_z^n \times \mathbb{R}_\zeta^n$  are linear combinations of vector fields of the form  $z^i \partial_{z_j}, \zeta^k \partial_{\zeta_\ell}$ ). The scattering pseudodifferential algebra  $\Psi_{sc}^{m,l}(N)$  on a manifold  $N$  with boundary is defined by locally identifying it with  $\Psi_{sc}^{m,l}(\overline{\mathbb{R}^n})$ .

Our scattering pseudodifferential operators will be applied to vector valued 1-forms, it is necessary to introduce the (co)tangent bundle that is suitable for the scattering calculus. If we denote  $r = x^{-1}$  the standard radial variable, under the polar coordinates there is a natural change of basis for  $T\mathbb{R}^n$ ,

$$\partial_{z_1}, \dots, \partial_{z_n} \rightarrow \partial_r, r^{-1}\partial_{\theta_1}, \dots, r^{-1}\partial_{\theta_{n-1}}$$

where  $\theta_1, \dots, \theta_{n-1}$  are local coordinates on the sphere. We consider the spheres as the level sets of  $x$ , and to be consistent with the notation of the paper we use  $y_1, \dots, y_{n-1}$  as the local coordinates of the level sets, then it is straightforward to check that  $\partial_r = -x^2\partial_x$  and  $r^{-1}\partial_{\theta_j} = x\partial_{y_j}$ . In particular these vector fields can be smoothly extended to  $\overline{\mathbb{R}^n}$ , and form a local basis for  $T_{sc}\overline{\mathbb{R}^n}$  near  $x = 0$ . Consequently, the dual bundle,  $T_{sc}^*\overline{\mathbb{R}^n}$  has a local basis  $x^{-2}dx, x^{-1}dy_1, \dots, x^{-1}dy_{n-1}$ , which are exactly the local basis  $-dr, rd\theta_1, \dots, rd\theta_{n-1}$  under the standard polar coordinates. For a manifold with boundary  $N$ , the same local bases work for  $T_{sc}N$  and  $T_{sc}^*N$ , at least near the boundary, with a boundary defining function  $x$ . Here  $T_{sc}N$  is the *scattering tangent bundle* whose smooth sections are locally linear combinations of  $x^2\partial_x, x\partial_{y_1}, \dots, x\partial_{y_{n-1}}$ , and  $T_{sc}^*N$ , the *scattering cotangent bundle*, is the dual bundle whose smooth sections are locally linear combinations of  $x^{-2}dx, x^{-1}dy_1, \dots, x^{-1}dy_{n-1}$ . This also gives rise to the *scattering metrics*  $g_{sc}$ , as a positive definite symmetric 2-tensor, which has the form  $g_{sc} = x^{-4}dx^2 + x^{-2}h$  in local coordinates with  $h$  a metric on the level sets of  $x$ .

The principal symbol of a scattering pseudodifferential operator in  $\Psi_{sc}^{m,l}(\overline{\mathbb{R}^n})$  is the equivalent class of symbols  $a \in S^{m,l}$ , defined above, modulo  $S^{m-1,l-1}$ . The ellipticity in the scattering calculus, also called *full ellipticity*, is in the sense that the principal symbol  $\tilde{a} \in S^{m,l}/S^{m-1,l-1}$  satisfies a lower bound,  $|\tilde{a}(z, \zeta)| \geq C\langle z \rangle^l \langle \zeta \rangle^m$ , for  $|z| + |\zeta|$  sufficiently large, in contrast to the standard pseudodifferential algebra, where only  $|\zeta|$  is required to be large. In terms of the boundary defining function  $x$  by the identification (13), this means that we need to verify two cases: i)  $|\zeta|$  is sufficiently large, which is similar to the standard ellipticity for pseudodifferential operators; ii)  $x$  is sufficiently close to 0, while  $|\zeta|$  is relatively small comparing with  $x^{-1}$ . Full ellipticity is needed for showing Fredholm properties of scattering pseudodifferential operators between appropriate Sobolev spaces. The principal symbol of an element in  $\Psi_{sc}^{m,l}(N)$ , which is living on  $T_{sc}^*N$ , is defined again by locally identifying it with  $\tilde{a}$  above for the case of  $\overline{\mathbb{R}^n}$ .

Now following the approach of [52, 51], let  $\chi$  be a smooth non-negative even function on  $\mathbb{R}$  with compact support, which will be specified later, and

$$W = \begin{pmatrix} x^{-1}Id_{n \times n} & 0 \\ 0 & Id_{n \times n} \end{pmatrix},$$

we consider the following operator, for  $F > 0$ ,

$$\begin{aligned} & N_{AB}[\varphi, \Phi](x, y) \\ &= W^{-1}e^{-F/x} \int \int \begin{pmatrix} x^{-2}B^* \\ g_{sc}(\lambda\partial_x + \omega\partial_y) \end{pmatrix} A^* \chi(\lambda/x) \left( I_{AB}e^{F/x}W \right) \begin{pmatrix} \varphi \\ \Phi \end{pmatrix} (x, y, \lambda, \omega) d\lambda d\omega \\ &= \begin{pmatrix} N_{00} & N_{01} \\ N_{10} & N_{11} \end{pmatrix} \begin{pmatrix} \varphi \\ \Phi \end{pmatrix} (x, y) \end{aligned}$$

with

$$\begin{aligned} N_{00}\varphi &= xe^{-F/x} \int \int x^{-2}\chi(\lambda/x)B^*A^* \left( \int AB e^{F/x}x^{-1}\varphi dt \right) d\lambda d\omega, \\ N_{01}\Phi &= xe^{-F/x} \int \int x^{-2}\chi(\lambda/x)B^*A^* \left( \int Ae^{F/x}\Phi dt \right) d\lambda d\omega, \\ N_{10}\varphi &= e^{-F/x} \int \int \chi(\lambda/x)g_{sc}(\lambda\partial_x + \omega\partial_y)A^* \left( \int AB e^{F/x}x^{-1}\varphi dt \right) d\lambda d\omega, \\ N_{11}\Phi &= e^{-F/x} \int \int \chi(\lambda/x)g_{sc}(\lambda\partial_x + \omega\partial_y)A^* \left( \int Ae^{F/x}\Phi dt \right) d\lambda d\omega, \end{aligned}$$

where the scattering metric  $g_{sc}$  is used to convert vectors to covectors.

Comparing with the exponentially conjugated operators introduced in [52, 51], we add an additional conjugacy  $W$ . This additional conjugacy helps to make  $N_{ij}$ ,  $i, j = 1, 2$  of the same order, see Lemma 3.1. Similar idea appears in [35, 56]. From now on, we assume that both  $A$  and  $B$  are smooth invertible matrix-valued function on  $SU$ , and  $B$  is even on the momentum variable, i.e.  $B(z, v) = B(z, -v)$ . We will show that  $N_{AB}$  is an elliptic scattering pseudodifferential operator for such choice of  $A, B$ .

It is well known, see e.g. [12], that the maps (notice that  $\widetilde{M}$  is complete)

$$\Gamma_+ : S\widetilde{M} \times [0, \infty) \rightarrow [\widetilde{M} \times \widetilde{M}; diag], \Gamma_+(z, v, t) = (z, \gamma_{z,v}(t))$$

and

$$\Gamma_- : S\widetilde{M} \times (-\infty, 0] \rightarrow [\widetilde{M} \times \widetilde{M}; diag], \Gamma_-(z, v, t) = (z, \gamma_{z,v}(t))$$

are two diffeomorphisms near  $S\widetilde{M} \times \{0\}$ . Here, by denoting  $z' := \gamma_{z,v}(t)$ ,  $[\widetilde{M} \times \widetilde{M}; diag]$  is the *blow-up* of  $\widetilde{M}$  at the diagonal  $z = z'$ , which essentially means the introduction of polar coordinates around the diagonal, so that  $\Gamma_{\pm}(z, v, 0) \neq \Gamma_{\pm}(z, v', 0)$  if  $v \neq v'$ . In particular, for  $t \geq 0$  sufficiently small, the following local (polar) coordinates

$$(z, |z' - z|, \frac{z' - z}{|z' - z|})$$

are valid on the image of  $\Gamma_{\pm}$ , where  $|\cdot|$  is the Euclidean norm.

Recall the local coordinates  $(x, y)$  near the strictly convex boundary point  $p$ , we write  $z = (x, y)$  and  $z' = (x', y')$ , then similar to [52], it's convenient to use

$$(15) \quad (x, y, |y' - y|, \frac{x' - x}{|y' - y|}, \frac{y' - y}{|y' - y|})$$

as the local coordinates on the images of  $\Gamma_{\pm}$  for  $t \geq 0$  small, when  $|y' - y|$  is large relative to  $|x' - x|$ , i.e. in our region of interest.

On the other hand, the analysis is carried out near  $\partial U = \{x = 0\}$  in  $U$ , we recall the scattering coordinates introduced in [52] for dealing with manifolds with boundary

$$X = \frac{x' - x}{x^2}, Y = \frac{y' - y}{x},$$

under which (15) becomes

$$(16) \quad (x, y, x|Y|, \frac{xX}{|Y|}, \hat{Y})$$

with  $\hat{Y} = \frac{Y}{|Y|}$ .

We denote  $(\gamma(t), \dot{\gamma}(t)) = (x(t), y(t), k(t)\lambda(t), k(t)\omega(t))$  in short by  $(x', y', k\lambda', k\omega')$ , the multiple  $k$ , which is a function on  $t$ , is added to make  $\|k(\lambda'\partial_x + \omega'\partial_y)\|_g = 1$ . For the main proof of this section, it is convenient to make a change of parameters of the curve so that, if  $s$  is the new parameter,  $\dot{\gamma}(s) = \lambda'\partial_x + \omega'\partial_y$ . As a result, smooth positive weights are introduced to the ray transform  $I_{AB}$ . However, as one can see from the analysis of the ellipticity of  $N_{AB}$  below, the introduction of a smooth positive weight will not affect the argument, see also [50, 35]. For the sake of simplicity, we totally drop the multiple  $k$  from now on, so work as if the curve is parameterized by the non-unit speed one, but still denoted by  $t$ .

By the diffeomorphisms  $\Gamma_{\pm}$  near  $t = 0$ ,

$$t \circ \Gamma_{\pm}^{-1} = \pm|y' - y| + \mathcal{O}(|y' - y|^2),$$

$$\lambda \circ \Gamma_{\pm}^{-1} = \pm \frac{x' - x}{|y' - y|} + \mathcal{O}(|y' - y|), \quad \omega \circ \Gamma_{\pm}^{-1} = \pm \frac{y' - y}{|y' - y|} + \mathcal{O}(|y' - y|),$$

The coefficients in the remainder terms are all smooth under the coordinates (16). Then applying the scattering coordinates,

$$(17) \quad (\Gamma_{\pm}^{-1})^* dt d\lambda d\omega = x^2|Y|^{1-n} J_{\pm}(x, y, X, Y) dXdY$$

with the smooth positive density function  $J_{\pm}$ ,  $J_{\pm}|_{x=0} = 1$ , see also [52, Equation (3.15)].

Now given a curve  $\gamma = \gamma_{x,y,\lambda,\omega}$ , we have near  $t = 0$

$$(18) \quad \begin{aligned} x' &= x + \lambda t + \alpha t^2 + \mathcal{O}(t^3), & y' &= y + \omega t + \mathcal{O}(t^2), \\ \lambda' &= \lambda + 2\alpha t + \mathcal{O}(t^2), & \omega' &= \omega + \mathcal{O}(t), \end{aligned}$$

where  $\alpha = \alpha(x, y, \lambda, \omega, 0)$  defined at the beginning of section 2 is proportional to the second derivative of  $x'$  with respect to  $t$ . Notice that unlike the geodesic case,  $\alpha$  is no longer a quadratic form. From now on we denote  $\alpha(x, y, 0, \pm \frac{xX}{|Y|}, \pm \hat{Y})$  by  $\alpha_{\pm}$  and  $\frac{X - \alpha_{\pm}|Y|^2}{|Y|}$  by  $S_{\pm}$ , so  $\frac{X + \alpha_{\pm}|Y|^2}{|Y|} = S_{\pm} + 2\alpha_{\pm}|Y|$ . Then using (18) one can show that, see [51, Section 3] for more details, under the scattering tangent and cotangent bases

$$(19) \quad \begin{aligned} & g_{sc} \left( (\lambda \circ \Gamma_{\pm}^{-1}) \partial_x + (\omega \circ \Gamma_{\pm}^{-1}) \partial_y \right) \\ &= x^{-1} \left( \left( \pm S_{\pm} + x \tilde{\Lambda}_{\pm} \right) \frac{dx}{x^2} + \left( \pm \hat{Y} + x|Y| \tilde{\Omega}_{\pm} \right) \frac{h(\partial_y)}{x} \right) \end{aligned}$$

and

$$(20) \quad \begin{aligned} & (\lambda' \circ \Gamma_{\pm}^{-1}) \partial_x + (\omega' \circ \Gamma_{\pm}^{-1}) \partial_y \\ &= x^{-1} \left( \left( \pm (S_{\pm} + 2\alpha_{\pm}|Y|) + x|Y|^2 \tilde{\Lambda}'_{\pm} \right) x^2 \partial_x + \left( \pm \hat{Y} + x|Y| \tilde{\Omega}'_{\pm} \right) x \partial_y \right). \end{aligned}$$

Here  $\tilde{\Lambda}_{\pm}, \tilde{\Lambda}'_{\pm}, \tilde{\Omega}_{\pm}$  and  $\tilde{\Omega}'_{\pm}$  are smooth in terms of coordinates (16).

We plug the formulas (17), (19) and (20) into the integral expression of  $N_{AB}$  to get that the Schwartz kernel of  $N_{AB}$  is essentially of the following form

$$K_{AB}(x, y, X, Y) = K_{AB}(x, y, |Y|, \frac{X}{|Y|}, \hat{Y}) = e^{-\frac{FX}{1+X}} |Y|^{1-n} K(x, y, |Y|, \frac{X}{|Y|}, \hat{Y}),$$

where  $K$  is a smooth term, non-zero down to  $x = 0$ . Now it is easy to see that  $K_{AB}$  is smooth in  $(x, y)$  down to  $x = 0$ , with values in functions Schwartz in  $(X, Y)$  (due to the exponential weight) for  $(X, Y) \neq 0$ , and is conormal to the diagonal  $(X, Y) = 0$ . This shows that  $N_{AB}$  is a scattering pseudodifferential operator on  $U$  of order  $(-1, 0)$ , i.e.  $N_{AB} \in \Psi_{sc}^{-1,0}(U)$ , see [52, Section 2] for additional details.

Next we analyze the principal symbol of  $N_{AB}$  on  $(z, \zeta) = (x, y, \xi, \eta) \in T_{sc}^*U$ . This analysis is pointwise, we can assume that at one point  $z = (x, y)$  the metric  $g_{sc}$  has the trivial form  $g_{sc} = x^{-4}dx^2 + x^{-2}dy^2$ , while the fiber  $T_{sc,z}^*U$  is equivalent to  $\mathbb{R}^n$ . As mentioned in the introduction of the scattering calculus in Section 3.1, the analysis includes two cases: i) the first case is when  $|\zeta| \rightarrow \infty$ , i.e. near the fiber infinity; ii) the second case is at finite points of the fiber  $T_{sc,z}^*U$ , in particular near  $\zeta = 0$ . Since the Schwartz kernel  $K_{AB}$ , so is the principal symbol, is smooth in  $(x, y)$  down to  $x = 0$ , it suffices to investigate the principal symbol at  $x = 0$ . Once we show the full ellipticity at  $x = 0$ , by smoothness on  $x$ , the same result holds in a neighborhood of  $\bar{O} = U \cap M$  assuming that  $c > 0$  is small enough.

The Schwartz kernel  $K_{AB}$  contains both contributions from  $\Gamma_+$  and  $\Gamma_-$ . In particular,  $K$  is even in  $(X, Y)$ , the contributions of  $\Gamma_+$  and  $\Gamma_-$  to the principal symbol (through the  $(X, Y)$ -Fourier transform) are indeed equal. For the sake of simplicity, we drop the  $+, -$  signs (i.e. omit the contribution from  $\Gamma_-$ ) for the rest of the paper.

**Lemma 3.1.** *The Schwartz kernel of  $N_{AB}$  at  $x = 0$  is of the following form*

$$K_{AB}(0, y, X, Y) = e^{-FX} |Y|^{1-n} \chi(S) \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix},$$

where

$$\begin{aligned} K_{00} &= B^* A^* AB, \\ K_{01} &= B^* A^* A \left( (S + 2\alpha|Y|)(x^2 \partial_x) + \hat{Y}(x \partial_y) \right), \\ K_{10} &= \left( S \frac{dx}{x^2} + \hat{Y} \frac{dy}{x} \right) A^* AB, \\ K_{11} &= \left( S \frac{dx}{x^2} + \hat{Y} \frac{dy}{x} \right) A^* A \left( (S + 2\alpha|Y|)(x^2 \partial_x) + \hat{Y}(x \partial_y) \right). \end{aligned}$$

Here  $(S + 2\alpha|Y|)(x^2 \partial_x) + \hat{Y}(x \partial_y)$  and  $S \frac{dx}{x^2} + \hat{Y} \frac{dy}{x}$  are treated as diagonal matrices, i.e.  $\left( (S + 2\alpha|Y|)(x^2 \partial_x) + \hat{Y}(x \partial_y) \right) I_{n \times n}$  and  $\left( S \frac{dx}{x^2} + \hat{Y} \frac{dy}{x} \right) I_{n \times n}$ . Given a pair  $[\varphi, \Phi]$ ,

$$K_{AB}[\varphi, \Phi]|_{x=0} = e^{-FX} |Y|^{1-n} \chi(S) \begin{pmatrix} K_{00}\varphi + K_{01}\Phi \\ K_{10}\varphi + K_{11}\Phi \end{pmatrix}.$$

It is more convenient to separate the components with respect to  $\frac{dx}{x^2} (x^2 \partial_x)$  and  $\frac{dy}{x} (x \partial_y)$  respectively, and notationally write it as below

$$\begin{aligned} \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix} &= \begin{pmatrix} B^* A^* AB & B^* A^* A(S + 2\alpha|Y|) & B^* A^* A\hat{Y} \\ SA^* AB & SA^* A(S + 2\alpha|Y|) & SA^* A\hat{Y} \\ \hat{Y}A^* AB & \hat{Y}A^* A(S + 2\alpha|Y|) & \hat{Y}A^* A\hat{Y} \end{pmatrix} \\ &= \begin{pmatrix} B^* \\ S \\ \hat{Y} \end{pmatrix} A^* A \begin{pmatrix} B & S + 2\alpha|Y| & \hat{Y} \end{pmatrix}. \end{aligned}$$

**3.2. Ellipticity of  $N_{AB}$ .** To show that  $I_{AB}$  is invertible (locally near  $p \in \partial M$ ) on the space  $\{[\varphi, \Phi] : \varphi, \Phi \text{ supported in } M, \langle \Phi(u), u \rangle_g = 0, \forall (z, u) \in TU\}$  (notice that in the non-linear problem,  $\Phi = Y - \tilde{Y}$ , since  $g$  and  $\tilde{g}$  are conformal, we get  $\langle \Phi(u), u \rangle_g = 0$  for any  $u$ ), we study the ellipticity of  $N_{AB}$  which consists the main part of this subsection.

We first make a remark on the vector valued 1-form  $\Phi$ . Under the trivialization of the metric at one point, we can treat  $\Psi$  as a 2-tensor

$$\Phi = (\Phi_{xx} dx + \Phi_{xy} dy) \otimes dx + (\Phi_{yx} dx + \Phi_{yy} dy) \otimes dy,$$

or equivalently a square matrix

$$\Phi = \begin{pmatrix} \Phi_{xx} & \Phi_{xy} \\ \Phi_{yx} & \Phi_{yy} \end{pmatrix}.$$

Then  $\langle \Phi(u), u \rangle = 0$  for any vector  $u$  implies that the matrix  $\Phi$  is antisymmetric (i.e. it is a 2-form). Since we are now working in the scattering cotangent basis, we may rewrite  $\Phi$  as

$$\Phi^{sc} = x^4 \Phi_{xx} \frac{dx}{x^2} \otimes \frac{dx}{x^2} + x^3 \Phi_{xy} \frac{dy}{x} \otimes \frac{dx}{x^2} + x^3 \Phi_{yx} \frac{dx}{x^2} \otimes \frac{dy}{x} + x^2 \Phi_{yy} \frac{dy}{x} \otimes \frac{dy}{x},$$

or as a matrix

$$\Phi^{sc} = \begin{pmatrix} x^4 \Phi_{xx} & x^3 \Phi_{xy} \\ x^3 \Phi_{yx} & x^2 \Phi_{yy} \end{pmatrix},$$

which is still antisymmetric, i.e.  $\langle \Phi^{sc}(u), u \rangle_{g_{sc}} = 0, \forall u \in T_{sc,z}U$ . We denote the space of scattering antisymmetric 2-tensors on  $U$  by  $\Lambda_{sc}^2 U$ .

**Lemma 3.2.** *For any  $F > 0$ ,  $N_{AB}$  is elliptic near the fiber infinity of  $T_{sc}^*U$ , acting on  $C^\infty(U)^n \times \Lambda_{sc}^2 U$ .*

*Proof.* The analysis of the principal symbol of  $N_{AB}$  at fiber infinity is quite similar to the standard microlocal analysis, i.e. the analysis of the conormal singularity of the standard principal symbol of  $N_{AB}$  at the diagonal,  $X = Y = 0$ , see e.g. [51, Lemma 3.4]. Following the discussion above, we only need to study the behavior at  $x = 0$ .

Let  $\zeta = (\xi, \eta)$  be the Fourier dual variables of  $(X, Y)$ , we evaluate the  $(X, Y)$ -Fourier transform of the Schwartz kernel as  $|\zeta| \rightarrow \infty$ . Changing the coordinates  $(X, Y) \rightarrow (|Y|, \tilde{S}, \hat{Y})$  with  $\tilde{S} = X/|Y|$ , the principal symbol is given by the Fourier transform of (notice that  $X = \tilde{S}|Y| = 0$  at the front face)

$$|Y|^{1-n} \chi(\tilde{S}) \begin{pmatrix} B^* \\ \tilde{S} \\ \hat{Y} \end{pmatrix} A^* A(B \ \tilde{S} \ \hat{Y}).$$

Similar to the argument in [52] after equation (3.8), we may insert an even cutoff function  $\phi(t)$  with compact support on  $\mathbb{R}$ , identically 1 near 0, without affecting the principal symbol. The resulted integral has the expression

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{S}^{n-2}} \left( \int_0^\infty e^{-i(\xi \tilde{S} + \eta \cdot \hat{Y})|Y|} \phi(|Y|) d|Y| \right) \chi(\tilde{S}) \begin{pmatrix} B^* \\ \tilde{S} \\ \hat{Y} \end{pmatrix} A^* A(B \ \tilde{S} \ \hat{Y}) d\tilde{S} d\hat{Y} \\ &= \int_{\mathbb{R} \times \mathbb{S}^{n-2}} \left( \frac{1}{2} \int_{\mathbb{R}} e^{-i(\xi \tilde{S} + \eta \cdot \hat{Y})t} \phi(t) dt \right) \chi(\tilde{S}) \begin{pmatrix} B^* \\ \tilde{S} \\ \hat{Y} \end{pmatrix} A^* A(B \ \tilde{S} \ \hat{Y}) d\tilde{S} d\hat{Y} \\ &= \frac{1}{2} \int_{\mathbb{R} \times \mathbb{S}^{n-2}} \hat{\phi}(\xi \tilde{S} + \eta \cdot \hat{Y}) \chi(\tilde{S}) \begin{pmatrix} B^* \\ \tilde{S} \\ \hat{Y} \end{pmatrix} A^* A(B \ \tilde{S} \ \hat{Y}) d\tilde{S} d\hat{Y}. \end{aligned}$$

Here  $\hat{\phi}$ , the Fourier transform of  $\phi$ , is Schwartz, and  $|\zeta| \hat{\phi}(\xi \tilde{S} + \eta \cdot \hat{Y}) \rightarrow 2\pi \delta_0$  in distributions as  $|\zeta| \rightarrow \infty$ , thus the principal symbol of  $N_{AB}$  is a non-zero multiple of

$$\begin{aligned} & |\zeta|^{-1} \int_{\mathbb{R} \times \mathbb{S}^{n-2}} \delta_0(\xi \tilde{S} + \eta \cdot \hat{Y}) \chi(\tilde{S}) \begin{pmatrix} B^* \\ \tilde{S} \\ \hat{Y} \end{pmatrix} A^* A(B \ \tilde{S} \ \hat{Y}) d\tilde{S} d\hat{Y} \\ &= |\zeta|^{-1} \int_{\zeta^\perp \cap (\mathbb{R} \times \mathbb{S}^{n-2})} \chi(\tilde{S}) \begin{pmatrix} B^* \\ \tilde{S} \\ \hat{Y} \end{pmatrix} A^* A(B \ \tilde{S} \ \hat{Y}) d\tilde{S} d\hat{Y}. \end{aligned}$$

Given any non-zero pair  $[\varphi, \Phi]$ ,  $\Phi = (\Phi^0, \Phi')$  (as a vector valued 1-form) with  $\Phi^0$  and  $\Phi'$  corresponding to the coefficients for the covectors  $\frac{dx}{x^2}$  and  $\frac{dy}{x}$  respectively,

$$\begin{aligned} & (\sigma_p(N_{AB})[\varphi, \Phi], [\varphi, \Phi]) \\ &= c |\zeta|^{-1} \int_{\zeta^\perp \cap (\mathbb{R} \times \mathbb{S}^{n-2})} \chi(\tilde{S}) \left| A(B(0, y, \tilde{S}, \hat{Y})\varphi + \tilde{S}\Phi^0 + \hat{Y} \cdot \Phi') \right|^2 d\tilde{S} d\hat{Y}. \end{aligned}$$

Now to prove the ellipticity of  $N_{AB}$ , it suffices to show that there is  $(\tilde{S}, \hat{Y}) \in \zeta^\perp \cap (\mathbb{R} \times \mathbb{S}^{n-2})$  such that  $\chi(\tilde{S}) > 0$  and  $B\varphi + \tilde{S}\Phi^0 + \hat{Y} \cdot \Phi' \neq 0$  (notice that  $A$  is invertible). We prove by contradiction, assume that for any  $(\tilde{S}, \hat{Y}) \in \zeta^\perp \cap (\mathbb{R} \times \mathbb{S}^{n-2})$  with  $\chi(\tilde{S}) > 0$ ,  $B\varphi + \tilde{S}\Phi^0 + \hat{Y} \cdot \Phi' = 0$ . Notice that  $\chi$  is even, if  $\chi(\tilde{S}) > 0$ , then

$\chi(-\tilde{S}) > 0$ , thus  $B\varphi - \tilde{S}\Phi^0 - \hat{Y} \cdot \Phi' = 0$  (since  $B(z, v) = B(z, -v)$ ), which implies that  $\tilde{S}\Phi^0 + \hat{Y} \cdot \Phi' = 0$  and  $B\varphi = 0$ . As  $B$  is invertible too, we get  $\varphi = 0$ .

On the other hand, we can find  $n - 1$  linearly independent elements from the set  $\{(\tilde{S}, \hat{Y}) : \xi\tilde{S} + \eta \cdot \hat{Y} = 0, \chi(\tilde{S}) > 0\}$  (here we need the dimension  $n$  be at least 3, if  $n = 2$  the set might be empty) such that  $\tilde{S}\Phi^0 + \hat{Y} \cdot \Phi' = 0$ , which implies that each row of  $\Phi$  (as a matrix) is parallel to  $\zeta$ . Now take into account the antisymmetry of  $\Phi$ , a simple algebraic computation shows that  $\Phi$  has to be zero, which is a contradiction as the pair  $[\varphi, \Phi]$  is non-zero. Thus the lemma is proved.  $\square$

Then we study the principal symbol of  $N_{AB}$  at finite points of the fiber, especially near the zero section.

**Lemma 3.3.** *For any  $F > 0$ , there exists non-negative  $\chi \in C_c^\infty(\mathbb{R})$  with  $\chi(0) > 0$ , such that  $N_{AB}$  is elliptic at finite points of  $T_{sc}^*U$ , acting on  $C^\infty(U)^n \times \Lambda_{sc}^2 U$ .*

*Proof.* To study the principal symbol near  $\zeta = 0$ , we evaluate the  $(X, Y)$ -Fourier transform of  $K_{AB}$  in full. In order to find a suitable  $\chi$  to make  $N_{AB}$  elliptic, we follow the strategy of [52], namely we do calculation for Gaussian function  $\chi(s) = e^{-s^2/(2F^{-1}\alpha)}$  with  $F > 0$  first. Here  $\chi$  does not have compact support, thus an approximation argument will be necessary at the end.

The calculation of the Fourier transform of  $K_{AB}$  with Gaussian like  $\chi$  is similar to [52, Lemma 4.1] and [51, Lemma 3.5]. Notice that at  $x = 0$  the matrices  $A$  and  $B$  only depend on  $(y, \hat{Y})$ , i.e.  $A = A(0, y, 0, \hat{Y})$  and  $B = B(0, y, 0, \hat{Y})$ . Denoting  $F^{-1}\alpha$  by  $\mu$ , the Fourier transform of  $K_{AB}$  in  $X$  equals a non-zero multiple of

$$(21) \quad |Y|^{2-n} \sqrt{\mu} e^{i\alpha(\xi+iF)|Y|^2} \times \begin{pmatrix} B^* A^* AB & B^* A^* A(-D_\nu + 2\alpha|Y|) & B^* A^* A\hat{Y} \\ -D_\nu A^* AB & -D_\nu A^* A(-D_\nu + 2\alpha|Y|) & -D_\nu A^* A\hat{Y} \\ \hat{Y} A^* AB & \hat{Y} A^* A(-D_\nu + 2\alpha|Y|) & \hat{Y} A^* A\hat{Y} \end{pmatrix} e^{-\mu(\xi+iF)^2|Y|^2/2},$$

where  $D_\nu$  is the differentiation with respect to the variable of  $\hat{\chi}$ , i.e.  $-(\xi + iF)|Y|$ . Notice that  $\hat{\chi}(-(\xi + iF)|Y|) = c\sqrt{\mu}e^{-\mu(\xi+iF)^2|Y|^2/2}$  for some non-zero constant  $c$ . After taking the derivatives, the  $Y$ -Fourier transform of (21) in polar coordinates has the form

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{S}^{n-2}} e^{i|Y|\hat{Y} \cdot \eta} \sqrt{\mu} \times \\ & \begin{pmatrix} B^* A^* AB & B^* A^* Ai\mu(\xi - iF)|Y| & B^* A^* A\hat{Y} \\ i\mu(\xi + iF)|Y|A^* AB & A^* A(i\mu(\xi + iF)i\mu(\xi - iF)|Y|^2 + \mu) & i\mu(\xi + iF)|Y|A^* A\hat{Y} \\ \hat{Y} A^* AB & \hat{Y} A^* Ai\mu(\xi - iF)|Y| & \hat{Y} A^* A\hat{Y} \end{pmatrix} \\ & \times e^{-\mu(\xi^2+F^2)|Y|^2/2} d|Y|d\hat{Y} \\ & = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{S}^{n-2}} e^{i(\hat{Y} \cdot \eta)t} \sqrt{\mu} \times \\ & \begin{pmatrix} B^* A^* AB & B^* A^* Ai\mu(\xi - iF)t & B^* A^* A\hat{Y} \\ i\mu(\xi + iF)tA^* AB & A^* A(i\mu(\xi + iF)i\mu(\xi - iF)t^2 + \mu) & i\mu(\xi + iF)tA^* A\hat{Y} \\ \hat{Y} A^* AB & \hat{Y} A^* Ai\mu(\xi - iF)t & \hat{Y} A^* A\hat{Y} \end{pmatrix} \\ & \times e^{-\mu(\xi^2+F^2)t^2/2} dt d\hat{Y} \\ & = c \int_{\mathbb{S}^{n-2}} \frac{1}{\sqrt{\xi^2 + F^2}} \times \end{aligned}$$

$$\begin{pmatrix} B^*A^*AB & B^*A^*Ai\mu(\xi - iF)D_{\hat{Y}\cdot\eta} & B^*A^*A\hat{Y} \\ i\mu(\xi + iF)D_{\hat{Y}\cdot\eta}A^*AB & A^*A\left(i\mu(\xi + iF)i\mu(\xi - iF)D_{\hat{Y}\cdot\eta}^2 + \mu\right) & i\mu(\xi + iF)D_{\hat{Y}\cdot\eta}A^*A\hat{Y} \\ \hat{Y}A^*AB & \hat{Y}A^*Ai\mu(\xi - iF)D_{\hat{Y}\cdot\eta} & \hat{Y}A^*A\hat{Y} \end{pmatrix} \times e^{-|\hat{Y}\cdot\eta|^2/2\mu(\xi^2+F^2)} d\hat{Y}.$$

Finally, we apply the derivative  $D_{\hat{Y}\cdot\eta}$  to the exponential term, then it is easy to check that the scattering principal symbol of  $N_{AB}$ ,  $\sigma_{sc}(N_{AB})$ , is a non-zero multiple of

$$\int_{\mathbb{S}^{n-2}} \frac{1}{\sqrt{\xi^2 + F^2}} \begin{pmatrix} B^* \\ -\frac{(\xi+iF)\hat{Y}\cdot\eta}{\xi^2+F^2} \\ \hat{Y} \end{pmatrix} A^*A \begin{pmatrix} B & -\frac{(\xi-iF)\hat{Y}\cdot\eta}{\xi^2+F^2} \\ & \hat{Y} \end{pmatrix} e^{-|\hat{Y}\cdot\eta|^2/2F^{-1}\alpha(\xi^2+F^2)} d\hat{Y}.$$

Given any non-zero pair  $[\varphi, \Phi]$ ,  $\Phi = (\Phi^0, \Phi')$ ,

$$\begin{aligned} & (\sigma_{sc}(N_{AB})[\varphi, \Phi], [\varphi, \Phi]) \\ &= \frac{c}{\sqrt{\xi^2 + F^2}} \int_{\mathbb{S}^{n-2}} \left| A \left( B\varphi - \frac{(\xi - iF)\hat{Y}\cdot\eta}{\xi^2 + F^2} \Phi^0 + \hat{Y}\cdot\Phi' \right) \right|^2 e^{-|\hat{Y}\cdot\eta|^2/2F^{-1}\alpha(\xi^2+F^2)} d\hat{Y}. \end{aligned}$$

Similar to Lemma 3.2, to prove the ellipticity, it suffices to show that there is  $\hat{Y}$  such that  $B(0, y, 0, \hat{Y})\varphi - \frac{(\xi-iF)\hat{Y}\cdot\eta}{\xi^2+F^2}\Phi^0 + \hat{Y}\cdot\Phi' \neq 0$ . Again, we prove by contradiction, assume that for any  $\hat{Y} \in \mathbb{S}^{n-2}$ ,  $B(0, y, 0, \hat{Y})\varphi - \frac{(\xi-iF)\hat{Y}\cdot\eta}{\xi^2+F^2}\Phi^0 + \hat{Y}\cdot\Phi'$  always vanishes. Then by the evenness of  $B$ ,  $B(0, y, 0, \hat{Y})\varphi + \frac{(\xi-iF)\hat{Y}\cdot\eta}{\xi^2+F^2}\Phi^0 - \hat{Y}\cdot\Phi' = 0$  too, which implies that  $\varphi = 0$  and  $-\frac{(\xi-iF)\hat{Y}\cdot\eta}{\xi^2+F^2}\Phi^0 + \hat{Y}\cdot\Phi' = 0$  for all  $\hat{Y}$ . Similar to Lemma 3.2, this implies that each row of  $\Phi$  is parallel to  $(\xi + iF, \eta)$ . Combine with the fact that  $\Phi$  is antisymmetric, we show that  $\Phi = 0$ , which is a contradiction. This establishes the ellipticity of  $N_{AB}$  for Gaussian like  $\chi$ . Moreover, one can derive the following lower bound for the principal symbol

$$(\sigma_{sc}(N_{AB})[\varphi, \Phi], [\varphi, \Phi]) \geq C\langle(\xi, \eta)\rangle^{-1} \left\| [\varphi, \Phi] \right\|^2.$$

Finally by an approximation argument, one can show that there exists a compactly supported even function  $\chi$  on  $\mathbb{R}$ , close to a Gaussian function (so the principal symbol of  $\chi$  is also close to that of a Gaussian one in distributions), such that  $N_{AB}$  defined by such  $\chi$  is still elliptic as desired.  $\square$

Combining Lemma 3.2 and 3.3 gives the following ellipticity result

**Proposition 3.4.** *For any  $F > 0$ , given  $\Omega$  a neighborhood of  $U \cap M$  in  $U$ , there exists  $\chi \in C_c^\infty(\mathbb{R})$  such that  $N_{AB}$  is fully elliptic in  $\Omega$  acting on  $C^\infty(U)^n \times \Lambda_{sc}^2 U$ .*

**3.3. Injectivity of  $I_{AB}$ .** We denote  $H_{sc}^{s,r}(U)$  the scattering Sobolev space of order  $(s, r)$  on  $U$ , which locally is just the standard weighted Sobolev space  $H^{s,r}(\mathbb{R}^n) = \langle z \rangle^{-r} H^s(\mathbb{R}^n)$ , see [52, Section 2]. By Proposition 3.4,  $N_{AB}$  satisfies the Fredholm property. Moreover, due to the localness nature of the problem, see also [52, Section 2], we get that given any pair  $[\varphi, \Phi] \in H_{sc}^{s,r}(U)$ ,

$$\|[\varphi, \Phi]\|_{H_{sc}^{s,r}} \leq C\|N_{AB}[\varphi, \Phi]\|_{H_{sc}^{s+1,r}},$$

without an error term on the right hand side of the estimate. This gives the full invertibility, thus if  $I_{AB}[\varphi, \Phi] = 0$  for magnetic geodesics close to  $S_p\partial M$ , we get  $[\varphi, \Phi] = 0$  near  $p$  as well. Moreover, such determination is stable under usual Sobolev norms on any compact subset of  $O = M \cap U^{int}$ . Notice that when restricted

uniformly away from  $\partial U$ , the scattering Sobolev norm is equivalent to the usual Sobolev norm. In summary, we have the following result

**Proposition 3.5.** *Let  $\dim M \geq 3$  and  $\partial M$  is strictly magnetic convex at  $p \in \partial M$ . Given smooth invertible matrix function  $A$  and  $B$  on  $SM$  with  $B(z, v) = B(z, -v)$  for any  $(z, v) \in SM$ , there exists a smooth function  $\tilde{x}$  near  $p$  with  $O_p = \{\tilde{x} > -c\} \cap M$  for sufficiently small  $c > 0$ , such that given  $[\varphi, \Phi] \in L^2_{loc}(O_p)$  with  $\langle \Phi(u), u \rangle_g = 0$  for any  $u \in TO_p$ ,  $[\varphi, \Phi]$  can be stably determined by the weighted magnetic ray transform  $I_{AB}[\varphi, \Phi]$  restricted to  $O_p$ -local magnetic geodesics with the following stability estimates*

$$\|[\varphi, \Phi]\|_{H^{s-1}(K)} \leq C \|I_{AB}[\varphi, \Phi]\|_{H^s(\mathcal{M}_{O_p})}$$

for any compact subset  $K$  of  $O_p$  and  $s \geq 0$ , if  $[\varphi, \Phi] \in H^s_{loc}(O_p)$ . Here  $\mathcal{M}_{O_p}$  is the set of  $O_p$ -local magnetic geodesics.

**4. Injectivity of the non-linear rigidity problems.** We prove the main rigidity results for both the local and global non-linear problems in this section.

**4.1. Lens rigidity.** We first prove the local magnetic lens rigidity and its application in the global case.

*Proof of Theorem 1.1.* Since  $(c^2g_0, \Omega)$  and  $(\tilde{c}^2g_0, \tilde{\Omega})$  have the same lens data near  $S_p(\partial M)$ , by Proposition 2.4,  $I_{AB}[\varphi, \Phi](\gamma) = 0$  for magnetic geodesics  $\gamma$  close enough to the ones tangent to  $\partial M$  at  $p$ , as in identity (12), with  $\varphi = d(\ln \frac{\tilde{c}}{c})$ ,  $\Phi = Y - \tilde{Y}$ . Then Proposition 3.5 implies that  $d(\ln \frac{\tilde{c}}{c})$  and  $Y - \tilde{Y}$  vanish in  $M$  near  $p$ . Since  $c = \tilde{c}$  near  $p$  on  $\partial M$ , which implies that  $\ln \frac{\tilde{c}}{c}(z) = 0$  for  $z \in \partial M$  near  $p$ , then we have actually  $\ln \frac{\tilde{c}}{c}(z) = 0$  for  $z \in M$  near  $p$ , i.e.  $c = \tilde{c}$  in  $M$  near  $p$ . On the other hand,  $c = \tilde{c}$  and  $Y = \tilde{Y}$  in  $M$  near  $p$  imply that  $\Omega = \tilde{\Omega}$  there, which proves the theorem.  $\square$

*Proof of Theorem 1.5.* The proof is essentially using the layer stripping argument as in [50], however, since  $\partial M$  is strictly magnetic convex, it is not necessary to be a level set of the foliation  $f$ , and  $\partial M \cap (M \setminus f^{-1}((a, b)))$  might not be empty. A similar case was considered in [35] for ordinary geodesics.

It is not difficult to see that  $f(M) \subset (-\infty, b]$  and  $f^{-1}(b) \subset \partial M$  (if not,  $M \setminus f^{-1}((a, b))$  can not have empty interior). Let

$$\tau := \inf\{t \leq b : \text{supp}(c - \tilde{c}) \cup \text{supp}(\Omega - \tilde{\Omega}) \subset f^{-1}((-\infty, t])\}.$$

From now on we denote  $\text{supp}(c - \tilde{c}) \cup \text{supp}(\Omega - \tilde{\Omega})$  by  $K$ . Since  $\text{supp}(c - \tilde{c})$  and  $\text{supp}(\Omega - \tilde{\Omega})$  are compact, the infimum actually can be reached, i.e.  $K \subset f^{-1}((-\infty, \tau])$  and  $K \cap f^{-1}(\tau) \neq \emptyset$ . We claim that  $\tau \leq a$ .

First, since  $\partial M$  is strictly magnetic convex with respect to both magnetic systems, by Theorem 1.1 there exists an open neighborhood  $U$  of  $\partial M$  in  $M$  such that  $K \subset M \setminus U$  (it also implies that  $\tau < b$ ). Assume  $\tau > a$ , since  $c = \tilde{c}$  and  $\Omega = \tilde{\Omega}$  in  $f^{-1}((\tau, b])$ , by taking the limit we have that  $c = \tilde{c}$  and  $\Omega = \tilde{\Omega}$  on  $f^{-1}(\tau)$  too. Thus  $f^{-1}(t)$  is strictly magnetic convex with respect to both systems for  $t \geq \tau$ . Now let  $p \in K \cap f^{-1}(\tau)$ , since  $K \cap \partial M = \emptyset$ ,  $p \notin \partial M$ , i.e.  $p$  is an interior point. By convexity for  $t \geq \tau$ , there exists an open neighborhood  $U_p$  of  $p$  such that for any  $(z, v) \in \partial_+ S(f^{-1}((-\infty, \tau]))$  sufficiently close to  $S_p(f^{-1}(\tau))$ ,  $\gamma_{z,v}$  and  $\tilde{\gamma}_{z,v}$  stay in  $U_p \cup f^{-1}((\tau, b])$  until they hit  $\partial M$  in positive and negative finite times. Moreover, since  $L = \tilde{L}$ ,  $\ell = \tilde{\ell}$  and  $(c^2g_0, \Omega) = (\tilde{c}^2g_0, \tilde{\Omega})$  in  $f^{-1}([\tau, b])$ , it is

easy to see that  $\gamma_{z,v}$  and  $\tilde{\gamma}_{z,v}$  exit  $f^{-1}((-\infty, \tau])$  in the same time (denoted by  $s$ ), and  $(\gamma_{z,v}(s), \dot{\gamma}_{z,v}(s)) = (\tilde{\gamma}_{z,v}(s), \dot{\tilde{\gamma}}_{z,v}(s)) \in \partial_- S(f^{-1}((-\infty, \tau]))$ , i.e. the lens data of  $(c^2 g_0, \Omega)$  and  $(\tilde{c}^2 g_0, \tilde{\Omega})$  on  $\partial_+ S(f^{-1}((-\infty, \tau]))$  are equal near  $S_p(f^{-1}(\tau))$ . Then Theorem 1.1 implies that  $c = \tilde{c}$  and  $\Omega = \tilde{\Omega}$  near  $p$ . We apply the result to all  $p \in K \cap f^{-1}(\tau)$  (note it is closed) to conclude that there exist some  $\kappa < \tau$  such that  $K \subset f^{-1}((-\infty, \kappa])$ , which is a contradiction, so  $\tau \leq a$ .

Thus  $K \subset M \setminus f^{-1}((a, b])$  with empty interior, by smoothness this implies that actually  $K = \emptyset$  and  $c = \tilde{c}$ ,  $\Omega = \tilde{\Omega}$  on  $M$ . □

**4.2. Boundary rigidity with travel time data.** Now we prove the local boundary rigidity result with the help of the travel time data.

*Proof of Theorem 1.3.* Since  $\mathbb{A} = \tilde{\mathbb{A}}$  near  $(p, p)$ , by the argument of [12, Lemma 2.1],  $c = \tilde{c}$ ,  $\iota^* \alpha = \iota^* \tilde{\alpha}$  on  $\partial M$  near  $p$ . Moreover,  $\iota^* \Omega - \iota^* \tilde{\Omega} = \iota^* d(\iota^* \alpha - \iota^* \tilde{\alpha}) = 0$ . Then [12, Lemma 2.5] implies that  $L = \tilde{L}$ . Now as mentioned in Remark 1.4,  $T = \tilde{T}$  near  $(p, p)$  is equivalent to  $\ell = \tilde{\ell}$  near  $S_p \partial M$ , note that  $c = \tilde{c}$  on  $\partial M$  near  $p$ . Thus the uniqueness result is an immediate consequence of Theorem 1.1. □

**5. Lens rigidity for general smooth curves.** The method discussed in section 2 and 3 can be applied to the study of the lens rigidity problem for more general systems. We consider smooth parametrized curves  $\gamma$ ,  $|\dot{\gamma}| \neq 0$ , satisfying the following equation

$$(22) \quad \nabla_{\dot{\gamma}} \dot{\gamma} = G(\gamma, \dot{\gamma}),$$

with  $G(z, v) \in T_z M$  smooth on  $TM$ .  $\gamma = \gamma_{z,v}$  depends smoothly on  $(z, v) = (\gamma(0), \dot{\gamma}(0))$ . We call the collection of such smooth curves, denoted by  $\mathcal{G}$ , a *general family of curves*. Note that if  $G \equiv 0$ ,  $\mathcal{G}$  is the set of usual geodesics; if  $G$  is the Lorentz force corresponding to some magnetic field, then  $\mathcal{G}$  consists of magnetic geodesics. It is easy to see that the generating vector field of  $\mathcal{G}$  is

$$\mathfrak{G}(z, v) = \mathbf{G}(z, v) + G^j(z, v) \frac{\partial}{\partial v^j}.$$

Generally curves  $\gamma$  are not necessarily of constant speed (unless  $\langle G(\gamma, \dot{\gamma}), \dot{\gamma} \rangle = 0$  along  $\gamma$ ), and the dependence of  $G$  on the second variable  $v$  could be non-linear.

Since  $\gamma$  could be of non-unit speed, we should consider the scattering relation defined on  $TM$ , instead of  $SM$ . We define the following set

$$\partial_{\pm} TM \setminus 0 := \{(z, v) \in TM \mid z \in \partial M, v \in T_z M, v \neq 0, \pm \langle v, \nu(z) \rangle \geq 0\}.$$

For the sake of simplicity, we assume that any  $\gamma \in \mathcal{G}$  satisfies the following property: if  $(\gamma(s), \dot{\gamma}(s)) \in \partial_+ TM \setminus 0$  for some  $s$ , then  $|\dot{\gamma}(s)| = 1$ , i.e.  $(\gamma(s), \dot{\gamma}(s)) \in \partial_+ SM$ . Thus we have the scattering relation with respect to  $\mathcal{G}$

$$L : \partial_+ SM \rightarrow \partial_- TM \setminus 0,$$

with  $L(z, v) = (z', v')$  not necessarily belonging to  $\partial_- SM$ . The travel time

$$\ell : \partial_+ SM \rightarrow \mathbb{R} \cup \infty,$$

$\ell(z, v)$  does not necessarily coincide with the length of  $\gamma_{z,v}$  again. Given  $z \in \partial M$ , we say that  $M$  (or  $\partial M$ ) is *strictly convex* at  $z$  with respect to  $\mathcal{G}$  if

$$\Lambda(z, v) > \langle G(z, v), \nu(z) \rangle_g$$

for any  $v \in S_z \partial M$ . It is easy to see that the geometric meaning of our definition is similar to the usual convexity with respect to the metric (geodesics). It is also consistent with the definition of the magnetic convexity.

Now assume  $M$  is strictly convex at  $p \in \partial M$  with respect to  $\mathcal{G}$ , we consider the partial data lens rigidity problem near  $p$  in a fixed conformal class. As mentioned above, generally  $G(z, v)$  depends on  $v$  non-linearly, it is not expectable that one can simultaneously recover both the conformal factor  $c$  and the bundle map  $G$ . In this section, we assume that  $G$  is given, we recover  $c$  from the lens data. This is somehow similar to the case of the ordinary lens rigidity problem.

Similar to the integral identity (11), we have the following identity for general curves.

**Proposition 5.1.** *Given two systems  $(c^2 g_0, G)$  and  $(\tilde{c}^2 g_0, G)$ , assume  $L(\sigma) = \tilde{L}(\sigma)$ ,  $\ell(\sigma) = \tilde{\ell}(\sigma)$  for some  $\sigma \in \partial_+ SM$ , then*

$$(23) \quad \int_0^{\ell(\sigma)} \frac{\partial \tilde{\Xi}^l}{\partial v^i}(\ell(\sigma) - s, X(s, \sigma))(2v^i v^j - g_0^{ij} |v|_{g_0}^2) \frac{\partial h}{\partial z^j}(X(s, \sigma)) ds = 0,$$

where  $h = \ln(\tilde{c}/c)$ .

**Remark.** If one considers the general case where the bundle map is unknown, then the identity becomes

$$\int_0^{\ell(\sigma)} \frac{\partial \tilde{\Xi}^l}{\partial v^i}(\ell(\sigma) - s, X(s, \sigma)) \left( (2v^i v^j - g_0^{ij} |v|_{g_0}^2) \frac{\partial h}{\partial z^j} + (G^i - \tilde{G}^i) \right) (X(s, \sigma)) ds = 0,$$

with  $G(z, v) = G^i(z, v) \frac{\partial}{\partial z^i}$ ,  $\tilde{G}(z, v) = \tilde{G}^i(z, v) \frac{\partial}{\partial z^i}$  for  $(z, v) \in TM$ .

Identity (23) induces the following weighted X-ray transform

$$(24) \quad I_{AB} \varphi(\gamma) = \int A(\gamma(s), \dot{\gamma}(s)) B(\gamma(s), \dot{\gamma}(s)) \varphi(\gamma(s)) ds = 0,$$

with invertible matrices  $A$  and  $B$ . X-ray transforms along general smooth curves were considered in [15] and [52, Appendix]. Notice that  $\gamma(s)$  does not necessarily have unit speed, by reparametrization, there exists a smooth positive function  $\omega$  on  $SM$  such that

$$(25) \quad I_{AB} \varphi(\gamma) = \int A(\gamma(t), \omega \dot{\gamma}(t)) B(\gamma(t), \omega \dot{\gamma}(t)) \varphi(\gamma(t)) \omega^{-1}(\gamma(t), \dot{\gamma}(t)) dt = 0,$$

with  $|\dot{\gamma}(t)|_g \equiv 1$ .

Denote the product of the terms other than  $\varphi$  in the integrand of (25) by  $\mathcal{W}$ , which is invertible on  $SM$ , thus

$$(26) \quad I_{\mathcal{W}} \varphi(\gamma) := \int \mathcal{W}(\gamma(t), \dot{\gamma}(t)) \varphi(\gamma(t)) dt = 0.$$

Such matrix weighted X-ray transform was also considered in [35] for geodesic flows. By similar arguments as in section 3, we can prove that  $I_{\mathcal{W}}$  is locally invertible. Thus the following weaker lens rigidity result holds for a general family of smooth curves.

**Theorem 5.2.** *Let  $n = \dim M \geq 3$ , let  $c, \tilde{c} > 0$  be smooth functions,  $G : TM \rightarrow TM$  be a smooth bundle map and let  $\partial M$  be strictly convex with respect to both  $(c^2 g_0, G)$  and  $(\tilde{c}^2 g_0, G)$  near a fixed  $p \in \partial M$ . Assume that on  $\partial M$  near  $p$ ,  $c, \tilde{c}$  has the same boundary jet. If  $L = \tilde{L}$ ,  $\ell = \tilde{\ell}$  near  $S_p \partial M$ , then  $c = \tilde{c}$  in  $M$  near  $p$ .*

A global result can be derived also under proper foliation conditions. Notice that in Theorem 5.2 we assume that the boundary jets of  $c$  and  $\tilde{c}$  are equal. It is unclear that whether one can make the assumption weaker, for example just assuming  $c = \tilde{c}$  on  $\partial M$  near  $p$ . The usual proof of the determination of boundary jets relies on the existence of some ‘distance’ minimizing functional (not necessarily equalling the travel time) on  $\partial M \times \partial M$  which satisfies an eikonal equation. However given a general system, the existence of such minimizing functional is unknown. One might want to first consider it on a general family of curves whose induced flow is Hamiltonian.

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