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The weighted X-ray transform and applications

Abstract: In this article, we survey recent developments of weighted geodesic X-ray transforms. A special case of weighted X-ray transform is the attenuated X-ray transform. We review both attenuated X-ray transforms and X-ray transforms with general weights, in particular the matrix version, with emphasis on the approaches using microlocal analysis. We also discuss applications of weighted X-ray transforms to nonlinear inverse problems, such as the non-abelian X-ray transform and the lens rigidity problem.

Keywords: weighted X-ray transform, attenuated X-ray transform, microlocal analysis, non-abelian transform, lens rigidity

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1 Introduction

In medical imaging techniques such as Computed Tomography (CT) and Positron Emission Tomography (PET), the inner structure of tissues is reconstructed from the collected data of the radiation of X-rays or gamma rays through the human body. The theoretical underpinning for these medical imaging methods is the Radon/X-ray transform [48] in the plane, which consists of recovering a function in some bounded domain from its integrals along straight lines through this domain. Mathematically, the (Euclidean) X-ray transform of a function f, over the set of straight lines in $\mathbb{R}^n$, is defined as

$$If(x,\theta) := \int f(x+s\theta)\,ds, \quad (x,\theta)\in\mathbb{R}^n\times\mathbb{S}^{n-1}.$$

Here θ defines the direction of the straight line. We refer to [35, 58] for thorough accounts of the Euclidean X-ray transform. In particular, Explicit inversion formulas are available in any dimensions ≥ 2.

The standard X-ray transform is over straight lines. In practical applications, generalizations of X-ray transforms are often needed. For instance, in seismic and ultrasound imaging, the waves do not always propagate along

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straight lines due to the variable index of refraction, often modeled by the geodesics of a Riemannian metric.

Let (M, g) be a compact non-trapping Riemannian manifold of dimension ≥ 2, with strictly convex boundary ∂M. A compact manifold with boundary is non-trapping if every geodesic (with a starting point) exits the manifold in a finite time. We denote

$$SM := \{(x, v) \in TM : \|v\|_{g(x)} = 1\}$$

the unit tangent bundle. Given any $(x, v) \in SM$, denote $\tau(x, v)$ the (forward) exit time of the unit speed geodesic $\gamma_{x,v}$ starting at x in direction v, i.e. $(x, v) = (\gamma(0), \dot\gamma(0))$. τ is a finite function on SM due to the non-trapping assumption. Let ∂_+SM and ∂_-SM be the incoming and outgoing boundaries of SM respectively, which are defined by

$$\partial_\pm SM = \{(x, v) \in SM : x \in \partial M, \pm\langle \nu(x), v\rangle_g \geq 0\}$$

with $\nu(x)$ the unit inward pointing normal vector at $x \in \partial M$. The geodesic X-ray transform of a function f on SM is defined as

$$If(x, v) = \int_0^{\tau(x,v)} f(\gamma_{x,v}(t), \dot\gamma_{x,v}(t))\, dt, \quad (x, v) \in \partial_+SM.$$

The inverse problem is concerned with determining f from If, defined on ∂_+SM.

In some applications, one needs to consider the weighted version of the X-ray transform

$$I_w f(x, v) = \int_0^{\tau(x,v)} w(\gamma_{x,v}(t), \dot\gamma_{x,v}(t)) f(\gamma_{x,v}(t), \dot\gamma_{x,v}(t))\, dt.$$

where the weight w is a function on SM too. One such example is the attenuated X-ray transform, where the weight w has the form

$$w(\gamma_{x,v}(t), \dot\gamma_{x,v}(t)) := e^{\int_0^t a(\gamma_{x,v}(s), \dot\gamma_{x,v}(s))\, ds}, \quad (x, v) \in \partial_+SM$$

for some attenuation coefficient a. The attenuated X-ray transform finds its applications in medical imaging modalities, such as the Single Photon Emission Computerized Tomography (SPECT). Besides the medical imaging applications, the attenuated ray transform also arises in inverse problems for radiative transport equations [5]. More recently, such transforms also demonstrate their connection with the Calderón's inverse conductivity problem in anisotropic media [8].

In this survey paper, we will focus on the case that f is a function or a vector field/1-form on M, or their combinations. We denote the X-ray transform of scalar functions by I_0 and of 1-forms by I_1. There is an obvious difference between I_0 and I_1 in terms of their kernels. Notice that by the fundamental theorem of calculus, the kernel of I_1 contains all the potential 1-forms $f = dp$, where p is a scalar function with $p|_{\partial M} = 0$. Therefore, one can only expect to recover f from $I_1 f$ up to such potential 1-forms. Such non-trivial kernel also appears in weighted X-ray transforms.

In Section 2, we review results on the attenuated X-ray transform, in particular, the matrix version. There are multiple approaches to the problem, under different geometric conditions. We emphasize the studies using microlocal analysis. Section 3 is devoted to X-ray transforms with general weights. In Section 4, we discuss applications of weighted X-ray transforms to several non-linear inverse problems, including the non-abelian X-ray transform and the lens rigidity problem.

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2 Attenuated X-ray transform

Unlike the Euclidean case, there is no global parameterization of the geodesics on general Riemannian manifolds, additional geometric assumptions are always needed. Substantial progress has been made on the X-ray transform in the case where the metric is simple, i.e. (M, g) is simply connected, free of conjugate points, and the boundary is strictly convex. It is known that I_0 and I_1 are injective (up to natural gauge) on simple manifolds [33, 34, 1] with stability estimates [54, 30, 3, 39]. Inversion formulas for I_0 and I_1 on simple surfaces when the curvature is close to constant were given in [46, 24], and numerical implementations in [28]. It is worth mentioning that there is also much progress on the geodesic X-ray transform of higher order tensor fields, see the books [50, 43] and survey papers [42, 21] for recent developments.

2.1 Attenuated X-ray transform on simple manifolds

Recall the attenuated X-ray transform

$$I_a f(x,v) = \int_0^{\tau(x,v)} f(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)) e^{\int_0^t a(\gamma_{x,v}(s), \dot{\gamma}_{x,v}(s))\, ds}\, dt, \quad (x,v) \in \partial_+ SM.$$

We refer to [10, 25] for surveys of the progress on the Euclidean attenuated X-ray transform. In particular, inversion formulas are derived for the 2D attenuated X-ray transform [36, 23]. In the case of the attenuated geodesic ray transform on Riemannian manifolds, injectivity results have been accomplished on simple surfaces [49], simple manifolds with small attenuation [8] and for real-analytic data [12, 18]. Inversions of such transforms on simple surfaces were given in [29, 2].

Notice that the attenuation weight w satisfies

$$Xw = wa, \quad w|_{\partial_+ SM} = 1,$$

where X is the generating vector field of the geodesic flow on SM so given $f \in C^\infty(SM)$

$$Xf(x,v) = \frac{d}{dt} f(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))|_{t=0}.$$

In local coordinates

$$X(x,v) = v^i \frac{\partial}{\partial x^i} - \Gamma^i_{jk} v^j v^k \frac{\partial}{\partial v^i}$$

where Γ^i_{jk} are the Christoffel symbols of the metric g. More generally, one can consider the matrix attenuation $\mathcal{A} \in C^\infty(SM; \mathbb{C}^{N\times N})$. Similar to the scalar case, $\mathcal{A}$ uniquely determines a matrix weight $W_{\mathcal{A}}$ through the transport equation

$$XW_{\mathcal{A}} = W_{\mathcal{A}} \mathcal{A}, \quad W_{\mathcal{A}}|_{\partial_+ SM} = Id.$$

We consider $\mathcal{A}(x,v) = A_x(v) + \Phi(x)$ where A is a smooth matrix-valued 1-form and Φ is a smooth matrix-valued function on M. Let $f(x,v) = h(x) + \alpha_x(v)$ where h is a smooth vector-valued function and α is a smooth vector-valued 1-form. We denote the corresponding attenuated X-ray transform by $I_{\mathcal{A}} f$. Similar to the standard X-ray transform, the kernel of $I_{\mathcal{A}}$ is nontrivial in general. It's easy to check that $I_{\mathcal{A}}(dp + \mathcal{A}p) \equiv 0$ for any $p \in C^\infty(M; \mathbb{C}^N)$ with $p|_{\partial M} = 0$. For the sake of simplicity, we denote the operator $d + \mathcal{A}$ by $d_{\mathcal{A}}$ from now on. Note that since $f = h + \alpha$, $f = d_{\mathcal{A}} p$ is equivalent to say that $h = \Phi p$ and $\alpha = dp + Ap$.

Motivation for considering matrix attenuation comes from the question of identifying a connection A from the parallel transport with respect to A along geodesics, see Section 4.1 for more details. Such question also arises in gauge theories in explaining the dynamics of elementary particles, as many of the gauge theories are best described in the language of vector bundles and connections on such bundles (connections define the parallel transport in a vector bundle). Locally a connection is represented by a matrix valued 1-form, and it was shown in [41, 44] that this identification question could be reduced to the geodesic X-ray transform with connection attenuations through a pseudo-linearization process. In light of this reduction, the scalar version I_a is associated with the electromagnetism, where a represents a magnetic potential.

$I_{\mathcal{A}}$ is injective (up to nature gauge) on simple surfaces [41, 40] with reconstruction formulas (up to a Fredholm error) [31].

Theorem 2.1. *[40, Theorem 1.2] Let M be a simple surface. If $I_{\mathcal{A}}f = 0$, then $f = d_{\mathcal{A}}p$, where $p : M \to \mathbb{C}^N$ is a smooth function with $p|_{\partial M} = 0$.*

In dimension three and higher, it is known that $I_{\mathcal{A}}$ is injective on simple manifolds with negative (sectional) curvature [14, 44, 38]. Most of aforementioned results of the attenuated X-ray transform are based on L^2 energy estimates initiated in [33], and further developed by many mathematicians, which is now called Pestov identity. It is known that the solution to the transport equation

$$Xu + (A + \Phi)u = -f, \quad u|_{\partial_- SM} = 0$$

is given by

$$u(x, v) = \int_0^{\tau(x,v)} W_{\mathcal{A}}(\gamma_{x,v}(t), \dot\gamma_{x,v}(t)) f(\gamma_{x,v}(t), \dot\gamma_{x,v}(t))\, dt, \quad (x, v) \in SM.$$

Therefore, if $I_{\mathcal{A}}f = 0$, then $u|_{\partial SM} = 0$, i.e. u has trivial boundary value. As a consequence, under the negative curvature assumption, the Pestov identity implies that if f is the sum of a (vector-valued) function and a (vector-valued) one-form, u must be independent of the direction v, i.e. $u = u(x)$, so

$$f = -Xu - (A + \Phi)u = -du - (A + \Phi)u = d_{\mathcal{A}}u.$$

See [41, 14, 38] for more details.

The question of the injectivity of $I_{\mathcal{A}}$ on general simple manifolds is still open in general. In [68], the author proves that $I_{\mathcal{A}}$ is injective for generic simple metrics and generic $\mathcal{A}$. The proof is based on the following two theorems. The first theorem is on the injectivity of $I_{\mathcal{A}}$ in the real-analytic category.

Theorem 2.2. *[68, Theorem 1.2] Let M be a real-analytic simple manifold with real-analytic metric g, let $\mathcal{A}$ be real-analytic, then $I_{\mathcal{A}}$ is injective.*

We say that $\mathcal{A}$ is real-analytic if both the real and imaginary parts of $\mathcal{A}$ are real-analytic. The proof of Theorem 2.2 relies on the analytic microlocal analysis approach initiated in [55, 56] in the study of the standard X-ray transform. Recall that a distribution f is called *microlocally analytic* at (x_0, ξ_0), $\xi_0 \neq 0$, if in local coordinates

$$\int e^{i\lambda(x-y)\cdot\xi - \lambda|x-y|^2/2} \chi(x) f(x)\, dx = O(e^{-\lambda/C}) \tag{1}$$

for some $\chi \in C_0^\infty$ with $\chi(x_0) \neq 0$ and for (y, ξ) near (x_0, ξ_0). The complement of all such (x_0, ξ_0) defines the *analytic wave front set* $\mathrm{WF}_a(f)$.

Proposition 2.3. *[68, Proposition 4.1] Assume that g and $\mathcal{A}$ are analytic. Given nonzero covector (x_0, ξ_0), let γ_0 be a geodesic through x_0 conormal to ξ_0. If for some $f \in L^2$, $I_{\mathcal{A}} f(\gamma) = 0$ for γ in a neighborhood of γ_0 and $\delta_{\mathcal{A}} f = 0$, then $(x_0, \xi_0) \notin WF_a(f)$.*

Here $\delta_{\mathcal{A}}$ is the adjoint of $d_{\mathcal{A}}$ under the L^2 inner product.

To prove Proposition 2.3, we notice that in a tubular neighborhood of the geodesic γ_0, one can define analytic coordinates $x = (x', t)$ with t the parameter for γ_0 and x' transversal to γ_0. One then multiplies $I_{\mathcal{A}} f = 0$ by $e^{i\lambda\varphi}$ with some properly chosen complex phase φ, and integrates with respect to the x' variable. Now we can apply the complex stationary phase method in Sjöstrand [52] to obtain an FBI type transform as in (1) to resolve $\mathrm{WF}_a(f)$, assuming that $\delta_{\mathcal{A}} f = 0$.

Applying Proposition 2.3, it follows that f, modulo the gauge, is real-analytic. If we extend f by zero outside M, so $I_{\mathcal{A}} f = 0$ still holds, the analytic continuation shows that f, modulo the gauge, must be zero, which proves Theorem 2.2. There are related studies in the analytic category of attenuated X-ray transforms in [16, 18], they either only consider the function case or impose extra conditions on the 1-forms which make the kernel of the ray transform trivial and the arguments simpler too.

Notice that we need to extend f by zero outside M. In particular, let M_1 be a slightly larger open manifold, whose closure is simple as well. Consider the normal operator $N_{\mathcal{A}} = I_{\mathcal{A}}^* I_{\mathcal{A}}$ on M_1, where $I_{\mathcal{A}}^*$ is the adjoint of $I_{\mathcal{A}}$ under the L^2 inner product. One can show that $N_{\mathcal{A}}$ is a pseudodifferential operator of order -1. Moreover, $N_{\mathcal{A}}$ is elliptic when restricted on those f satisfying $\delta_{\mathcal{A}} f = 0$, i.e. the L^2 orthogonal complement of the natural gauge of $I_{\mathcal{A}}$. Then the Fredholm property follows, namely there are pseudodifferential operators P and Q of order 1 and -1 respectively, such that

$$PN_{\mathcal{A}} + Q\delta_{\mathcal{A}} = Id + K \qquad (2)$$

for some smoothing operator K. As a consequence, one obtains the following stability estimate

$$\|f^s\|_{L^2(M)} \leq C(\|N_{\mathcal{A}} f\|_{H^1(M_1)} + \|f\|_{H^{-1}(M_1)}),$$

in particular, the error term $\|f\|_{H^{-1}(M_1)}$ can be absorbed if $I_{\mathcal{A}}$, so $N_{\mathcal{A}}$, is injective (which holds in the real-analytic case by Theorem 2.2). Here f^s is the projection of f onto the L^2 orthogonal complement of the space of natural gauges on M. See [17] for more detailed microlocal analysis of the attenuated X-ray transform. Moreover, the above estimate is stable under small perturbations of the metric and the attenuation $\mathcal{A}$. We therefore reach the following generic results.

Theorem 2.4. *[68, Theorem 1.3] Let (M, g) be a simple manifold, assume that $I_{\mathcal{A}}$ is injective up to natural gauge, then the following stability estimate for $N_{\mathcal{A}}$ holds*

$$\|f^s\|_{L^2(M)} \leq C\|N_{\mathcal{A}} f\|_{H^1(M_1)}.$$

Moreover, there exists $0 < \epsilon \ll 1$ such that the above estimate remains true if g and $\mathcal{A}$ are replaced by $\tilde{g}$ and $\tilde{\mathcal{A}}$ satisfying $\|\tilde{g} - g\|_{C^4(M_1)} \leq \epsilon$, $\|\tilde{\mathcal{A}} - \mathcal{A}\|_{C^3(M_1)} \leq \epsilon$. The constant $C > 0$ can be chosen uniformly, only depending on g, $\mathcal{A}$.

Remark 2.1. *Theorem 2.2 and 2.4 will still hold on a compact manifold satisfying certain microlocal conditions, which essentially says that the union of the conormal bundles of nontrapped geodesics that are free of conjugate points covers the cotangent bundle T^*M. In dimension 2, this microlocal condition excludes the existence of conjugate points, and it is known that conjugate points could be a problem [32]. In dimension 3 and higher this condition allows the existence of trapped geodesics and conjugate points, so one only has access to partial data, and the boundary is not necessarily convex, see also [12, 56].*

2.2 Attenuated X-ray transform on non-simple manifolds

Results for the attenuated X-ray transform on non-simple manifolds are more limited. As mentioned above, the generic injectivity [68] of $I_{\mathcal{A}}$ also holds on non-simple manifolds satisfying certain microlocal conditions. Injectivity of $I_{\mathcal{A}}$ on negatively curved manifolds with hyperbolic trapping sets can be found in [14]. In recent work [44], the author with collaborators establish injectivity results on manifolds of dimension ≥ 3 admitting a convex foliation.

Theorem 2.5. *[44] Let (M, g) be a compact manifold with strictly convex boundary and dim$(M) \geq 3$, admitting a smooth strictly convex function. If $I_{\mathcal{A}} f = 0$, then $f = d_{\mathcal{A}} p$ for some $p \in C^\infty(M; \mathbb{C}^N)$ with $p|_{\partial M} = 0$.*

The condition on the existence of strictly convex function on M was extensively studied in [44] (see also the references there). In particular Lemma 2.1 of [44] shows that such a function exists if any one of the following conditions holds:

(1) The sectional curvature is non-negative;
(2) M is simply connected with no focal points;
(3) M is simply connected and the curvature is non-positive.

It is easy to see that the existence of a strictly convex function implies that (M, g) is nontrapping. The class of manifolds of non-negative curvature shows that, in contrast to many earlier results, Theorem 2.5 allows for the metric to have conjugate points, so not necessarily simple. It is an open question that whether simple manifolds admit strictly convex functions.

To prove Theorem 2.5, which is a global result, we follow the microlocal approach leading to local results in dimension ≥ 3 initiated in [64] for the standard X-ray transform of functions and further exploited in [59, 60, 61]. The local uniqueness results can be iterated, through a layer stripping argument, to obtain global results provided that (M, g) can be foliated by strictly convex hypersurfaces, see [44, Section 6] for details. In particular, we establish the following local injectivity result near a strictly convex boundary point.

Theorem 2.6. *[44] Assume ∂M is strictly convex at $p \in \partial M$. There exists a function $x \in C^\infty(\Omega)$ with $O = \{x > 0\} \cap M$ nonempty, and $u : O \to \mathbb{C}^N$ with $u|_{O \cap \partial M} = 0$, such that $f - d_{\mathcal{A}} u$ can be stably determined from $I_{\mathcal{A}} f$ restricted to O-local geodesics.*

Here Ω is a small smooth extension of (M, g), and O-local geodesics are those geodesics segments in O with endpoints in ∂M.

The proofs of the local theorem is microlocal and we set things up so that a suitably localized version of the normal operator $I_{\mathcal{A}}^* I_{\mathcal{A}}$ fits into Melrose's scattering calculus [26], after conjugation by an exponential weight. As in the previous references, a key ingredient is the introduction of an artificial boundary $x = 0$ which is a little bit less convex than the actual boundary, see the Figure 1 below. The level sets of the function x give a local foliation near the convex boundary point p, and $p \in \{x = c\}$ for some $c > 0$. Let y be the local coordinates along the level sets, so (x, y) are valid local coordinates in the domain of interest O. Then O-local geodesics correspond to those whose tangent vectors are $\lambda\partial_x + \omega\partial_y$, $|\lambda| \ll 1$. The localized operator has the form

$$N_{\mathcal{A},F} f = Q^{-1} e^{-F/x} \int\int \widetilde{W} \chi(\lambda/x) I_{\mathcal{A}}(e^{F/x} Q f)\, d\lambda d\omega, \quad F > 0$$

where Q is a matrix function only depending on x and $\widetilde{W}$ is a modified conjugate of the attenuation weight $W_{\mathcal{A}}$. The cutoff function $\chi \in C_c^\infty(\mathbb{R})$ is even and non-negative, thus when x is small, only integrals along those geodesics parameterized by sufficiently small λ will contribute to $N_{\mathcal{A},F} f$.

To obtain the Fredholm property for $N_{\mathcal{A},F}$ in Melrose's scattering calculus, one needs to verify its ellipticity at both the fiber infinity (the same as the ellipticity for standard pseudodifferential operators) and the spacial infinity $x^{-1} = \infty$ (i.e. $x = 0$, so the spacial boundary is pushed to infinity) when restricted on the kernel of the principal symbol of $\delta_{\mathcal{A},F} := e^{F/x} \delta_{\mathcal{A}} Q^{-1} e^{-F/x}$. This implies that there exists some operator P, such that

$$N_{\mathcal{A},F} + d_{\mathcal{A},F} P \delta_{\mathcal{A},F}$$

is an elliptic scattering pseudodifferential operator of order $(-1, 0)$. Here the indices -1 and 0 correspond to the decay of the principal symbol at fiber infinity and the boundary $x = 0$ respectively. $d_{\mathcal{A},F} := e^{-F/x} Q^{-1} d_{\mathcal{A}} e^{F/x}$ is the conjugate of $\delta_{\mathcal{A},F}$. Similar to (2), the ellipticity essentially implies that there exist operators B and D, such that

$$B N_{\mathcal{A},F} + D \delta_{\mathcal{A},F} = Id + K$$

for some compact operator K. In addition, if $c > 0$ is sufficiently small, i.e. the domain of interest O is small enough, the operator K becomes a contraction, so $Id + K$ is invertible by Neumann series, therefore $N_{\mathcal{A},F}$ is injective and stable in proper weighted Sobolev norms when restricted to the kernel of $\delta_{\mathcal{A},F}$. Theorem 2.6 then follows after transferring the above conclusion in a gauge free way.

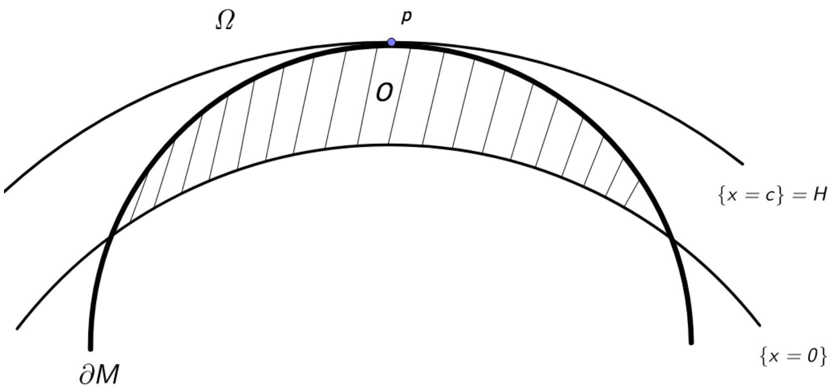

Fig. 1: The region below ∂M is the interior of M, while the region $\{x > 0\}$ is above $\{x = 0\}$. The neighborhood $O = M \cap \{x > 0\}$ is the shadowed area. The hypersurface $H = \{x = c\}$ is tangent to ∂M at the point p.

3 X-ray transform with general weights

Let $W \in C^\infty(SM; \mathbb{C}^{N\times N})$ be invertible and $f \in C^\infty(M; \mathbb{C}^N)$, we consider the X-ray transform with a general matrix weight

$$I_W f(x,v) = \int_0^{\tau(x,v)} W(\gamma_{x,v}(t), \dot\gamma_{x,v}(t)) f(\gamma_{x,v}(t))\, dt, \quad (x,v) \in \partial_+ SM.$$

In the Euclidean space of dimension ≥ 3, I_W is injective for $C^{1,\alpha}$ weight [20], but there are counterexamples for C^α weights [13]. In the two dimensional plane, there are counterexamples with smooth weights [7].

By applying the analytic microlocal analysis, we can prove generic injectivity similar to Theorem 2.2 and 2.4, which works in any dimensions.

Theorem 3.1. *[68] Let M be a real-analytic simple manifolds with real-analytic metric g, let W be real-analytic and invertible, then I_W is injective.*

Theorem 3.2. *[68] Let (M, g) be a simple manifold, assume that I_W is injective, then the following stability estimate for $N_W = I_W^* I_W$ holds*

$$\|f\|_{L^2(M)} \leq C\|N_W f\|_{H^1(M_1)}.$$

Moreover, there exists $0 < \epsilon \ll 1$ *such that the above estimate remains true if* g *and* W *are replaced by* $\tilde{g}$ *and* $\tilde{W}$ *satisfying* $\|\tilde{g} - g\|_{C^4(M_1)} \leq \epsilon$, $\|\tilde{W} - W\|_{C^3(M_1)} \leq \epsilon$. *The constant* $C > 0$ *can be chosen uniformly, only depending on* g, W.

The scalar case was considered in [12], and a version for Radon transforms was studied in [19].

When the manifold admits a strictly convex function, using the scattering calculus and the layer stripping argument, one can prove the following injectivity result similar to Theorem 2.5.

Theorem 3.3. *[44] Let* (M, g) *be a compact manifold with strictly convex boundary and dim*$(M) \geq 3$*, admitting a smooth strictly convex function, then* I_W *is injective.*

Theorem 3.3 generalizes earlier results of the scalar case [64, Appendix]. A quantitative version regarding the stability of I_W under the convex foliation condition can be found in [6]. Under similar geometric assumptions, [22] establishes injectivity for piecewise constant functions where the weight W is only continuous on SM. In [57], the authors apply microlocal analysis to study weighted X-ray transform on manifolds with fold caustics.

Remark 3.1. *We just consider the weighted X-ray transform along ordinary geodesics in this section. However, the results can be generalized to general smooth curves, even with nonconstant speed, see [12] and [64, Appendix], also Section 4.2.*

4 Applications

In this section, we discuss several applications of the weighted X-ray transforms to various inverse problems.

4.1 The non-abelian X-ray transform

We consider the non-abelian X-ray transform, which is a non-linear inverse problem. However, it can be reduced to the linear question of inverting certain attenuated X-ray transform.

4.1.1 Non-linear problem for connections

Let A be a $GL(N,\mathbb{C})$-connection, this simply means that A is an $N\times N$ matrix whose entries are smooth 1-forms with values in $\mathbb{C}$. It is natural to incorporate a potential or Higgs field into the problem by considering a *pair* (A,Φ), where A is a $GL(N,\mathbb{C})$-connection and Φ is a smooth map $M\to\mathbb{C}^{N\times N}$.

Given $(x,v)\in\partial_+SM$ and smooth geodesic $\gamma=\gamma_{x,v}:[0,\tau(x,v)]\to M$, we can solve a transport equation along geodesics:

$$\begin{cases} \dot U+[A_{\gamma(t)}(\dot\gamma(t))+\Phi(\gamma(t))]U=0,\\ U(0)=\mathrm{id}\end{cases}$$

and define the *scattering data* $C_{A,\Phi}:\partial_+SM\to GL(N,\mathbb{C})$ by

$$C_{A,\Phi}(x,v):=U(\tau(x,v)).$$

$C_{A,\Phi}$ is also called the *non-abelian X-ray transform* of (A,Φ). It encapsulates the parallel transport information along geodesics connecting boundary points. The inverse problem of recovering the pair (A,Φ) from $C_{A,\Phi}$ has a natural gauge equivalence: if $u:M\to GL(N,\mathbb{C})$ is smooth and $u|_{\partial M}=\mathrm{id}$ then

$$C_{A,\Phi}=C_{u^{-1}du+u^{-1}Au,u^{-1}\Phi u}.$$

Recently, the author and collaborators [44] proved the following unique determination result up to gauge transformations.

Theorem 4.1. *[44, Theorem 1.1] Let (M,g) be a compact Riemannian manifold of dimension ≥ 3 with strictly convex boundary, and suppose (M,g) admits a smooth strictly convex function. Let (A,Φ) and (B,Ψ) be two pairs such that $C_{A,\Phi}=C_{B,\Psi}$. Then there is a smooth map $u:M\to GL(N,\mathbb{C})$ such that $u|_{\partial M}=\mathrm{id}$, $B=u^{-1}du+u^{-1}Au$ and $\Psi=u^{-1}\Phi u$.*

Theorem 4.1 is proved by introducing a pseudo-linearization that reduces the non-linear problem to a linear one related to the attenuated X-ray transform. A similar scenario arises in *polarization tomography* [37] and *quantum state tomography* [20], see [44, Section 8].

One virtue of Theorem 4.1 is that there is no restriction on the pair (A,Φ). In 2D, similar result holds on simple surfaces [40]. In previous works [11, 51, 41, 14] it was assumed that the structure group was the unitary group. There are also recent works dealing with $GL(N,\mathbb{C})$-connections, but under additional assumptions [31, 38]. In [68], the author shows that the rigidity result (up to the natural gauge) holds for generic simple metrics and generic connections and Higgs fields, including the real-analytic ones.

Theorem 4.2. *[68, Theorem 1.1] Let M be a real-analytic simple manifold with real-analytic metric g_0. Let $\mathcal{A}_0$, $\mathcal{B}_0$ be real-analytic, there exists $\epsilon > 0$ such that whenever there are another metric g and pairs $\mathcal{A} = (A, \Phi)$, $\mathcal{B} = (B, \Psi)$ satisfying*

$$\|g - g_0\|_{C^4(M)} \le \epsilon, \quad \|\mathcal{A} - \mathcal{A}_0\|_{C^3(M)} + \|\mathcal{B} - \mathcal{B}_0\|_{C^3(M)} \le \epsilon,$$

(1) if $C_{\mathcal{A}} = C_{\mathcal{B}}$ w.r.t. the metric g, then there is $p : M \to GL(N, \mathbb{C})$ with $p|_{\partial M} = id$, such that $\mathcal{B} = p^{-1} d_{\mathcal{A}} p$;
(2) if $\|\mathcal{A}_0 - \mathcal{B}_0\|_{C^2(M)} \le \epsilon$ and $\iota^* \mathcal{A} = \iota^* \mathcal{B}$ with $\iota : \partial M \to M$ the canonical inclusion, then there exists $p : M \to GL(N, \mathbb{C})$ with $p|_{\partial M} = id$ such that the following stability estimate holds w.r.t. the metric g

$$\|\mathcal{B} - p^{-1} d_{\mathcal{A}} p\|_{\mathbf{L^2(M)}} \le C \|C_{\mathcal{B}} - C_{\mathcal{A}}\|_{H^1(\partial_- SM)}$$

for some uniform constant $C > 0$ which depends only on g_0, $\mathcal{A}_0$, $\mathcal{B}_0$.

4.1.2 Reduction to the linear problem

As mentioned above, the pseudo-linearization of the scattering data $C_{A,\Phi}$ gives certain weighted geodesic X-ray transform. Here we give the reduction, see also [41, 44].

Let $\mathcal{A} = A + \Phi$ and $\mathcal{B} = B + \Psi$. Given a geodesic $\gamma : [0, T] \to M$, let $\phi(t) = (\gamma(t), \dot{\gamma}(t))$ be the corresponding geodesic flow on SM. Define the matrix-valued function

$$F(t) = W_{\mathcal{B}}(\phi(t)) W_{\mathcal{A}}^{-1}(\phi(t)),$$

by the fundamental theorem of calculus and the definitions of $W_{\mathcal{A}}$, $W_{\mathcal{B}}$ (see Section 2.1)

$$F(T) - F(0) = \int_0^T W_{\mathcal{B}}(\phi)(\mathcal{B}(\phi) - \mathcal{A}(\phi)) W_{\mathcal{A}}^{-1}(\phi)\, dt. \tag{1}$$

We define $\hat{W}$ by

$$\hat{W} U = W_{\mathcal{B}} U W_{\mathcal{A}}^{-1}, \qquad U \in C^\infty(SM; \mathbb{C}^{N \times N}),$$

then the right-hand side of (1) indeed gives the following weighted geodesic ray transform of $\mathcal{B} - \mathcal{A}$:

$$\int_\gamma \hat{W}(\mathcal{B} - \mathcal{A})\, dt. \tag{2}$$

By the definition of $\hat{W}$, it is obvious that $\hat{W}|_{\partial_+ SM} = \mathrm{id}$. Given a matrix-valued function U we have

$$X(W_{\mathcal{B}} U W_{\mathcal{A}}^{-1}) = W_{\mathcal{B}} \mathcal{B} U W_{\mathcal{A}}^{-1} + W_{\mathcal{B}}(XU) W_{\mathcal{A}}^{-1} - W_{\mathcal{B}} U \mathcal{A} W_{\mathcal{A}}^{-1},$$

which implies that

$$(X\hat{W})U = \hat{W}(\mathcal{B}U - U\mathcal{A}).$$

If we define $\hat{\mathcal{A}}$ by $\hat{\mathcal{A}}U = \mathcal{B}U - U\mathcal{A}$, we get exactly $X\hat{W} = \hat{W}\hat{\mathcal{A}}$, i.e. (2) is the attenuated geodesic ray transform with the attenuation $\hat{\mathcal{A}}$, we denote it by $I_{\hat{\mathcal{A}}}$.

Now it's easy to see that Theorem 2.5 and 2.4 imply Theorem 4.1 and 4.2 respectively.

4.2 Lens rigidity problem

In this section, we consider several non-linear inverse problems associated with various Hamiltonian flows. In particular, one encounters X-ray transforms along general family of curves.

4.2.1 Lens rigidity for geodesic flows

An important inverse problem arose in geophysics in an attempt to determine the inner structure of the Earth, such as the sound speed or index of refraction, from measurements on the surface of travel times of seismic waves, which is called travel time tomography in seismology. From a mathematical point of view, the sound speed of the Earth is modeled by a Riemannian metric, and the travel times by the lengths of geodesics between boundary points.

Consider the behavior of all the geodesics going through the manifold, in particular on non-simple manifolds, there could be more than one geodesics connecting two points. In addition to the length information, we could also consider the data of the geodesic flow in the phase space. This induces another type of information: the scattering relation, introduced by Guillemin

[15], is a map which sends the point and direction of entrance of a geodesic to the point and direction of exit at the boundary. The scattering relation

$$\mathcal{S} : \partial_+ SM \to \partial_- SM, \quad \mathcal{S}(x, v) := (\gamma_{x,v}(\tau(x, v)), \dot{\gamma}_{x,v}(\tau(x, v)))$$

together with information of lengths (travel time) $\ell = \tau|_{\partial_+ SM}$ of corresponding geodesics gives the lens data $(\mathcal{S}, \ell)$. Notice that the lens data is unchanged under any isometry which fixes the boundary. The lens rigidity problem is concerned with the determination of a Riemannian metric up to isometry (the natural obstruction), from the lens data.

On simple manifolds, the lens data and travel time data are equivalent [27], and is called the boundary rigidity problem. It is known that simple surfaces are boundary distance rigid [47]. For dimensions ≥ 3, the boundary rigidity problem on simple manifolds is still open in general. Uniqueness and stability were proven for simple metrics in the same conformal class [33, 34], and for generic simple metrics, including the real-analytic ones [55]. Recent work [63] addresses the statistical inversion of travel time tomography. We refer to survey papers [66, 62] for summaries of recent developments on the lens/boundary rigidity problem.

The lens rigidity problem is closely related to the geodesic X-ray transform. Indeed, the linearization of the lens/travel time data gives the X-ray transform of symmetric tensors of order two. In [53], the lens rigidity problem on a Euclidean domain in $\mathbb{R}^n$ was reduced to the invertibility of some weighted X-ray transform through a pseudo-linearization argument. A space-time version of the problem was studied in [65]. Let $\phi_i(t, x, v)$ and X_i be the geodesic flow and geodesic vector field associated with Riemannian metric g_i, $i = 1, 2$. They are viewed as vectors in $\mathbb{R}^{2n}$.

Lemma 4.3. *[53, equation 2.10] Assume that two metrics g_1 and g_2 have the same lens data, then*

$$\int_0^{\ell(x_0,v_0)} \frac{\partial \phi_2}{\partial(x, v)}(t - s, \phi_1(s, x_0, v_0))(X_1 - X_2)(\phi_1(s, x_0, v_0))\, ds = 0. \tag{3}$$

The left hand side of (3) is a weighted X-ray transform of $X_1 - X_2$. Notice that $X_1 - X_2$ implicitly measures the difference between g_1 and g_2. On the other hand, locally near a strictly convex boundary point, the weight $\frac{\partial \phi_2}{\partial(x,v)}(t - s, \phi_1(s, x_0, v_0))$ is close to the identity matrix, so invertible, for (x_0, v_0) almost tangent to ∂M.

Applying the scattering calculus to the transform from Lemma 4.3, local lens rigidity was proven in [59, 61]. Consequently, the layer stripping argument gives the following global lens rigidity.

Theorem 4.4. *[61, Theorem 1.3] Suppose that (M, g) is a compact n-dimensional Riemannian manifold, $n \geq 3$, with strictly convex boundary, and x is a smooth function with non-vanishing differential whose level sets are strictly concave from the superlevel sets, and $\{x \geq 0\} \cap M \subset \partial M$. Suppose also that $\hat{g}$ is a Riemannian metric on M and suppose that the lens data of g and $\hat{g}$ are the same. Then there exists a diffeomorphism $\psi : M \to M$ fixing ∂M such that $g = \psi^* \hat{g}$.*

The isotropic version of Theorem 4.4 was shown earlier in [59].

4.2.2 Lens rigidity for Magnetic flows

Let G be the Lorentz force, a 1-1 tensor, associated with some magnetic field Ω, that is a closed 2-form, through the equality

$$\Omega_x(v, w) = \langle G_x(v), w \rangle_g, \quad v, w \in T_x M.$$

A solution γ of

$$\nabla_{\dot{\gamma}} \dot{\gamma} = G(\dot{\gamma})$$

is called a *magnetic geodesic*, where ∇ is the Levi-Civita connection of g. Then $\phi_t : t \to (\gamma(t), \dot{\gamma}(t))$ defines a *magnetic flow* on TM, which is a Hamiltonian flow. The triple (M, g, Ω) defines a magnetic system. It is easy to check that every magnetic geodesic has constant speed, here we only consider the unit speed magnetic geodesics.

Let $x \in \partial M$, $S\partial M$ be the unit sphere bundle of the boundary ∂M, we say M is *strictly magnetic convex* at x if

$$\Lambda(x, v) > \langle Y_x(v), \nu(x) \rangle_g$$

for all $v \in S_x \partial M$, where Λ is the second fundamental form of ∂M, $\nu(x)$ is the inward unit vector normal to ∂M at x. When $Y = 0$, this is consistent with the ordinary definition of convexity.

Similar to the definition of the lens data of usual geodesics, one can define the lens data of magnetic geodesics and consider the corresponding lens rigidity problem. In [70], the conformal case of this non-linear problem was studied.

Theorem 4.5. *[70, Theorem 1.5] Let dim $M \geq 3$, g_0 be a given Riemannian metric, let $c, \tilde{c} > 0$ be smooth functions, $\Omega, \tilde{\Omega}$ be smooth closed 2-forms and let ∂M be strictly magnetic convex with respect to both $(c^2 g_0, \Omega)$ and $(\tilde{c}^2 g_0, \tilde{\Omega})$. Assume that $c = \tilde{c}$ and $\iota^*\Omega = \iota^*\tilde{\Omega}$ on ∂M, and M can be foliated by strictly magnetic convex hypersurfaces for $(M, c^2 g_0, \Omega)$. If $\mathcal{S} = \tilde{\mathcal{S}}$, $\ell = \tilde{\ell}$, then $c = \tilde{c}$ and $\Omega = \tilde{\Omega}$ in M.*

The proof of Theorem 4.5 is also a combination of the pseudo-linearization and a layer stripping process. It is worth mentioning that integral identity (3) holds for general Hamiltonian flows. Earlier results under the simplicity assumption can be found in e.g. [9, 4]. On the other hand, the linearized problem was considered in [9, 69].

4.2.3 Lens rigidity for particles in a Yang-Mills field

In this section, we consider a nonlinear inverse problem associated with the motion of a classical colored spinless particle under the influence of an external Yang-Mills potential.

Let G be a compact Lie group of matrices with Lie algebra $\mathfrak{g}$. We think of (M, g) as the configuration space where our classical colored particle travels and we think of the $\mathfrak{g}$ (or its dual $\mathfrak{g}^*$) as the space of "color charges" or internal degrees of freedom. In this case, a connection A (the external Yang-Mills potential) is just an element $A \in C^\infty(M, T^*M \otimes \mathfrak{g}) = \Lambda^1(M, \mathfrak{g})$. Since $\mathfrak{g}$ is a Lie algebra of matrices, we can think of A as a matrix of 1-forms in that Lie algebra. We define $F := F_A = dA + A \wedge A \in \Lambda^2(M, \mathfrak{g})$ the *curvature* or field strength of A. Using the metric g, given $\xi \in \mathfrak{g}$, we can define a $(1,1)$-tensor $\mathbb{F}^\xi : TM \to TM$ uniquely by

$$g_x(\mathbb{F}^\xi_x(v), w) = \langle F_x(v, w), \xi \rangle$$

for all $x \in M$ and $v, w \in T_x M$. The field $\mathbb{F}$ will play the role of a generalized Lorentz force. The connection A induces a covariant derivative in the adjoint bundle which we denote by D.

The system lives in $TM \times \mathfrak{g}$ and the ODEs determining the trajectories $t \mapsto (\gamma(t), \dot{\gamma}(t), \xi(t)) \in TM \times \mathfrak{g}$ are given by

$$\begin{cases} \nabla_{\dot{\gamma}} \dot{\gamma} = \mathbb{F}^\xi_\gamma(\dot{\gamma}), \\ D_{\dot{\gamma}} \xi = 0, \end{cases} \tag{4}$$

which are called *Wong's equations* [67]. The equations reduce to the Lorentz equation of magnetic geodesics in the abelian case $G = U(1)$. A quick analysis of (4) reveals two kinematic constraints: γ must travel at constant speed and ξ must remain in the adjoint orbit it started on. For this reason, it makes sense from now on to restrict our motion to the compact phase space $SM \times \mathcal{O}$.

Definition 4.1. *A smooth function $f : M \to \mathbb{R}$ is said to be strictly YM-convex if*

$$Hess_x(f)(v,v) + \langle F_x(v, \nabla f(x)), \xi \rangle > 0$$

for all $(x, v, \xi) \in SM \times \mathcal{O}$. Similarly, we shall say that $x \in \partial M$ is strictly YM-convex if

$$\Lambda_x(v,v) + \langle F_x(v, \nu(x)), \xi \rangle > 0$$

for any $v \in S_x \partial M$ and $\xi \in \mathcal{O}$, where Λ is the second fundamental form of ∂M. If this holds for all $x \in \partial M$, then we say that ∂M is strictly YM-convex.

Now under suitable conditions, it is possible to recover the potential A, up to gauge transformations, from the lens data $(\mathcal{S}, \ell)$ of the system (i.e. scattering data plus travel times).

Theorem 4.6. *[45, Theorem 1.2] Let (M, g) be a compact Riemannian manifold with boundary and dimension ≥ 3 and let $\mathcal{O}$ be an adjoint orbit that contains a basis of $\mathfrak{g}$. Let A and $\tilde{A}$ be two Yang-Mills potentials such that*
1. *∂M is strictly YM-convex with respect to both (g, A) and $(g, \tilde{A})$;*
2. *$i^*A = i^*\tilde{A}$ where $i : \partial M \to M$ is the canonical inclusion.*

If (g, A) admits a strictly YM-convex function and $(\mathcal{S}_A, \ell_A) = (\mathcal{S}_{\tilde{A}}, \ell_{\tilde{A}})$, then there exists a smooth function $u : M \to G$ such that $\tilde{A} = u^{-1}du + u^{-1}Au$ and $u|_{\partial M} = e$.

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