SIAM J. Math. Anal.
Vol. 50, No. 2, pp. 1753–1778

THE LOCAL MAGNETIC RAY TRANSFORM OF TENSOR FIELDS*

HANMING ZHOU†

Abstract. In this paper we study the local magnetic ray transform of symmetric tensor fields up to rank two on a Riemannian manifold of dimension ≥ 3 with boundary. In particular, we consider the magnetic ray transform of the combinations of tensors of different orders due to the nature of magnetic flows. We show that such magnetic ray transforms can be stably inverted, up to natural obstructions, near a strictly convex (with respect to magnetic geodesics) boundary point. Moreover, a global invertibility result follows on a compact Riemannian manifold with strictly convex boundary assuming that some global foliation condition is satisfied.

Key words. tensor tomography, magnetic geodesics, partial data, X-ray transform

AMS subject classifications. 53C65, 35R30, 35S05

DOI. 10.1137/16M1093963

1. Introduction.

Given a Riemannian manifold (M, g) and a magnetic field Ω, which is a closed 2-form, we consider the law of motion described by

$$\nabla_{\dot\gamma}\dot\gamma = E(\dot\gamma), \tag{1}$$

where ∇ is the Levi–Civita connection of g with the Christoffel symbols $\{\Gamma^i_{jk}\}$ and $E : TM \to TM$ is the *Lorentz force*, which is the bundle map uniquely determined by

$$\Omega_z(v, w) = \langle E_z(v), w\rangle$$

for all $z \in M$ and $v, w \in T_zM$. A curve $\gamma : \mathbb{R} \to M$, satisfying (1) is called a *magnetic geodesic*. The flow on TM defined by $\phi_t : t \to (\gamma(t), \dot\gamma(t))$ is called a *magnetic flow*. One can check that the generator $\mathbf{G}_\mu$ of the magnetic flow is

$$\mathbf{G}_\mu(z, v) = \mathbf{G}(z, v) + E^j_i(z)v^i\frac{\partial}{\partial v^j},$$

where $\mathbf{G}(z, v) = v^i\frac{\partial}{\partial x^i} - \Gamma^i_{jk}v^jv^k\frac{\partial}{\partial v^i}$ is the generator of the geodesic flow. Note that time is not reversible on the magnetic geodesics, unless $\Omega = 0$. When $\Omega = 0$ we obtain the ordinary geodesic flow. We call the triple (M, g, Ω) a *magnetic system*.

From a dynamical system point of view, the magnetic flow is the Hamiltonian flow of $H(v) = \frac{1}{2}|v|^2_g$, $v \in TM$ w.r.t. the symplectic form $\beta = \beta_0 + \pi^*\Omega$, where β_0 is the canonical symplectic form on TM and $\pi : TM \to M$ is the canonical projection. Thus the magnetic flow preserves the level sets of the Hamiltonian function H, i.e., every magnetic geodesic has constant speed. Unlike the usual geodesics, the behavior of magnetic geodesics depends on the choice of the energy level. Throughout the paper we fix the energy level $H^{-1}(\frac{1}{2})$, so we only consider the unit speed magnetic geodesics. However, this is not a constraint at all; it is easy to check from (1) that

---

[*] Received by the editors September 14, 2016; accepted for publication (in revised form) January 22, 2018; published electronically March 29, 2018.
http://www.siam.org/journals/sima/50-2/M109396.html
Funding: The work of the author was supported by RCUK/EPSRC: EP/M023842/1.

[†] Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge CB3 0WB, UK. Current address: Department of Mathematics, University of California Santa Barbara, Santa Barbara, CA 93106-3080 (hzhou@math.ucsb.edu).

one can obtain the behavior (up to time scale) of magnetic geodesics at any energy level by rescaling the magnetic field Ω.

Given a magnetic geodesic γ and a smooth function f on SM, the unit sphere bundle of M, the *magnetic ray transform* of f along γ is defined by

$$If(\gamma) = \int f(\gamma(t), \dot\gamma(t))\, dt.$$

It is easy to check that the kernel of I contains elements of the following form:

$$\{f(z,v) = \mathbf{G}_\mu \psi(z,v) : \psi \in C^\infty(SM), \psi|_{\partial SM} = 0\}.$$

In applications one often considers ray transforms of f which correspond to symmetric tensor fields, i.e., $f(z,v) = f_{i_1\cdots i_m}(z)v^{i_1}\cdots v^{i_m}$, denoted by $I_m f$ for nonnegative integers m. A basic inverse problem regarding the magnetic ray transform on a compact Riemannian manifold with boundary is to recover the tensor field f, up to natural obstructions, from $If(\gamma)$ along all magnetic geodesics γ joining boundary points.

If $\Omega = 0$, this reduces to the usual geodesic ray transform of tensor fields, known as the *tensor tomography* problem. In this case, the natural elements in the kernel of I_m are of the form $d^s\psi$, where d^s is the symmetric differentiation and ψ is a symmetric $(m-1)$-tensor vanishing on the boundary. These natural elements of the kernel are called *potential* tensor fields. So the question is whether the whole kernel consists of purely potential tensors, and when this is the case we say that I_m is *s-injective* (when $m = 0$, this just means injective). The problem is wide open on compact *nontrapping* manifolds with strictly convex boundary. Note that a compact manifold with boundary is nontrapping if every geodesic exits the manifold within a finite time.

More progress has been made on manifolds under the stronger assumption of being *simple*. A compact manifold with boundary is simple if it is simply connected and free of conjugate points and ∂M is strictly convex. It is known that I_0 [18, 19] and I_1 [2] are s-injective on simple manifolds with sharp stability estimates [36]. For I_m, $m \geq 2$, the tensor tomography problem on simple manifolds is still open in general, except the two-dimensional (2D) case. I_m is s-injective on simple surfaces for any $m \geq 2$ [34, 21]. In higher dimensions, I_m is s-injective for generic simple metrics including the analytic ones [37] and a sharp stability estimate holds [35]. The equivalence between the s-injectivity of I_m and the surjectivity of its adjoint is known on simple manifolds [25]. See also the recent survey [22] and the references therein. For nonsimple manifolds, there are studies under various assumptions [26, 31, 32, 33, 4] and possibly with conjugate points [39, 40, 17, 12] or trapped geodesics [8]. Reconstruction formulas and numerical implementations of the geodesic ray transform on surfaces can be found, e.g., in [27, 16, 9].

For the magnetic ray transform, potential tensors might not stay in the kernel of I (except I_0 and I_1). Generally the natural elements in the kernel of the magnetic ray transform are combinations of tensors of different orders. For example, the magnetic ray transform of $d^s\beta - E(\beta) + d\varphi = \mathbf{G}_\mu(\beta + \varphi)$ always vanishes, where β is a 1-form and φ is a function on M, both vanishing on the boundary. In the current paper, we focus on the magnetic ray transform of tensor fields of orders up to 2. In particular, we are interested in the magnetic ray transform of tensor fields which are sums of 1-forms and symmetric 2-tensors. Note that for the geodesic ray transform, it is unnecessary to consider the combination of 1-forms and 2-tensors, since one can decouple the integral by the fact that geodesic flows are symmetric (or time reversible).

The tensor tomography problem is closely related to another well-known geometric inverse problem, namely, the boundary rigidity problem, which is concerned with the recovery of a Riemannian metric on a compact smooth manifold with boundary from the length data of distance minimizing geodesics connecting boundary points. In particular, the linearization of the boundary rigidity problem is the geodesic ray transform of symmetric 2-tensors. It has been proved that simple surfaces are boundary rigid [28]; in higher dimensions generic simple manifolds are boundary rigid [37] including the analytic ones. See also recent surveys [3, 38, 44] and the references therein. There is also a boundary rigidity problem on magnetic systems [5], whose linearization is exactly the magnetic ray transform of $h + \beta$, where h is a symmetric 2-tensor and β is a 1-form. This provides another motivation for considering such magnetic ray transforms.

A new approach to the tensor tomography problem on compact manifolds of dimension ≥ 3 with strictly convex boundary has been developed recently in [43, 42] under a global foliation assumption, based on corresponding local invertibility results. It was also applied to the boundary rigidity problem [41] through a pseudo-linearization argument. As a generalization, we study the local invertibility of the magnetic ray transform of tensor fields in the current paper. We ask the following question: can one recover f, up to natural obstructions, near a boundary point p from its integrals If along magnetic geodesics near p? By saying magnetic geodesics near p we mean that all magnetic geodesic segments that are completely contained in some small neighborhood O of p with end points on ∂M close to p, which we call *O-local* magnetic geodesics, denoted by $\mathcal{M}_O$. Of course, such a set might be empty if there is no additional geometric assumption of the boundary.

In order to state our main theorems in concrete terms, we describe briefly the setting for our problem. Let M be a compact Riemannian manifold with boundary. Letting $z \in \partial M$, we say M is *strictly magnetic convex (concave)* at z if

$$\Lambda(z, v) - \langle E_z(v), \nu(z)\rangle_g > 0\,(< 0)$$

for all $v \in S_z(\partial M)$, where Λ is the second fundamental form of ∂M, and $\nu(z)$ is the inward unit vector normal to ∂M at z. We can extend M to a complete manifold $\widetilde{M}$ and denote the extended metric and magnetic field still by g and Ω. Obviously, magnetic geodesics γ on M can be uniquely extended to a magnetic geodesic on $\widetilde{M}$, and we still denote it by γ. Then intuitively the strict magnetic convexity at $z \in \partial M$ means that any magnetic geodesic γ which is tangent to the boundary ∂M at z will stay away from M except at z locally.

Now let $\rho \in C^\infty(\widetilde{M})$ be a boundary defining function of ∂M, so that $\rho \geq 0$ on M. Suppose ∂M is strictly magnetic convex at $p \in \partial M$; then given a magnetic geodesic γ on $\widetilde{M}$ with $\gamma(0) = p$, $\dot\gamma(0) \in S_p(\partial M)$, one has

$$\frac{d^2\rho}{dt^2}(\gamma(t))|_{t=0} = -\Lambda(p, \dot\gamma(0)) + \langle E_p(\dot\gamma(0)), \nu(p)\rangle_g < 0. \tag{2}$$

Similar to [43] we consider the function $\tilde{x}(z) = -\rho(z) - \epsilon|z - p|^2$, where $|\cdot|$ can be taken as the Euclidean norm locally, for some small enough $\epsilon > 0$, so that the level set $\{\tilde{x} = -c\}$ (as a local hypersurface) is strictly magnetic concave from $U_c = \{\tilde{x} > -c\} \subset \widetilde{M}$ for some sufficiently small $c > 0$. For the sake of simplicity, we drop the subscript c, i.e., $U_c = U$, and $O = U \cap M$ with compact closure.

From now on, we assume that M is of dimension $\geq$3. We first consider a simpler case, namely, $f = \beta + \varphi$, where β is a 1-form and φ is a function. In fact such a magnetic ray transform is the linearization of the magnetic boundary rigidity problem in a fixed conformal class. Integrals of such tensors also appear in attenuated ray transforms [11, 30, 20, 24]. For magnetic systems, a similar weighted magnetic ray transform was considered in [46] for studying the magnetic lens rigidity problem in a fixed conformal class. Here we consider purely the integral of f, without extra weights. The global case was considered in [5] for simple systems. When f is just a function on M, i.e., $\beta = 0$, the local invertibility of I was studied by the author for even a general family of smooth curves; see the appendix of [43].

Theorem 1.1. *Let $n = \dim M \geq 3$. Assume that ∂M is strictly magnetic convex at $p \in \partial M$. Given $f \in L^2(T^*\overline{O}) \times L^2(\overline{O})$, there is $q \in H^1_{loc}(O)$ with $q|_{O\cap\partial M} = 0$ such that $f - dq \in L^2_{loc}(T^*O) \times L^2_{loc}(O)$ can be determined from If restricted to O-local magnetic geodesics. Moreover, the stability estimate for $s \geq 0$*

$$\|f - dq\|_{H^{s-1}(K)} \leq C\|If\|_{H^s(\mathcal{M}_O)}$$

holds on any compact subset K of O, assuming f is in H^s instead of L^2.

Next we consider the local magnetic ray transform of $f = h + \beta$ with h a symmetric 2-tensor and β a 1-form. As mentioned above, such ray transforms might find their application in the boundary rigidity problem for magnetic systems. The global case was considered in [5] for simple magnetic systems satisfying some curvature assumption or real analytic magnetic systems, and later on simple 2D magnetic systems [1].

Theorem 1.2. *Let $n = \dim M \geq 3$. Assume that ∂M is strictly magnetic convex at $p \in \partial M$. Given $f \in L^2(Sym^2T^*\overline{O}) \times L^2(T^*\overline{O})$, there exist $u \in H^1_{loc}(T^*O)$ and $q \in H^1_{loc}(O)$ with $u|_{O\cap\partial M} = 0$, $q|_{O\cap\partial M} = 0$ such that $f - (d^su - E(u) + dq) \in L^2_{loc}(Sym^2T^*O) \times L^2_{loc}(T^*O)$ can be determined from If restricted to O-local magnetic geodesics. Moreover, the stability estimate for $s \geq 0$*

$$\|f - (d^su - E(u) + dq)\|_{H^{s-1}(K)} \leq C\|If\|_{H^s(\mathcal{M}_O)}$$

holds on any compact subset K of O, assuming f is in H^s instead of L^2.

Theorems 1.1 and 1.2 generalize the Helgason's type of support theorems for the tensor tomography problem of the geodesic flow in the real-analytic category [13, 14] and the smooth category [43, 42] to the magnetic case. Reconstruction formulas can also be derived in the spirit of [42, Theorem 4.15].

As an immediate consequence and application of our local invertibility theorems, we consider the global s-injectivity of the magnetic ray transform on tensors. Given a compact Riemannian manifold (M, g) with smooth boundary and a magnetic field Ω, we say that M can be *foliated by strictly magnetic convex hypersurfaces* w.r.t. the magnetic system (M, g, Ω) if there exist a smooth function $\tau : M \to \mathbb{R}$ and $a < b$, such that $M \subset \{\tau \leq b\}$, the level set $\tau^{-1}(t)$ is strictly magnetic convex from $\{\tau \leq t\}$ for any $t \in (a, b]$, $d\tau$ is nonzero on these level sets, and $\{\tau \leq a\}$ has empty interior. Note that ∂M is not necessarily a level set of τ.

Theorem 1.3. *Let M be compact with smooth boundary and $\dim M \geq 3$, ∂M is strictly magnetic convex. Assume that M can be foliated by strictly magnetic convex hypersurfaces and the set $\{\tau \leq a\}$ is nontrapping.*

(a) *Given $f \in C^\infty(T^*M) \times C^\infty(M)$, if $If \equiv 0$, there exists $q \in C^\infty(M)$ with $q|_{\partial M} = 0$ such that $f = dq$.*

(b) *Given $f \in C^\infty(Sym^2T^*M) \times C^\infty(T^*M)$, if $If \equiv 0$, there exist $u \in C^\infty(T^*M)$ and $q \in C^\infty(M)$ with $u|_{\partial M} = 0$, $q|_{\partial M} = 0$, such that $f = d^su - E(u) + dq$.*

The proof of the global result is based on a layer stripping argument similar to that in [43, 42, 24], combined with a regularity result of the solutions of some transport equation w.r.t. the magnetic flow on the unit sphere bundle. A global stability estimate can be derived in a similar way.

In the case of absence of magnetic fields, the foliation condition is an analogue of the Herglotz [10] and Wiechert and Zoeppritz [45] condition $\frac{\partial}{\partial r}\frac{r}{c(r)} > 0$ for radial symmetric metrics; $c(r)e$ on a disk with e the Euclidean metric; see also [41, section 6]. Examples of manifolds satisfying the foliation conditions are compact submanifolds of complete manifolds with positive curvature [7], compact manifolds with nonnegative sectional curvature [6], and compact manifolds with no focal points [29]. Our foliation condition defined above is the corresponding version for magnetic systems. It implies the absence of trapped magnetic geodesics in $\{\tau > a\}$ but allows the existence of conjugate points (w.r.t. the magnetic geodesics). Given a compact Riemannian manifold (M, g) with strictly convex boundary, satisfying the foliation condition (w.r.t. the usual geodesics), a simple way of constructing examples of magnetic systems admitting the magnetic foliation condition is by adding a magnetic field Ω supported away from $\{\tau \leq a\}$ (here τ and a are w.r.t. the geodesic case) with sufficiently small norm, e.g., $|\Omega_z(u, v)|$ is small enough for any $z \in M$ and $u, v \in S_zM$. Then it is easy to check by definition that the magnetic system (M, g, Ω) satisfies the magnetic foliation condition, and the boundary is also strictly magnetic convex.

As mentioned above, the main difference of the magnetic tensor tomography problem compared with the geodesic case is the coupling of tensors of different orders. Similar to [43, 42], we introduce some localized version of I^*I near $p \in \partial M$ to fit into Melrose's scattering calculus [15]. However, in addition to the exponential conjugacy that appeared in the geodesic papers, we add an extra pair of conjugacy to address the issue arising from the coupling of tensors of different orders; see sections 2 and 3 for details. Another technical difficulty comes up during the decoupling of the effects from tensors of different orders when studying the ellipticity of the localized operator near the artificial boundary $\tilde{x} = -c$. In particular, the nature of the magnetic flow appears in the symmetric 2-tensor case (section 3.2), our algebraic argument for the ellipticity of the localized operator is different from [42], and it has potential applications to the boundary rigidity problem for magnetic systems and the invertibility of ray transforms along more general curves.

The paper is organized as follows. In section 2, we give a brief introduction of the scattering calculus and define the localized operators and the proper gauges for our problem. Section 3 is devoted to the proof of the ellipticity of the localized operator, up to the gauges, which addresses the key technical issue of the paper. The proofs of Theorems 1.1 and 1.2 are given in section 4. Finally, we give the proof of Theorem 1.3 in section 5.

2. The localized operators.

For fixed small $c > 0$, let $x = \tilde{x} + c$; thus $U = \{x > 0\}$ with the artificial boundary $x = 0$. One can complete x to a coordinate system (x, y) on a neighborhood of p, with y the coordinates on ∂U. For each point (x, y) we can parameterize magnetic geodesics through this point which are "almost tangent" to level sets of x (these are the curves that we are interested in) by $\lambda\partial_x + \omega\partial_y \in T\widetilde{M}$, $\omega \in \mathbb{S}^{n-2}$ and λ is relatively small. Given a magnetic geodesic $\gamma_{x,y,\lambda,\omega}(t) = (x(t), y(t))$ with $\gamma_{x,y,\lambda,\omega}(0) = (x, y)$, generally $\lambda\partial_x + \omega\partial_y$ is not of unit length, but we still assume that t is the unit speed parameter for γ, so $\dot{\gamma}_{x,y,\lambda,\omega}(0)$ is a positive multiple of $\lambda\partial_x + \omega\partial_y$. We define $\alpha(x, y, \lambda, \omega, t) = \frac{1}{2}\frac{d^2}{dt^2}x(t)$, in particular, $\alpha(x, y, 0, \omega, 0) > 0$ for x small by the concavity of x. Furthermore, it was shown in [43] that there exist $\delta_0 > 0$

small and $C > 0$ such that for $|\lambda| \le C\sqrt{x}$ (and $|\lambda| < \delta_0$), $x(t) \ge 0$ for $t \in (-\delta_0, \delta_0)$, the magnetic geodesics remain in $\{x \ge 0\}$ at least for $|t| < \delta_0$, i.e., they are O-local magnetic geodesics for sufficiently small c. Note that [43] considers ordinary geodesics, but the settings work for general curves; see the appendix of [43].

Our inverse problem is now that assuming

$$(If)(x, y, \lambda, \omega) = \int_{\mathbb{R}} f(\gamma_{x,y,\lambda,\omega}(t), \dot{\gamma}_{x,y,\lambda,\omega}(t))\, dt$$

($f = \beta + \varphi$ or $h + \beta$) is known for all $\gamma_{x,y,\lambda,\omega} \in \mathcal{M}_O$, the set of O-local geodesics, we would like to recover f from If up to some gauge. Originally f was defined on M; we can extend f by zero to $\widetilde{M}$ so that the integral is actually defined on a finite interval.

We will construct some localized version of the normal operator I^*I and study the microlocal properties of it. The main microlocal analysis will be carried out near $\partial U = \{x = 0\}$ in $\overline{U} = \{x \ge 0\}$, which is a manifold with boundary. Since the standard pseudodifferential calculus is not suitable for working near the boundary of a manifold, we will apply the scattering pseudodifferential calculus (scattering calculus, for short) introduced by Melrose [15]. Below we give a brief introduction of the scattering calculus and related terminologies; a more thorough discussion can be find in [43, section 2]. As one can see from the discussion below, the scattering calculus is really a natural generalization of the standard pseudodifferential calculus. We start with the scattering pseudodifferential algebra in the Euclidean case; as the scattering calculus is typically used to study the behavior "at infinity," it is convenient to compactify the Euclidean space $\mathbb{R}^n$ by gluing a sphere at the infinity, so topologically the compactified space $\overline{\mathbb{R}^n}$ is a ball. More precisely, we can use the following identification (inverse polar coordinates) near the boundary (i.e., the infinity) of $\overline{\mathbb{R}^n}$:

$$(x, \theta) \in [0, \epsilon) \times \mathbb{S}^{n-1} \to z = x^{-1}\theta \in \overline{\mathbb{R}^n}. \tag{3}$$

The scattering pseudodifferential algebra $\Psi^{m,l}_{sc}(\overline{\mathbb{R}^n})$ is the generalization of the standard pseudodifferential algebra by quantizing symbols $a \in S^{m,l}$, $m, l \in \mathbb{Z}$, which are elements in $C^\infty(\mathbb{R}^n_z \times \mathbb{R}^n_\zeta)$ satisfying

$$|D^\alpha_z D^\beta_\zeta a(z, \zeta)| \le C_{\alpha\beta} \langle z\rangle^{l-|\alpha|} \langle \zeta\rangle^{m-|\beta|}$$

for any multiindices α, β, where $D_z = -i\partial_z$, $\langle z\rangle = \sqrt{1 + |z|^2}$, similarly for D_ζ and $\langle\zeta\rangle$, respectively, and $C_{\alpha\beta}$ is some positive constant only depending on α, β. We also require that a can be extended smoothly to $\overline{\mathbb{R}^n_z} \times \mathbb{R}^n_\zeta$ through the identification (3). Now the scattering pseudodifferential algebra $\Psi^{m,l}_{sc}(N)$ on a manifold with boundary N is defined by locally identifying with $\Psi^{m,l}_{sc}(\overline{\mathbb{R}^n})$.

Our scattering pseudodifferential operators will be applied to tensors, which are sections of vector bundles; it is necessary to introduce the (co)tangent bundle that is suitable for the scattering calculus. If we denote $r = x^{-1}$ the standard radial variable, under the polar coordinates there is a natural change of basis for $T\mathbb{R}^n$,

$$\partial_{z_1}, \dots, \partial_{z_n} \to \partial_r, r^{-1}\partial_{\theta_1}, \dots, r^{-1}\partial_{\theta_{n-1}},$$

where $\theta_1, \dots, \theta_{n-1}$ are local coordinates on the sphere. We consider the sphere as the level sets of x, and to be consistent with the notation of the paper we use $y_1, \dots, y_{n-1}$ as the local coordinates of the level sets; then it is straightforward to check that $\partial_r = -x^2\partial_x$ and $r^{-1}\partial_{\theta_j} = x\partial_{y_j}$. In particular these vector fields can be smoothly extended to $\overline{\mathbb{R}^n}$ and form a local basis for $T_{sc}\overline{\mathbb{R}^n}$ near $x = 0$. Consequently, the dual bundle, $T^*_{sc}\overline{\mathbb{R}^n}$, has a local basis $x^{-2}dx, x^{-1}dy_1, \dots, x^{-1}dy_{n-1}$, which are exactly the local bases $-dr, rd\theta_1, \dots, rd\theta_{n-1}$ under the standard polar coordinates. For a manifold with boundary N, the same local bases work for $T_{sc}N$ and $T^*_{sc}N$, at least near the boundary, with a boundary defining function x. Similarly, one can define $Sym^2T^*_{sc}N$, the bundle of symmetric scattering 2-tensors. This gives rise to the scattering metrics g_{sc}, as positive definite sections of $Sym^2T^*_{sc}N$, which has the form in local coordinates $g_{sc} = x^{-4}dx^2 + x^{-2}h$ with h a metric on the level sets of x.

The principal symbol of a scattering pseudodifferential operator in $\Psi^{m,l}_{sc}(\overline{\mathbb{R}^n})$ is the equivalent class of symbols $a \in S^{m,l}$, defined above, modulo $S^{m-1,l-1}$. The ellipticity in the scattering calculus, also called *full ellipticity*, is in the sense that the principal symbol $\tilde{a} \in S^{m,l}/S^{m-1,l-1}$ satisfies a lower bound, $|\tilde{a}(z,\zeta)| \geq C\langle z\rangle^l\langle\zeta\rangle^m$, for $|z| + |\zeta|$ sufficiently large, in contrast to the standard pseudodifferential algebra, where only $|\zeta|$ is required to be large. In terms of the boundary defining function x by the identification (3), this means that we need to verify two cases: (i) $|\zeta|$ is sufficiently large, which is similar to the standard ellipticity for pseudodifferential operators; (ii) x is sufficiently close to 0, while $|\zeta|$ is relatively small comparing with x^{-1}. Full ellipticity is needed for showing Fredholm properties of scattering pseudodifferential operators between appropriate Sobolev spaces. The principal symbol of an element in $\Psi^{m,l}_{sc}(N)$, which is living on $T^*_{sc}N$, is defined again by locally identifying it with $\tilde{a}$ above for the case of $\overline{\mathbb{R}^n}$.

In this paper we are working with tensor fields, which are sections of corresponding vector bundles. Under local trivializations (i.e., given local coordinates and bases), scattering pseudodifferential operators acting on sections of bundles are given by matrices of scalar scattering pseudodifferential operators. The principal symbols are also matrix valued in this case.

Now following the approach of [43, 42], let χ be a smooth nonnegative even function on $\mathbb{R}$ with compact support, which will be specified later. Given a function v defined on $\mathcal{M}_O$, or more specifically $\{(x,y,\lambda,\omega) : \lambda/x \in \operatorname{supp}\chi\}$, we define

$$\begin{aligned}
J_0v(x,y) &= x^{-2}\int v(x,y,\lambda,\omega)\chi(\lambda/x)\,d\lambda d\omega;\\
J_1v(x,y) &= \int v(x,y,\lambda,\omega)g_{sc}(\lambda\partial_x + \omega\partial_y)\chi(\lambda/x)\,d\lambda d\omega;\\
J_2v(x,y) &= x^2\int v(x,y,\lambda,\omega)g_{sc}(\lambda\partial_x + \omega\partial_y)\otimes g_{sc}(\lambda\partial_x + \omega\partial_y)\chi(\lambda/x)\,d\lambda d\omega,
\end{aligned}$$

where g_{sc} is a *scattering metric*; as discussed above locally it can be written as $g_{sc} = x^{-4}dx^2 + x^{-2}h$ with h the metric on the level sets of x. As a symmetric 2-tensor, g_{sc} sends vectors to 1-forms. Note that the images of J_i, $i = 0,1,2$, are functions, 1-forms and symmetric 2-tensors on U, respectively.

We denote $W := \left(\begin{smallmatrix} 1 & 0 \\ 0 & x^{-1}\end{smallmatrix}\right)$; for $F > 0$ we define

$$A_F[\beta,\varphi] = W^{-1}e^{-F/x}\begin{pmatrix} J_1 \\ J_0\end{pmatrix} I e^{F/x} W \begin{pmatrix}\beta\\ \varphi\end{pmatrix}; \tag{4}$$

$$B_F[h,\beta] = W^{-1}e^{-F/x}\begin{pmatrix} J_2 \\ J_1\end{pmatrix} I e^{F/x} W \begin{pmatrix} h\\ \beta\end{pmatrix}. \tag{5}$$

Comparing with the operators in [43, 42], we introduce an additional conjugacy $W^{-1}\cdot W$ in (4) and (5). The extra conjugacy helps to unify the microlocal properties of the components of A_F and B_F (as matrix operators), respectively; see section 3. This idea also appeared in [24, 46] for weighted X-ray transforms. Obviously when away from the boundary $x = 0$, A_F is a map between sections of $T^*U \times \underline{U}$, and B_F is a map between sections of $Sym^2T^*U \times T^*U$, where $\underline{U}$ is the trivial bundle. We will see in the next section that under proper coordinates the definition of A_F and B_F can be extended to include the boundary, i.e., one can replace T^*U by $T^*_{sc}\overline{U}$. More importantly, we will show that A_F, B_F are elliptic scattering pseudodifferential operators if one enforces some proper gauge conditions.

For the rest of this section, we study the gauge condition that suits the local magnetic ray transform. Let δ be the divergence on 1-forms, which is the adjoint of d relative to the scattering metric g_{sc}. Given a function φ and a 1-form β, define $\mathbf{d}\varphi = \left(\begin{smallmatrix} d \\ 0 \end{smallmatrix}\right)\varphi = \left(\begin{smallmatrix} d\varphi \\ 0 \end{smallmatrix}\right)$ and $\boldsymbol{\delta}[\beta, \varphi] = \left(\begin{smallmatrix} \delta & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \beta \\ \varphi \end{smallmatrix}\right) = \delta\beta$; we introduce the conjugated operators

$$\mathbf{d}_F = e^{-F/x}\mathbf{d}e^{F/x}$$

and $\boldsymbol{\delta}_F = e^{F/x}\boldsymbol{\delta}e^{-F/x}$ its adjoint with respect to the scattering metric g_{sc}. Note that by definition $\mathbf{d}_F = \left(\begin{smallmatrix} e^{-F/x}de^{F/x} \\ 0 \end{smallmatrix}\right)$, where the first component maps a function to a (scattering) 1-form; thus under the scattering basis $\frac{dx}{x^2}$, $\frac{dy}{x}$, we can further write $\mathbf{d}_F$ as

$$\begin{pmatrix} e^{-F/x}x^2\partial_x e^{F/x} \\ e^{-F/x}x\partial_y e^{F/x} \\ 0 \end{pmatrix}. \tag{6}$$

On the other hand, under the basis $\frac{dx}{x^2}$, $\frac{dy}{x}$, any $\zeta \in T^*_{sc}\overline{U}$ can be written as $\zeta = \xi\frac{dx}{x^2} + \eta\frac{dy}{x}$, or simply as $\zeta = (\xi, \eta)$.

Lemma 2.1. *The principal symbol of* $\mathbf{d}_F \in \mathrm{Diff}^{1,0}_{sc}(\underline{\overline{U}}, T^*_{sc}\overline{U} \times \{0\})$ *is* $(\xi + iF\ \eta\otimes\ 0)^T$*, while the principal symbol of* $\boldsymbol{\delta}_F \in \mathrm{Diff}^{1,0}_{sc}(T^*_{sc}\overline{U} \times \underline{\overline{U}}, \underline{\overline{U}})$ *is* $\left(\begin{smallmatrix} \xi - iF & \langle \eta, \cdot\rangle & 0 \end{smallmatrix}\right)$*, where the inner product* $\langle\cdot,\cdot\rangle$ *is induced by the dual metric* g_{sc}^{-1}*, i.e., given any* 1*-form* β*,* $\langle \eta, \beta\rangle = g_{sc}^{-1}(\eta, \beta) = g_{sc}^{ij}\eta_i\beta_j$.

Proof. It is not difficult to check (see also [42, Lemma 3.2]) that the principal symbol of $e^{-F/x}de^{F/x}$, under the basis $\frac{dx}{x^2}$, $\frac{dy}{x}$ for $T^*_{sc}\overline{U}$, is

$$\begin{pmatrix} \xi + iF \\ \eta\otimes \end{pmatrix}.$$

Thus in view of (6) the principal symbol of $\mathbf{d}_F$ is

$$\begin{pmatrix} \xi + iF \\ \eta\otimes \\ 0 \end{pmatrix}.$$

Since $\boldsymbol{\delta}_F$ is the adjoint of $\mathbf{d}_F$, its symbol is given by the adjoint of that of the latter with respect to g_{sc}, i.e.,

$$\begin{pmatrix} \xi - iF & \langle \eta, \cdot\rangle & 0 \end{pmatrix}. \qquad \square$$

Now let d^s be the symmetric differentiation acting on 1-forms, with the adjoint δ^s acting on symmetric 2-tensors with respect to g_{sc}. Define $\mathbf{d}^s = \left(\begin{smallmatrix} d^s & 0 \\ -E & d \end{smallmatrix}\right)$ and $\boldsymbol{\delta}^s = \left(\begin{smallmatrix} \delta^s & -E^* \\ 0 & \delta \end{smallmatrix}\right)$, where E is the Lorentz force. We introduce the operators $\mathbf{d}^s_F = e^{-F/x}W^{-1}\mathbf{d}^sWe^{F/x}$ and $\boldsymbol{\delta}^s_F = e^{F/x}W\boldsymbol{\delta}^sW^{-1}e^{-F/x}$; then similar to Lemma

2.1 we compute their principal symbols. Let the basis for the space of scattering 2-tensors be

$$\frac{dx}{x^2}\otimes\frac{dx}{x^2},\ \frac{dx}{x^2}\otimes\frac{dy}{x},\ \frac{dy}{x}\otimes\frac{dx}{x^2},\ \frac{dy}{x}\otimes\frac{dy}{x};$$

then the space of scattering symmetric 2-tensors, $Sym^2T^*_{sc}\overline{U}$, as a subspace, satisfies that the second and third components are the same under the above basis.

LEMMA 2.2. *The principal symbol of* $\boldsymbol{d}^s_F \in \mathrm{Diff}^{1,0}_{sc}(T^*_{sc}\overline{U}\times\underline{\overline{U}},\, Sym^2T^*_{sc}\overline{U}\times T^*_{sc}\overline{U})$ *is*

$$\begin{pmatrix} \xi+iF & 0 & 0\\ \frac12\eta\otimes & \frac12(\xi+iF) & 0\\ \frac12\eta\otimes & \frac12(\xi+iF) & 0\\ a & \eta\otimes_s & 0\\ 0 & 0 & \xi+iF\\ b & 0 & \eta\otimes \end{pmatrix},$$

while the principal symbol of $\boldsymbol{\delta}^s_F \in \mathrm{Diff}^{1,0}_{sc}(Sym^2T^*_{sc}\overline{U}\times T^*_{sc}\overline{U},\, T^*_{sc}\overline{U}\times\underline{\overline{U}})$ *is*

$$\begin{pmatrix} \xi-iF & \frac12\langle\eta,\cdot\rangle & \frac12\langle\eta,\cdot\rangle & \langle a,\cdot\rangle & 0 & \langle b,\cdot\rangle\\ 0 & \frac12(\xi-iF) & \frac12(\xi-iF) & \langle\eta,\cdot\rangle^s & 0 & 0\\ 0 & 0 & 0 & 0 & \xi-iF & \langle\eta,\cdot\rangle \end{pmatrix}.$$

Here a is a symmetric 2-tensor and b is a 1-form, both independent of F. The inner product $\langle a,\cdot\rangle$ is again with respect to the dual metric g^{-1}_{sc} such that for any symmetric 2-tensor f, $\langle a,f\rangle = g^{ij}_{sc}g^{kl}_{sc}a_{ik}f_{jl}$. The symmetrization of $\langle\eta,\cdot\rangle$ acting on symmetric 2-tensors f, $\langle\eta,f\rangle^s$, is a 1-form whose kth component is given by $\langle\eta,f\rangle^s_k = \frac12 g^{ij}_{sc}\eta_i(f_{jk}+f_{kj}) = g^{ij}_{sc}\eta_i f_{jk}$.

Proof. By definition

$$\mathbf{d}^s_F = \begin{pmatrix} e^{-F/x}d^se^{F/x} & 0\\ -xE & xe^{-F/x}de^{F/x}x^{-1} \end{pmatrix},$$

where $e^{-F/x}d^se^{F/x}$ maps 1-forms to symmetric 2-tensors. Similar to (6), we may write $e^{-F/x}d^se^{F/x}$ in the matrix form; by [42, Lemma 3.2] its symbol is

$$\begin{pmatrix} \xi+iF & 0\\ \frac12\eta\otimes & \frac12(\xi+iF)\\ \frac12\eta\otimes & \frac12(\xi+iF)\\ a & \eta\otimes_s \end{pmatrix},$$

where the second and third rows are the same due to the symmetrization. Here a is a suitable symmetric 2-tensor, which comes from the nontrivial contribution of the zeroth order term of the operator via the entry corresponding to $\frac{dy}{x}\otimes\frac{dy}{x}\otimes x^2\partial_x = dy\otimes dy\otimes\partial_x$; see [42, section 2] for more details. On the other hand, the principal symbol of $xe^{-F/x}de^{F/x}x^{-1}$ ($xe^{-F/x}x^2D_xe^{F/x}x^{-1} = x^2D_x + i(F+x)$; however, the term ix is of lower order) is

$$\begin{pmatrix} \xi+iF\\ \eta\otimes \end{pmatrix}.$$

Notice that in the scattering setting, the operator $-xE$ has the form

$$\begin{pmatrix} -xE^x_x & -x^2E^y_x\\ -E^x_y & -xE^y_y \end{pmatrix}$$

(note that E is a $(1,1)$-tensor, locally $E = E^x_x dx \otimes \partial_x + E^y_x dx \otimes \partial_y + E^x_y dy \otimes \partial_x + E^y_y dy \otimes \partial_y = E^x_x \frac{dx}{x^2} \otimes x^2\partial_x + xE^y_x \frac{dx}{x^2} \otimes x\partial_y + x^{-1}E^x_y \frac{dy}{x} \otimes x^2\partial_x + E^y_y \frac{dy}{x} \otimes x\partial_y$), and thus there is a nontrivial contribution from the term $-E^x_y$ at the boundary $x = 0$, denoted by b. Then we combine above arguments to give the principal symbol of $\mathbf{d}^s_F$. Moreover, the principal symbol of $\boldsymbol{\delta}^s_F$ is the adjoint of the principal symbol of $\mathbf{d}^s_F$ w.r.t. g_{sc}. $\square$

Now we introduce the Witten-type solenoidal gauge condition we will use in the paper in the spirit of [42]. The gauge for the operator A_F is

$$\boldsymbol{\delta}_F e^{-F/x} W^{-1}[\beta, \varphi] = \boldsymbol{\delta}_F [\beta, \varphi]_F = 0,$$

while the gauge for the operator B_F is

$$\boldsymbol{\delta}^s_F e^{-F/x} W^{-1}[h, \beta] = \boldsymbol{\delta}^s_F [h, \beta]_F = 0.$$

3. Ellipticity up to the gauge.

3.1. Blow-up coordinates. Before the proofs of main ellipticity statements, we introduce local coordinates that are suitable to the analysis of the microlocal properties of the operators A_F, B_F up to the artificial boundary $x = 0$. The introduction follows closely the corresponding discussion from [43, 42].

It is well known (see, e.g., [5]) that the maps (notice that $\widetilde{M}$ is complete)

$$\Gamma_+ : S\widetilde{M} \times [0, \infty) \to [\widetilde{M} \times \widetilde{M}; \mathrm{diag}],\ \Gamma_+(z, v, t) = (z, \gamma_{z,v}(t))$$

and

$$\Gamma_- : S\widetilde{M} \times (-\infty, 0] \to [\widetilde{M} \times \widetilde{M}; \mathrm{diag}],\ \Gamma_-(z, v, t) = (z, \gamma_{z,v}(t))$$

are two diffeomorphisms near $S\widetilde{M} \times \{0\}$. Here, by denoting $z' := \gamma_{z,v}(t)$, $[\widetilde{M} \times \widetilde{M}; \mathrm{diag}]$ is the *blow-up* of $\widetilde{M}$ at the diagonal $z = z'$, which essentially means the introduction of polar coordinates around the diagonal, so that $\Gamma_\pm(z, v, 0) \neq \Gamma_\pm(z, v', 0)$ if $v \neq v'$. In particular, for $t \geq 0$ sufficiently small, the local (polar) coordinates

$$\left(z, |z' - z|, \frac{z' - z}{|z' - z|}\right)$$

are valid on the image of $\Gamma_\pm$, where $|\cdot|$ is the Euclidean norm.

Recalling the local coordinates (x, y) near the strictly convex boundary point p, we write $z = (x, y)$ and $z' = (x', y')$; then similar to [43], it's convenient to use

$$\left(x, y, |y' - y|, \frac{x' - x}{|y' - y|}, \frac{y' - y}{|y' - y|}\right) \tag{7}$$

as the local coordinates on the images of $\Gamma_\pm$ for $t \geq 0$ small, when $|y' - y|$ is large relative to $|x' - x|$, i.e., in our region of interest.

On the other hand, the analysis is carried out on the region $x \geq 0$, which has the boundary $x = 0$. Notice that the integrand f of the ray transform If is initially defined on M, and the support of f and the boundary $x = 0$ are not necessarily disjoint; thus the standard pseudodifferential calculus does not work. This is where the scattering calculus comes in, and we recall the scattering coordinates introduced in [43],

$$X = \frac{x' - x}{x^2},\ Y = \frac{y' - y}{x},$$

under which (7) becomes

$$\left(x, y, x|Y|, \frac{xX}{|Y|}, \hat{Y}\right) \tag{8}$$

with $\hat{Y} = \frac{Y}{|Y|}$.

We denote $(\gamma(t), \dot{\gamma}(t)) = (x(t), y(t), k(t)\lambda(t), k(t)\omega(t))$ in short by $(x', y', k\lambda', k\omega')$; the multiple k, which is a function on t, is added to make $\|k(\lambda'\partial_x + \omega'\partial_y)\|_g = 1$. For the main proof of this section, it is convenient to make a change of parameters of the curve so that if s is the new parameter, $\dot{\gamma}(s) = \lambda'\partial_x + \omega'\partial_y$. As a result, smooth positive weights are introduced to the ray transform as follows:

$$If(x, y, \lambda, \omega) = \int f(\gamma(t), \dot{\gamma}(t))\, dt = \int f(\gamma(s), k(s)\dot{\gamma}(s)) \frac{1}{k(s)}\, ds.$$

However, as one can see from the analysis of the ellipticity (up to gauge) of A_F and B_F below, the introduction of a smooth positive weight will not affect the argument; see also [41, 24]. Moreover, since the key part of the microlocal analysis is at $x = 0$, where by the cut-off function χ, only curves $\gamma_{0,y,\lambda,\omega}$ with $\lambda = 0$ will contribute to the operators A_F and B_F. Then under the trivialization of the metric at one point (as the symbol calculation of pseudodifferential operators is pointwise), the vector $w\partial_y$ has unit length, i.e., $k = 1$ at that point. For the sake of simplicity, we totally drop the multiple k from now on and work as if the curve is parameterized by the nonunit speed one, but still denoted by t.

By the diffeomorphisms $\Gamma_\pm$ near $t = 0$,

$$\begin{gathered} t \circ \Gamma_\pm^{-1} = \pm|y' - y| + \mathcal{O}(|y' - y|^2), \\ \lambda \circ \Gamma_\pm^{-1} = \pm\frac{x' - x}{|y' - y|} + \mathcal{O}(|y' - y|), \quad \omega \circ \Gamma_\pm^{-1} = \pm\frac{y' - y}{|y' - y|} + \mathcal{O}(|y' - y|). \end{gathered} \tag{9}$$

The coefficients in the remainder terms are all smooth under the coordinates (8). Then applying the scattering coordinates,

$$(\Gamma_\pm^{-1})^* dt\, d\lambda\, d\omega = x^2|Y|^{1-n} J_\pm(x, y, X, Y)\, dX dY$$

with the smooth positive density function J, $J_\pm|_{x=0} = 1$.

Now given a curve $\gamma = \gamma_{x,y,\lambda,\omega}$, we have near $t = 0$

$$\begin{gathered} x' = x + \lambda t + \alpha t^2 + \mathcal{O}(t^3), \quad y' = y + \omega t + \mathcal{O}(t^2), \\ \lambda' = \lambda + 2\alpha t + \mathcal{O}(t^2), \quad \omega' = \omega + \mathcal{O}(t). \end{gathered} \tag{10}$$

Recall that $\alpha = \alpha(x, y, \lambda, \omega, 0)$ defined at the beginning of section 2 is proportional to the second derivative of x' with respect to t. Notice that unlike the geodesic case, α is no longer a quadratic form. From now on we work in the coordinates (8) and denote $\alpha(x, y, 0, \pm\frac{xX}{|Y|}, \pm\hat{Y})$ by $\alpha_\pm$ and $\frac{X - \alpha_\pm|Y|^2}{|Y|}$ by $S_\pm$, so $\frac{X + \alpha_\pm|Y|^2}{|Y|} = S_\pm + 2\alpha_\pm|Y|$. Then using (10) one can show that (see also [43, 42]) under the scattering tangent and cotangent bases

$$\begin{aligned} &g_{sc}\Big((\lambda \circ \Gamma_\pm^{-1})\partial_x + (\omega \circ \Gamma_\pm^{-1})\partial_y\Big) \\ &\quad = x^{-1}\bigg(\Big(\pm S_\pm + x\tilde{\Lambda}_\pm\Big)\frac{dx}{x^2} + \Big(\pm\hat{Y} + x|Y|\tilde{\Omega}_\pm\Big)\frac{h(\partial_y)}{x}\bigg) \end{aligned} \tag{11}$$

and

$$\begin{aligned}&(\lambda'\circ\Gamma_\pm^{-1})\partial_x+(\omega'\circ\Gamma_\pm^{-1})\partial_y\\ &=x^{-1}\Big(\Big(\pm(S_\pm+2\alpha_\pm|Y|)+x|Y|^2\tilde\Lambda'_\pm\Big)x^2\partial_x+\Big(\pm\hat Y+x|Y|\tilde\Omega'_\pm\Big)x\partial_y\Big).\end{aligned}\tag{12}$$

Here $\tilde\Lambda_\pm$, $\tilde\Lambda'_\pm$, $\tilde\Omega_\pm$, and $\tilde\Omega'_\pm$ are smooth in terms of coordinates (8).

3.2. Proofs of the ellipticity of A_F and B_F up to gauge.

To show that A_F, B_F are fully elliptic up to some gauge, we analyze the behavior of its principal symbol defined on $(z,\zeta)=(x,y,\xi,\eta)\in T^*_{sc}\overline{U}$. This analysis is pointwise; we can assume that at one point $z=(x,y)$ the metric g_{sc} has the trivial form $g_{sc}=x^{-4}dx^2+x^{-2}dy^2$, while the fiber $T^*_{sc,z}\overline{U}$ is equivalent to $\mathbb{R}^n$. As mentioned in the introduction of the scattering calculus in section 2, the analysis includes two cases: (i) the first case is when $|\zeta|\to\infty$, i.e., near the fiber infinity of $T^*_{sc,z}\overline{U}$ (see Lemmas 3.1 and 3.4); (ii) the second case is at finite points of the fiber $T^*_{sc,z}\overline{U}$, in particular near $\zeta=0$ (see Lemmas 3.2 and 3.5). Roughly speaking, to analyze the principal symbol, we compute the Fourier transform of the Schwartz kernel with respect to the (X,Y)-variables. We will show that the exponential weights and properly chosen cut-off function χ can help us eliminate possible issues of the principal symbol near the zero section of $T^*_{sc}\overline{U}$. As we will see that the Schwartz kernels of A_F, B_F are smooth in (x,y) down to $x=0$, it suffices to investigate the principal symbol at $x=0$. Once we show the full ellipticity at $x=0$, by smoothness on x, the same result holds in a neighborhood of $\overline{O}=\overline{U}\cap M$ assuming that $c>0$ is small enough.

3.2.1. Ellipticity of A_F.

According to the definition (4) and the expressions (11), (12), near $x=0$ the Schwartz kernel of A_F can be written as

$$K_A(x,y,X,Y)=e^{-\frac{FX}{1+xX}}|Y|^{1-n}\left(K_A^0\left(y,|Y|,\frac{X}{|Y|},\hat Y\right)+x\tilde K\left(x,y,|Y|,\frac{X}{|Y|},\hat Y\right)\right)\tag{13}$$

with smooth K_A^0 and $\tilde K$ (on their variables). Concretely, from (11) it is not difficult to see that $\lambda\circ\Gamma_\pm^{-1}/x=\pm S_\pm+x\tilde\Lambda_\pm=\pm\frac{X-\alpha_\pm|Y|^2}{|Y|}+x\tilde\Lambda_\pm$, so by letting $x=0$ we have

$$K_A^0(y,|Y|,\frac{X}{|Y|},\hat Y)=\chi(S_+)\begin{pmatrix}A_{11}^+ & A_{10}^+\\ A_{01}^+ & A_{00}^+\end{pmatrix}+\chi(-S_-)\begin{pmatrix}A_{11}^- & -A_{10}^-\\ -A_{01}^- & A_{00}^-\end{pmatrix},$$

where

$$\begin{aligned}A_{11}^\pm&=\Big(S_\pm\frac{dx}{x^2}+\hat Y\frac{dy}{x}\Big)\Big((S_\pm+2\alpha_\pm|Y|)(x^2\partial_x)+\hat Y(x\partial_y)\Big);\\ A_{10}^\pm&=S_\pm\frac{dx}{x^2}+\hat Y\frac{dy}{x};\\ A_{01}^\pm&=(S_\pm+2\alpha_\pm|Y|)(x^2\partial_x)+\hat Y(x\partial_y);\\ A_{00}^\pm&=1.\end{aligned}$$

When $x=0$, $\alpha_\pm$ is simply $\alpha(0,y,0,0,\pm\hat Y)$. Since χ is an even function, it is easy to see that K_A^0 is even in (X,Y). Now it is easy to see that K_A is smooth in (x,y) down to $x=0$, with values in functions Schwartz in (X,Y) (due to the exponential weight in (13)) for $(X,Y)\neq 0$, and is conormal to the diagonal $(X,Y)=0$. This shows that A_F is a scattering pseudodifferential operator on $\overline{U}$ of order $(-1,0)$, i.e., $A_F\in\Psi_{sc}^{-1,0}(\overline{U})$; see also [42, Proposition 3.1].

LEMMA 3.1. *For any $F > 0$, A_F is elliptic near the fiber infinity of $T^*_{sc}\overline{U}$ when restricted on the kernel of the standard principal symbol of $\boldsymbol{\delta}_F$.*

Proof. The analysis of the principal symbol of A_F near the fiber infinity is quite similar to the standard microlocal analysis of a pseudodifferential operator, i.e., the analysis of the conormal singularity of the principal symbol of A_F at the diagonal, $X = Y = 0$; see, e.g., [42, Lemma 3.4].

The restriction of the Schwartz kernel K_A at $x = 0$ is

$$K_A(0, y, X, Y) = e^{-FX}|Y|^{1-n}K_A^0.$$

It is more convenient to write the matrices in K_A^0 as cross products of vectors (for the sake of simplicity, we drop the $+, -$ signs); we treat A_{11} as a 2×2 matrix and A_{10} and A_{01} as vectors, and then

$$\begin{pmatrix} A_{11} & A_{10} \\ A_{01} & A_{00} \end{pmatrix} = \begin{pmatrix} S\frac{dx}{x^2} \\ \hat{Y}\frac{dy}{x} \\ 1 \end{pmatrix} \begin{pmatrix} (S + 2\alpha|Y|)(x^2\partial_x) & \hat{Y}(x\partial_y) & 1 \end{pmatrix}.$$

We may drop the bases and simplify it further as

$$\begin{pmatrix} S \\ \hat{Y} \\ 1 \end{pmatrix} \begin{pmatrix} S + 2\alpha|Y| & \hat{Y} & 1 \end{pmatrix} = \begin{pmatrix} S(S + 2\alpha|Y|) & S\hat{Y} & S \\ \hat{Y}(S + 2\alpha|Y|) & \hat{Y} \times \hat{Y} & \hat{Y} \\ S + 2\alpha|Y| & \hat{Y} & 1 \end{pmatrix}.$$

Under our settings, we need to evaluate the integration of K_A at $x = 0$ along the orthogonal equatorial sphere corresponding to $\zeta = (\xi, \eta)$, i.e., those $(\tilde{S}, \hat{Y})$ with $\xi\tilde{S} + \eta \cdot \hat{Y} = 0$. Here $\tilde{S}$ denotes $X/|Y|$. Notice that for this case the extra vanishing factor $|Y| = 0$ in χ and A_{ij}, and the exponential conjugacy (as $X = 0$) can be dropped. So by the evenness of K_A^0 the standard principal symbol of A_F is essentially of the following form, for some positive constant C:

$$\sigma_p(A_F)(0, y, \xi, \eta) = C|\zeta|^{-1} \int_{\zeta^\perp \cap (\mathbb{R} \times \mathbb{S}^{n-2})} \chi(\tilde{S}) \begin{pmatrix} \tilde{S} \\ \hat{Y} \\ 1 \end{pmatrix} \begin{pmatrix} \tilde{S} & \hat{Y} & 1 \end{pmatrix} d\tilde{S}d\hat{Y}.$$

Given any nonzero pair $[\beta, \varphi]$, $\beta = (\beta^0, \beta')$, in the kernel of the standard principal symbol of $\boldsymbol{\delta}_F$, i.e., $\xi\beta^0 + \eta \cdot \beta' = 0$,

$$(\sigma_p(A_F)[\beta, \varphi], [\beta, \varphi]) = C|\zeta|^{-1} \int_{\zeta^\perp \cap (\mathbb{R} \times \mathbb{S}^{n-2})} \chi(\tilde{S}) \left|\tilde{S}\beta^0 + \hat{Y} \cdot \beta' + \varphi\right|^2 d\tilde{S}d\hat{Y}.$$

Now to prove the ellipticity of A_F, it suffices to show that there is $(\tilde{S}, \hat{Y}) \in \zeta^\perp \cap (\mathbb{R} \times \mathbb{S}^{n-2})$ such that $\chi(\tilde{S}) > 0$ and $\tilde{S}\beta^0 + \hat{Y} \cdot \beta' + \varphi \neq 0$. We prove by contradiction. Assume that for any $(\tilde{S}, \hat{Y}) \in \zeta^\perp \cap (\mathbb{R} \times \mathbb{S}^{n-2})$ with $\chi(\tilde{S}) > 0$, $\tilde{S}\beta^0 + \hat{Y} \cdot \beta' + \varphi = 0$. Notice that if $\chi(\tilde{S}) > 0$, then $\chi(-\tilde{S}) > 0$, thus $-\tilde{S}\beta^0 - \hat{Y} \cdot \beta' + \varphi = 0$, which implies that $\tilde{S}\beta^0 + \hat{Y} \cdot \beta' = 0$ and $\varphi = 0$.

On the other hand, we can find generic $n - 1$ elements from the set $\{(\tilde{S}, \hat{Y}) : \xi\tilde{S} + \eta \cdot \hat{Y} = 0,\ \chi(\tilde{S}) > 0\}$ (notice that here we need the dimension n be at least 3, since if $n = 2$ the set might be empty) with $\tilde{S}\beta^0 + \hat{Y} \cdot \beta' = 0$; by linear algebra, this implies that $\beta = 0$ (since $\xi\beta^0 + \eta \cdot \beta' = 0$), which is a contradiction. This completes the proof. ∎

LEMMA 3.2. *For any $F > 0$, there exists $\chi = \chi_F \in C_c^\infty(\mathbb{R})$ such that A_F is elliptic at finite points of $T^*_{sc}\overline{U}$ when restricted on the kernel of the scattering principal symbol of $\boldsymbol{\delta}_F$.*

Proof. In order to find a suitable χ to make A_F elliptic acting on the kernel of $\sigma_{sc}(\boldsymbol{\delta}_F)$, we follow the strategy of [43], namely, we first do the calculation for a Gaussian function $\chi(s) = e^{-s^2/(2F^{-1}\alpha)}$ with $F > 0$, where α is again related to the second derivative of x with respect to t. Here χ does not have compact support, thus an approximation argument will be necessary at the end. The calculation of the Fourier transform of K_A is similar to [43, Lemma 4.1] and [42, Lemma 3.5]. For the sake of completeness, in the following we give the main steps.

Denoting $F^{-1}\alpha_\pm$ by $\mu_\pm$, the X-Fourier transform of K_A, $\mathcal{F}_X K_A(0, y, \xi, Y)$, with χ chosen as above, is a nonzero multiple of

$$
\begin{aligned}
|Y|^{2-n} \Bigg\{ & \sqrt{\mu_+} e^{i\alpha_+(\xi+iF)|Y|^2} \begin{pmatrix} D_\nu(D_\nu - 2\alpha_+|Y|) & -D_\nu\hat{Y} & -D_\nu \\ \hat{Y}(-D_\nu + 2\alpha_+|Y|) & \hat{Y}\times\hat{Y} & \hat{Y} \\ -D_\nu + 2\alpha_+|Y| & \hat{Y} & 1 \end{pmatrix} e^{-\mu_+(\xi+iF)^2|Y|^2/2} \\
& + \sqrt{\mu_-} e^{i\alpha_-(\xi+iF)|Y|^2} \begin{pmatrix} D_\nu(D_\nu - 2\alpha_-|Y|) & -D_\nu\hat{Y} & D_\nu \\ \hat{Y}(-D_\nu + 2\alpha_-|Y|) & \hat{Y}\times\hat{Y} & -\hat{Y} \\ D_\nu - 2\alpha_-|Y| & -\hat{Y} & 1 \end{pmatrix} e^{-\mu_-(\xi+iF)^2|Y|^2/2} \Bigg\},
\end{aligned}
$$

where D_ν is the differentiation with respect to the variable of $\hat{\chi}$, i.e., $-(\xi + iF)|Y|$. Taking the derivatives, by polar coordinates the Y-Fourier transform takes the form

$$
\begin{aligned}
&\int_{\mathbb{S}^{n-2}} \int_0^\infty e^{i|Y|\hat{Y}\cdot\eta} \\
&\times \Bigg\{ \sqrt{\mu_+} \begin{pmatrix} i\mu_+(\xi+iF)i\mu_+(\xi-iF)|Y|^2 + \mu_+ & i\mu_+(\xi+iF)|Y|\hat{Y} & i\mu_+(\xi+iF)|Y| \\ i\mu_+(\xi-iF)\hat{Y}|Y| & \hat{Y}\times\hat{Y} & \hat{Y} \\ i\mu_+(\xi-iF)|Y| & \hat{Y} & 1 \end{pmatrix} \\
&\times e^{-\mu_+(\xi^2+F^2)|Y|^2/2} \\
&+ \sqrt{\mu_-} \begin{pmatrix} i\mu_-(\xi+iF)i\mu_-(\xi-iF)|Y|^2 + \mu_- & i\mu_-(\xi+iF)|Y|\hat{Y} & -i\mu_-(\xi+iF)|Y| \\ i\mu_-(\xi-iF)\hat{Y}|Y| & \hat{Y}\times\hat{Y} & -\hat{Y} \\ -i\mu_-(\xi-iF)|Y| & -\hat{Y} & 1 \end{pmatrix} \\
&\times e^{-\mu_-(\xi^2+F^2)|Y|^2/2} \Bigg\} \, d\hat{Y} \, d|Y|.
\end{aligned}
$$

Since the integrand is invariant under the changes from $|Y|$ to $-|Y|$, $\hat{Y}$ to $-\hat{Y}$ (thanks to the evenness from K_A), we have that the integral above equals

$$
\begin{aligned}
&\int_{\mathbb{S}^{n-2}} \int_{\mathbb{R}} e^{i(\hat{Y}\cdot\eta)t} \sqrt{\mu_+} \\
&\quad \begin{pmatrix} i\mu_+(\xi+iF)i\mu_+(\xi-iF)t^2 + \mu_+ & i\mu_+(\xi+iF)t\hat{Y} & i\mu_+(\xi+iF)t \\ i\mu_+(\xi-iF)\hat{Y}t & \hat{Y}\times\hat{Y} & \hat{Y} \\ i\mu_+(\xi-iF)t & \hat{Y} & 1 \end{pmatrix} \\
&\quad \times e^{-\mu_+(\xi^2+F^2)t^2/2} \, d\hat{Y} \, dt,
\end{aligned}
$$

which gives a constant multiple of

$$\begin{aligned}&\int_{\mathbb{S}^{n-2}} \frac{1}{\sqrt{\xi^2+F^2}}\\ &\quad\begin{pmatrix} i\mu_+(\xi+iF)i\mu_+(\xi-iF)D^2_{\hat{Y}\cdot\eta}+\mu_+ & i\mu_+(\xi+iF)\hat{Y}D_{\hat{Y}\cdot\eta} & i\mu_+(\xi+iF)D_{\hat{Y}\cdot\eta}\\ i\mu_+(\xi-iF)\hat{Y}D_{\hat{Y}\cdot\eta} & \hat{Y}\times\hat{Y} & \hat{Y}\\ i\mu_+(\xi-iF)D_{\hat{Y}\cdot\eta} & \hat{Y} & 1 \end{pmatrix}\\ &\quad\times e^{-|\hat{Y}\cdot\eta|^2/2\mu_+(\xi^2+F^2)}\,d\hat{Y}.\end{aligned}$$

Finally, we apply the derivative $D_{\hat{Y}\cdot\eta}$ to the exponential term to get

$$\int_{\mathbb{S}^{n-2}} \frac{1}{\sqrt{\xi^2+F^2}} \begin{pmatrix} A_{xx} & A_{xy} & A_{x0}\\ A_{yx} & A_{yy} & A_{y0}\\ A_{0x} & A_{0y} & A_{00}\end{pmatrix} e^{-|\hat{Y}\cdot\eta|^2/2\mu(\xi^2+F^2)}\,d\hat{Y},$$

where

$$\begin{aligned}
A_{xx} &= (\xi+iF)(\xi-iF)\frac{|\hat{Y}\cdot\eta|^2}{(\xi^2+F^2)^2},\\
A_{xy} &= -(\xi+iF)\frac{\hat{Y}\cdot\eta}{(\xi^2+F^2)}\hat{Y},\\
A_{x0} &= -(\xi+iF)\frac{\hat{Y}\cdot\eta}{(\xi^2+F^2)},\\
A_{yx} &= -\hat{Y}(\xi-iF)\frac{\hat{Y}\cdot\eta}{(\xi^2+F^2)},\\
A_{yy} &= \hat{Y}\times\hat{Y},\\
A_{y0} &= \hat{Y},\\
A_{0x} &= -(\xi-iF)\frac{\hat{Y}\cdot\eta}{(\xi^2+F^2)},\\
A_{0y} &= \hat{Y},\\
A_{00} &= 1.
\end{aligned}$$

Therefore, the scattering principal symbol of A_F is

$$\begin{aligned}&\sigma_{sc}(A_F)(0,y,\xi,\eta)\\ &\quad= C\int_{\mathbb{S}^{n-2}} \frac{1}{\sqrt{\xi^2+F^2}} \begin{pmatrix} -\frac{(\xi+iF)\hat{Y}\cdot\eta}{\xi^2+F^2}\\ \hat{Y}\\ 1\end{pmatrix} \begin{pmatrix} -\frac{(\xi-iF)\hat{Y}\cdot\eta}{\xi^2+F^2} & \hat{Y} & 1\end{pmatrix} e^{-|\hat{Y}\cdot\eta|^2/2\mu(\xi^2+F^2)}\,d\hat{Y}\end{aligned}$$

for some positive constant C.

Given any nonzero pair $[\beta,\varphi]$, $\beta=(\beta^0,\beta')$, in the kernel of the scattering principal symbol of $\boldsymbol{\delta}_F$, i.e., $(\xi-iF)\beta^0+\eta\cdot\beta'=0$ by Lemma 2.1,

$$\begin{aligned}&(\sigma_{sc}(A_F)[\beta,\varphi],[\beta,\varphi])\\ &\quad=\frac{C}{\sqrt{\xi^2+F^2}}\int_{\mathbb{S}^{n-2}}\left|-\frac{(\xi-iF)\hat{Y}\cdot\eta}{\xi^2+F^2}\beta^0+\hat{Y}\cdot\beta'+\varphi\right|^2 e^{-|\hat{Y}\cdot\eta|^2/2\mu(\xi^2+F^2)}\,d\hat{Y}.\end{aligned}$$

To prove the ellipticity, it suffices to show that there is $\hat{Y}$ such that $-\frac{(\xi-iF)\hat{Y}\cdot\eta}{\xi^2+F^2}\beta^0+\hat{Y}\cdot\beta'+\varphi\neq 0$. Again, we prove by contradiction. Assume that for any $\hat{Y}\in\mathbb{S}^{n-2}$, $-\frac{(\xi-iF)\hat{Y}\cdot\eta}{\xi^2+F^2}\beta^0+\hat{Y}\cdot\beta'+\varphi$ always vanishes. Then $\frac{(\xi-iF)\hat{Y}\cdot\eta}{\xi^2+F^2}\beta^0-\hat{Y}\cdot\beta'+\varphi=0$ too, which implies that $\varphi=0$ and $-\frac{(\xi-iF)\hat{Y}\cdot\eta}{\xi^2+F^2}\beta^0+\hat{Y}\cdot\beta'=0$ for all $\hat{Y}$.

On the other hand, since $(\xi-iF)\beta^0+\eta\cdot\beta'=0$,

$$-\frac{(\xi-iF)\hat{Y}\cdot\eta}{\xi^2+F^2}\beta^0+\hat{Y}\cdot\beta'=\frac{1}{\xi^2+F^2}(\eta\cdot\beta')(\hat{Y}\cdot\eta)+\hat{Y}\cdot\beta'=0$$

for all $\hat{Y}$. It is not difficult to see that this implies that $\beta'=0$, so $\beta^0=-(\xi-iF)^{-1}\eta\cdot\beta'=0$ too. Thus we reach a contradiction as $[\beta,\varphi]$ is a nonzero pair, and this establishes the ellipticity of A_F for Gaussian type χ.

Finally we pick a sequence $\chi_n\in C_c^\infty(\mathbb{R})$ which converges to the Gaussian in Schwartz functions; then the Fourier transforms $\hat{\chi_n}$ converge to $\hat{\chi}$. One concludes that for some large enough n, if we use χ_n to define the operator A_F, then its principal symbol is still elliptic as desired. ∎

Combining Lemmas 3.1 and 3.2 we get the following ellipticity result

PROPOSITION 3.3. *For any $F>0$, given Ω a neighborhood of $\overline{O}$ in $\overline{U}$, there exist $\chi\in C_c^\infty(\mathbb{R})$ and $N\in\Psi_{sc}^{-3,0}(\overline{U};\underline{\overline{U}},\underline{\overline{U}})$ such that $A_F+\mathbf{d}_F N\boldsymbol{\delta}_F\in\Psi_{sc}^{-1,0}(\overline{U};T^*_{sc}\overline{U}\times\underline{\overline{U}},T^*_{sc}\overline{U}\times\underline{\overline{U}})$ is elliptic in Ω.*

3.2.2. Ellipticity of B_F.

The analysis of B_F is similar to the case of A_F but is more complicated. By an argument similar to the one for A_F, it is not difficult to check that B_F is a scattering pseudodifferential operator of order $(-1,0)$ too. Next we show that B_F is elliptic up to the gauge $\boldsymbol{\delta}^s_F$.

According to the definition (5) and the expressions (11), (12), the Schwartz kernel of B_F at $x=0$ is

$$K_B(0,y,X,Y)=e^{-FX}|Y|^{1-n}\left\{\chi(S_+)\begin{pmatrix}B_{22}^+ & B_{21}^+\\ B_{12}^+ & B_{11}^+\end{pmatrix}+\chi(-S_-)\begin{pmatrix}B_{22}^- & -B_{21}^-\\ -B_{12}^- & B_{11}^-\end{pmatrix}\right\},$$

where

$$\begin{aligned}
B_{22}^{\pm}=&\left(\left(S_\pm\frac{dx}{x^2}+\hat{Y}\frac{dy}{x}\right)\otimes\left(S_\pm\frac{dx}{x^2}+\hat{Y}\frac{dy}{x}\right)\right)\\
&\qquad\left(\left((S_\pm+2\alpha_\pm|Y|)(x^2\partial_x)+\hat{Y}(x\partial_y)\right)\otimes\left((S_\pm+2\alpha_\pm|Y|)(x^2\partial_x)+\hat{Y}(x\partial_y)\right)\right);\\
B_{21}^{\pm}=&\left(\left(S_\pm\frac{dx}{x^2}+\hat{Y}\frac{dy}{x}\right)\otimes\left(S_\pm\frac{dx}{x^2}+\hat{Y}\frac{dy}{x}\right)\right)\left((S_\pm+2\alpha_\pm|Y|)(x^2\partial_x)+\hat{Y}(x\partial_y)\right);\\
B_{12}^{\pm}=&\left(S_\pm\frac{dx}{x^2}+\hat{Y}\frac{dy}{x}\right)\\
&\qquad\left(\left((S_\pm+2\alpha_\pm|Y|)(x^2\partial_x)+\hat{Y}(x\partial_y)\right)\otimes\left((S_\pm+2\alpha_\pm|Y|)(x^2\partial_x)+\hat{Y}(x\partial_y)\right)\right);\\
B_{11}^{\pm}=&\left(S_\pm\frac{dx}{x^2}+\hat{Y}\frac{dy}{x}\right)\left((S_\pm+2\alpha_\pm|Y|)(x^2\partial_x)+\hat{Y}(x\partial_y)\right)=A_{11}^{\pm}.
\end{aligned}$$

Again we write the matrices in the Schwartz kernel as cross products of vectors, dropping the $+,-$ signs, to get

$$\begin{pmatrix} S^2 \\ S\hat{Y}_1 \\ S\hat{Y}_2 \\ \hat{Y}_1\otimes\hat{Y}_2 \\ S \\ \hat{Y} \end{pmatrix}\begin{pmatrix}(S+2\alpha|Y|)^2 & (S+2\alpha|Y|)\hat{Y}_1 & (S+2\alpha|Y|)\hat{Y}_2 & \hat{Y}_1\otimes\hat{Y}_2 & S+2\alpha|Y| & \hat{Y}\end{pmatrix}.$$

Here subscripts 1 and 2 of $\hat{Y}$ indicate the position of the factors of a 2-tensor it is acting on.

LEMMA 3.4. *For any $F>0$, B_F is elliptic near the fiber infinity of $T^*_{sc}\overline{U}$ when restricted on the kernel of the standard principal symbol of $\boldsymbol{\delta}^s_F$.*

Proof. Similar to the argument in Lemma 3.1, the standard principal symbol of B_F at $\zeta=(\xi,\eta)$ is essentially the following:

$$|\zeta|^{-1}\int_{\zeta^\perp\cap(\mathbb{R}\times\mathbb{S}^{n-2})}\chi(\tilde{S})\begin{pmatrix} \tilde{S}^2 \\ \tilde{S}\hat{Y}_1 \\ \tilde{S}\hat{Y}_2 \\ \hat{Y}_1\otimes\hat{Y}_2 \\ \tilde{S} \\ \hat{Y} \end{pmatrix}\begin{pmatrix}\tilde{S}^2 & \tilde{S}\hat{Y}_1 & \tilde{S}\hat{Y}_2 & \hat{Y}_1\otimes\hat{Y}_2 & \tilde{S} & \hat{Y}\end{pmatrix}d\tilde{S}d\hat{Y}.$$

Given a nonzero pair $[h,\beta]$, $h=(h_{xx},h_{xy},h_{yx},h_{yy})$ with $h_{xy}=h_{yx}^T$ and $\beta=(\beta_x,\beta_y)$, assuming $\sigma_p(\boldsymbol{\delta}^s_F)[h,\beta]=0$, i.e.,

$$\xi h_{xx}+\eta\cdot h_{xy}=0,\quad \xi h_{xy}+\frac{1}{2}(\eta_1+\eta_2)\cdot h_{yy}=0,\quad\text{and}\quad \xi\beta_x+\eta\cdot\beta_y=0, \tag{14}$$

then

$$\begin{aligned}&(\sigma_p(B_F)[h,\beta],[h,\beta])\\ &=C|\zeta|^{-1}\times\int_{\zeta^\perp\cap(\mathbb{R}\times\mathbb{S}^{n-2})}\chi(\tilde{S})|\tilde{S}^2h_{xx}+\tilde{S}(h_{xy}\cdot\hat{Y}_1+\hat{Y}_2\cdot h_{xy})+(\hat{Y}_1\otimes\hat{Y}_2)\cdot h_{yy}+\tilde{S}\beta_x\\ &\qquad+\hat{Y}\cdot\beta_y|^2\,d\tilde{S}d\hat{Y}.\end{aligned}$$

Now if the integral equals zero, we get that

$$\tilde{S}^2h_{xx}+\tilde{S}(h_{xy}\cdot\hat{Y}_1+\hat{Y}_2\cdot h_{xy})+(\hat{Y}_1\otimes\hat{Y}_2)\cdot h_{yy}+\tilde{S}\beta_x+\hat{Y}\cdot\beta_y=0$$

for all $(\tilde{S},\hat{Y})$ satisfying $\xi\tilde{S}+\eta\cdot\hat{Y}=0$, $\chi(\tilde{S})>0$. Notice that χ is even; this implies that

$$\tilde{S}^2h_{xx}+\tilde{S}(h_{xy}\cdot\hat{Y}_1+\hat{Y}_2\cdot h_{xy})+(\hat{Y}_1\otimes\hat{Y}_2)\cdot h_{yy}=0\quad\text{and}\quad\tilde{S}\beta_x+\hat{Y}\cdot\beta_y=0 \tag{15}$$

for such $(\tilde{S},\hat{Y})$. Since $\xi\beta_x+\eta\cdot\beta_y=0$, it is shown in the proof of Lemma 3.1 that $(\beta_x,\beta_y)=0$.

On the other hand, assume $\tilde{S}=0$; then by the first equality of (15), $\hat{Y}\cdot\eta=0$ implies that $(\hat{Y}_1\otimes\hat{Y}_2)\cdot h_{yy}=0$ for all $\hat{Y}\in\eta^\perp\cap\mathbb{S}^{n-2}$ (notice that h_{yy} is a symmetric $(n-1)\times(n-1)$ matrix). Then to show that $h_{yy}=0$, it suffices to verify that $(\eta\otimes\eta)\cdot h_{yy}=0$. If $\eta=0$, then it's done, so we assume that $\eta\neq0$. Since $|\tilde{S}|$ needs to be small to guarantee that $\chi(\tilde{S})>0$, we denote $\eta\cdot\hat{Y}=-\tilde{S}\xi$ by ε with $|\varepsilon|\ll1$; then

$\hat{Y}$ can be decomposed as $\hat{Y} = \frac{\varepsilon}{|\eta|}\frac{\eta}{|\eta|} + \hat{Y}^{\perp}$, where $\hat{Y}^{\perp}$ is the projection of $\hat{Y}$ in $\eta^{\perp}$. If $\xi = 0$, by (14) $(\eta_1 + \eta_2) \cdot h_{yy} = 0$, so is $(\eta \otimes \eta) \cdot h_{yy}$. If $\xi \neq 0$, by (14) again, we have

$$h_{xy} = -\frac{1}{2\xi}(\eta_1 + \eta_2) \cdot h_{yy}, \quad h_{xx} = -(\eta \cdot h_{xy})/\xi = \frac{1}{\xi^2}(\eta \otimes \eta) \cdot h_{yy}.$$

Plug the above equalities into the first part of (15); then

$$\left(\frac{\tilde{S}^2}{\xi^2}(\eta \otimes \eta) - \frac{\tilde{S}}{\xi}(\eta \otimes \hat{Y} + \hat{Y} \otimes \eta) + (\hat{Y} \otimes \hat{Y})\right) \cdot h_{yy} = 0,$$

or equivalently

$$\begin{aligned}&\left(\left(\frac{\varepsilon}{\xi^2}\eta + \hat{Y}\right) \otimes \left(\frac{\varepsilon}{\xi^2}\eta + \hat{Y}\right)\right) \cdot h_{yy}\\ &\quad = \left(\left(\varepsilon\left(\frac{1}{\xi^2} + \frac{1}{|\eta|^2}\right)\eta + \hat{Y}^{\perp}\right) \otimes \left(\varepsilon\left(\frac{1}{\xi^2} + \frac{1}{|\eta|^2}\right)\eta + \hat{Y}^{\perp}\right)\right) \cdot h_{yy} = 0.\end{aligned}$$

Since $(\hat{Y}^{\perp} \otimes \hat{Y}^{\perp}) \cdot h_{yy} = 0$, we have

$$\varepsilon^2\left(\frac{1}{\xi^2} + \frac{1}{|\eta|^2}\right)^2 (\eta \otimes \eta) \cdot h_{yy} = -\varepsilon\left(\frac{1}{\xi^2} + \frac{1}{|\eta|^2}\right)\left(\eta \otimes \hat{Y}^{\perp} + \hat{Y}^{\perp} \otimes \eta\right) \cdot h_{yy}. \tag{16}$$

Notice that for fixed $\hat{Y}^{\perp}$ and $\varepsilon \neq 0$, $\hat{Y} = -\frac{\varepsilon\eta}{|\eta|^2} + \hat{Y}^{\perp}$ will also work for the above equation. Thus both sides of (16) vanish, in particular $(\eta \otimes \eta) \cdot h_{yy} = 0$ and $(\eta \otimes Y + Y \otimes \eta) \cdot h_{yy}$ for any $Y \in \eta^{\perp}$. The above argument means that $(\hat{Y} \otimes \hat{Y}) \cdot h_{yy} = 0$ for all $\hat{Y} \in \mathbb{S}^{n-2}$. Taking into account the symmetricity of h_{yy}, it has to be zero.

Since $h_{yy} = 0$, by (14) if $\xi \neq 0$, we have $h_{xy} = 0$ and $h_{xx} = 0$. If $\xi = 0$, then $\tilde{S}^2 h_{xx} + \tilde{S}(h_{xy} \cdot \hat{Y}_1 + \hat{Y}_2 \cdot h_{xy}) = 0$, and thus $\tilde{S}h_{xx} + h_{xy} \cdot \hat{Y} + \hat{Y} \cdot h_{xy} = 0$ when $\tilde{S} \neq 0$ small, for any $\hat{Y} \in \eta^{\perp} \cap \mathbb{S}^{n-2}$. Take $\tilde{S}_i \neq 0$ with $\chi(\tilde{S}_i) > 0$, $i = 1, 2$; then $(\tilde{S}_1 - \tilde{S}_2)h_{xx} = 0$, which implies that $h_{xx} = 0$ and $h_{xy} \cdot \hat{Y} = 0$ for all $\hat{Y} \in \eta^{\perp} \cap \mathbb{S}^{n-2}$. However, since $\eta \cdot h_{xy} = 0$, we get $h_{xy} = 0$. Thus $h = (h_{xx}, h_{xy}, h_{yy}) = 0$, i.e., $[h, \beta] = 0$, which is a contradiction. This proves the lemma. ∎

Lemma 3.5. *There exists $F_0 > 0$; for any $F > F_0$, there is $\chi = \chi_F \in C_c^{\infty}(\mathbb{R})$ such that B_F is elliptic at finite points of $T^*_{sc}\overline{U}$ when restricted on the kernel of the scattering principal symbol of $\boldsymbol{\delta}^s_F$.*

Proof. If χ is a Gaussian function, i.e., $\chi(s) = e^{-s^2/2F^{-1}\alpha}$, by a computation similar to that of Lemma 3.2 we get that the scattering principal symbol of B_F is a nonzero multiple of

$$\int_{\mathbb{S}^{n-2}} \frac{1}{\sqrt{\xi^2 + F^2}} \begin{pmatrix} \bar{\theta}_2 \\ \hat{Y}_1\bar{\theta}_1 \\ \hat{Y}_2\bar{\theta}_1 \\ \hat{Y}_1 \otimes \hat{Y}_2 \\ \bar{\theta}_1 \\ \hat{Y} \end{pmatrix} \begin{pmatrix} \theta_2 & \theta_1\hat{Y}_1 & \theta_1\hat{Y}_2 & \hat{Y}_1 \otimes \hat{Y}_2 & \theta_1 & \hat{Y} \end{pmatrix} e^{-|\hat{Y}\cdot\eta|^2/2F^{-1}\alpha(\xi^2+F^2)}\, d\hat{Y},$$

where $\theta_1 = -\frac{\xi - iF}{\xi^2+F^2}(\hat{Y} \cdot \eta)$ and $\theta_2 = \frac{(\xi - iF)^2}{(\xi^2+F^2)^2}(\hat{Y} \cdot \eta)^2 + 2i\alpha\frac{\xi - iF}{\xi^2+F^2} = \theta_1^2 + 2i\alpha\frac{\xi - iF}{\xi^2+F^2}$.

Given a nonzero pair $[h, \beta]$ in the kernel of the scattering principal symbol of $\boldsymbol{\delta}^s_F$, by Lemma 2.2,

$$(\xi - iF)h_{xx} + \eta \cdot h_{xy} + a \cdot h_{yy} + b \cdot \beta_y = 0, \quad (\xi - iF)h_{xy} + \frac{1}{2}(\eta_1 + \eta_2) \cdot h_{yy} = 0 \tag{17}$$

and

$$(\xi - iF)\beta_x + \eta \cdot \beta_y = 0. \tag{18}$$

Then

$$\begin{aligned}(\sigma_{sc}(B_F)[h, \beta], [h, \beta]) = \frac{C}{\sqrt{\xi^2 + F^2}} \times \int_{\mathbb{S}^{n-2}} &|\theta_2 h_{xx} + 2\theta_1 \hat{Y} \cdot h_{xy} + (\hat{Y} \otimes \hat{Y}) \cdot h_{yy} \\ &+ \theta_1 \beta_x + \hat{Y} \cdot \beta_y|^2 e^{-|\hat{Y}\cdot\eta|^2/2F^{-1}\alpha(\xi^2+F^2)}\, d\hat{Y}.\end{aligned}$$

If the lemma is not true, then for any $N > 0$, there is $F > N$ such that the above integral vanishes for some nonzero $[h, \beta]$ in the kernel of $\sigma_{sc}(\boldsymbol{\delta}^s_F)$, and we get that $\theta_2 h_{xx} + 2\theta_1 \hat{Y} \cdot h_{xy} + (\hat{Y} \otimes \hat{Y}) \cdot h_{yy} + \theta_1 \beta_x + \hat{Y} \cdot \beta_y = 0$ for all $\hat{Y} \in \mathbb{S}^{n-2}$. Note that $\theta_1(-\hat{Y}) = -\theta_1(\hat{Y})$. On the other hand, by (2) it is not difficult to see that for magnetic geodesics $\alpha(\hat{Y}) = d^2x/dt^2|_{t=0} = \alpha^+(\hat{Y}) + \alpha^-(\hat{Y})$ with α^+ a positive definite quadratic form (similar to the geodesic case) and α^- a 1-form (related to E). Thus $\theta_2(-\hat{Y}) = \theta_1^2(\hat{Y}) + 2i(\alpha^+(\hat{Y}) - \alpha^-(\hat{Y}))\frac{\xi - iF}{\xi^2 + F^2}$, and

$$\begin{aligned}&\left(\theta_1^2(\hat{Y}) + 2i\alpha^+(\hat{Y})\frac{\xi - iF}{\xi^2 + F^2}\right) h_{xx} + 2\theta_1(\hat{Y})\hat{Y} \cdot h_{xy} + (\hat{Y} \otimes \hat{Y}) \cdot h_{yy} = 0, \\ &2i\alpha^- \cdot \hat{Y}\frac{\xi - iF}{\xi^2 + F^2} h_{xx} + \theta_1(\hat{Y})\beta_x + \hat{Y} \cdot \beta_y = 0\end{aligned} \tag{19}$$

for all $\hat{Y}$. In other words, there exist $\{F_k\}_{k=1}^\infty$, $F_k \to +\infty$ as $k \to \infty$, and $\{[h^k, \beta^k]\}_{k=1}^\infty$, $[h^k, \beta^k]$ in the kernel of $\sigma_{sc}(\boldsymbol{\delta}^s_{F_k})$ and nonzero, such that (19) holds for each pair $(F_k, [h^k, \beta^k])$.

First we claim that for large enough k, $h^k_{yy} \neq 0$. If not, then there exists a subsequence $\{F_{n_k}, [h^{n_k}, \beta^{n_k}]\}$ such that $h^{n_k}_{yy} = 0$ for all n_k. Then by (17) $h^{n_k}_{xy} = 0$ and $h^{n_k}_{xx} = -b \cdot \beta^{n_k}_y/(\xi - iF_{n_k})$. So by (18) and the second equation of (19),

$$\left(-2i\frac{b \cdot \beta^{n_k}_y}{\xi^2 + F^2_{n_k}}\alpha^- + \frac{\eta \cdot \beta^{n_k}_y}{\xi^2 + F^2_{n_k}}\eta + \beta^{n_k}_y\right) \cdot \hat{Y} = 0$$

for all $\hat{Y} \in \mathbb{S}^{n-2}$, i.e.,

$$-2i\frac{b \cdot \beta^{n_k}_y}{\xi^2 + F^2_{n_k}}\alpha^- + \frac{\eta \cdot \beta^{n_k}_y}{\xi^2 + F^2_{n_k}}\eta + \beta^{n_k}_y = 0. \tag{20}$$

If $\beta^{n_k}_y = 0$, then by (18) $\beta^{n_k}_x = 0$ and $h^{n_k}_{xx} = 0$, i.e., $[h^{n_k}, \beta^{n_k}] = 0$, which is a contradiction. Thus we can assume that $\beta^{n_k}_y$ has unit norm for all n_k (notice that at a fixed point the geometry is trivial). Let $F_{n_k} \to +\infty$, then by (20) $\beta^{n_k}_y \to 0$, which is again a contradiction.

Now we can assume that $h^k_{yy} \neq 0$ for all k. By (17) and (18), for any k

$$\begin{aligned} h^k_{xy} &= -\frac{\eta_1 + \eta_2}{2(\xi - iF_k)} \cdot h^k_{yy}, \\ h^k_{xx} &= -\frac{\eta \cdot h^k_{xy} + a \cdot h^k_{yy} + b \cdot \beta^k_y}{\xi - iF_k} = \frac{\eta \otimes \eta - (\xi - iF_k)a}{(\xi - iF_k)^2} \cdot h^k_{yy} - \frac{b}{\xi - iF_k} \cdot \beta^k_y, \\ \beta^k_x &= -\frac{\eta \cdot \beta^k_y}{\xi - iF_k}. \end{aligned}$$

Plugging the above equalities into (19) we get

$$\begin{aligned} &\Bigg(\frac{(\hat{Y} \cdot \eta)^2 + 2i\alpha^+(\xi + iF_k)}{(\xi^2 + F_k^2)^2} (\eta \otimes \eta - (\xi - iF_k)a) \\ &\quad + \frac{\hat{Y} \cdot \eta}{\xi^2 + F_k^2} (\eta \otimes \hat{Y} + \hat{Y} \otimes \eta) + \hat{Y} \otimes \hat{Y} \Bigg) \cdot h^k_{yy} \\ &\quad - \frac{\xi - iF_k}{(\xi^2 + F_k^2)^2} \Big((\hat{Y} \cdot \eta)^2 + 2i\alpha^+(\xi + iF_k) \Big) b \cdot \beta^k_y = 0 \end{aligned} \tag{21}$$

and

$$2i(\alpha^- \cdot \hat{Y}) \frac{\eta \otimes \eta - (\xi - iF_k)a}{(\xi^2 + F_k^2)(\xi - iF_k)} \cdot h^k_{yy} + \left(\hat{Y} + \frac{\hat{Y} \cdot \eta}{\xi^2 + F_k^2} \eta - \frac{2i(\alpha^- \cdot \hat{Y})}{\xi^2 + F_k^2} b \right) \cdot \beta^k_y = 0. \tag{22}$$

If there is a subsequence of $\{\beta^{n_k}_y\}_{n_k \to \infty}$ such that $\beta^{n_k}_y = 0$ for all n_k, since $h^k_{yy} \neq 0$, we may assume that $h^{n_k}_{yy}$ has unit norm for all n_k. Thus there exists further a subsequence $\{h^{n'_k}_{yy}\}_{n'_k \to \infty}$ of $\{h^{n_k}_{yy}\}$ and h^∞_{yy} satisfying $h^{n'_k}_{yy} \to h^\infty_{yy}$, $F_{n'_k} \to +\infty$ as $n'_k \to \infty$. As (ξ, η) is a finite point, we take the limit of (21) as $n'_k \to \infty$ to get that

$$(\hat{Y} \otimes \hat{Y}) \cdot h^\infty_{yy} = 0 \quad \forall \hat{Y} \in \mathbb{S}^{n-2}.$$

Since h^∞_{yy} is a symmetric tensor, the above equality forces it to be zero. However, since $h^{n_k}_{yy}$ has unit norm, the limit h^∞_{yy} cannot be zero, and we reach a contradiction.

So we can assume that $h^k_{yy} \neq 0$ and $\beta^k_y \neq 0$ for any k. Let $c_k = \max\{\|h^k_{yy}\|, \|\beta^k_y\|\} > 0$, and consider the sequence $\{[h^k/c_k, \beta^k/c_k]\}$; we still denote the new sequence by $\{[h^k, \beta^k]\}$, thus $\|h^k_{yy}\| \leq 1$ and $\|\beta^k_y\| \leq 1$. Then there exists a subsequence $\{(h^{n_k}, \beta^{n_k})\}_{n_k \to \infty}$ such that $h^{n_k}_{yy} \to h^\infty_{yy}$, $\beta^{n_k}_y \to \beta^\infty_y$, $F_{n_k} \to +\infty$ as $n_k \to \infty$. Now we take the limits of (21) and (22) with respect to the subsequence as $n_k \to \infty$ to get that

$$(\hat{Y} \otimes \hat{Y}) \cdot h^\infty_{yy} = 0, \quad \hat{Y} \cdot \beta^\infty_y = 0 \quad \forall \hat{Y} \in \mathbb{S}^{n-2}.$$

Again this implies that $h^\infty_{yy} = 0$ and $\beta^\infty_y = 0$. However, for each n_k, either $\|h^{n_k}_{yy}\| = 1$ or $\|\beta^{n_k}_y\| = 1$, so h^∞_{yy} and β^∞_y cannot both vanish. This is a contradiction too; thus our assumption for the contradiction argument is not true, i.e., there is some $F_0 > 0$ such that the lemma holds for a Gaussian like χ. Then we apply an approximation argument to complete the proof. ∎

Remark. The algebraic argument of the proof of Lemma 3.5 is different from the one of [42]. The magnetic case is more complicated than the geodesic case due to the coupling of tensors of different orders. In particular, α is no longer an even function of $\hat{Y}$ as in the geodesic case, which is the reason why we consider h_{yy} and β_y together in the main argument. On the other hand, our idea might work for the tensor tomography problem along general smooth curves, since generally one can decompose α into the even and odd parts with $\alpha = \alpha^+ + \alpha^-$, where $\alpha^+(\hat{Y}) = (\alpha(\hat{Y}) + \alpha(-\hat{Y}))/2$ and $\alpha^-(\hat{Y}) = (\alpha(\hat{Y}) - \alpha(-\hat{Y}))/2$.

Similar to Proposition 3.3, we have the following result for B_F.

PROPOSITION 3.6. *There exists* $F_0 > 0$ *such that for any* $F > F_0$, *given* Ω *a neighborhood of* $\overline{O}$ *in* $\overline{U}$, *there exist* $\chi \in C_c^\infty(\mathbb{R})$ *and* $N \in \Psi_{sc}^{-3,0}(\overline{U}; T^*_{sc}\overline{U} \times \overline{\underline{U}}, T^*_{sc}\overline{U} \times \overline{\underline{U}})$ *such that* $B_F + \mathbf{d}^s_F N \boldsymbol{\delta}^s_F \in \Psi_{sc}^{-1,0}(\overline{U}; Sym^2 T^*_{sc}\overline{U} \times T^*_{sc}\overline{U}, Sym^2 T^*_{sc}\overline{U} \times T^*_{sc}\overline{U})$ *is elliptic in* Ω.

4. Proofs of the main local results.

Now we rephrase the invertibility results of section 3 in a gauge-free way. This part is similar to [42, section 4]; the key ingredient is the local invertibility of some Witten-type Dirichlet Laplacian.

4.1. Proof of Theorem 1.1.

Note that if the "solenoidal Witten Laplacian" $\Delta_F = \boldsymbol{\delta}_F \mathbf{d}_F$ is invertible with the Dirichlet boundary condition, we can decompose $f_F := [\beta, \varphi]_F = e^{-F/x} W^{-1}[\beta, \varphi]$ into

$$f_F = \mathcal{S}_F f_F + \mathcal{P}_F f_F,$$

where $\mathcal{P}_F = \mathbf{d}_F \Delta_F^{-1} \boldsymbol{\delta}_F$. Thus we denote $\mathcal{P}_F f_F$ by $\mathbf{d}_F p_F = W^{-1} e^{-F/x} \mathbf{d} p$ with $p|_{\partial O \cap \partial M} = 0$; then given $f = [\beta, \varphi]$

$$If = I(f - \mathbf{d}p) = I(e^{F/x} W (f_F - \mathbf{d}_F p_F)) = I(e^{F/x} W \mathcal{S}_F f_F).$$

Notice that $\boldsymbol{\delta}_F(\mathcal{S}_F f_F) = 0$, and by Proposition 3.3 in O, $\mathcal{S}_F f_F$ or equivalently $e^{F/x} W \mathcal{S}_F f_F = f - \mathbf{d}p$ can be stably determined by $If = I(e^{F/x} W \mathcal{S}_F f_F)$; see [42, Theorem 4.15] and see [43, section 3.7] for the function case. Generally the stability estimate by ellipticity has an error term; however, for the local problem the error term is relatively small and can be absorbed to produce the full invertibility; see [43, section 2]. This proves Theorem 1.1. So one just needs to show that Δ_F is invertible with the Dirichlet boundary condition; however, this is immediate from the argument of [42, section 4]. Note that by the definition, Δ_F is the same as the Witten Laplacian of functions in [42].

4.2. Proof of Theorem 1.2.

Similar to the argument of section 4.1, if the Witten Laplacian $\Delta^s_F = \boldsymbol{\delta}^s_F \mathbf{d}^s_F$ is invertible with the Dirichlet boundary condition, let $f = [h, \beta]$; then by Proposition 3.6 there are some 1-form u and function p with $u|_{\partial O \cap \partial M} = 0$, $p|_{\partial O \cap \partial M} = 0$ such that $f - \mathbf{d}^s[u, p]$ can be stably determined by If. Notice that by Lemma 2.2, the principal symbol of Δ^s_F is

$$\begin{pmatrix} \langle\xi\rangle^2 + \frac{1}{2}|\eta|^2 & \frac{1}{2}(\xi + iF)\iota_\eta & 0 \\ \frac{1}{2}(\xi - iF)\eta\otimes & \frac{1}{2}\langle\xi\rangle^2 + |\eta|^2 & 0 \\ 0 & 0 & \langle\xi\rangle^2 + |\eta|^2 \end{pmatrix} + \begin{pmatrix} \langle a, \cdot\rangle a + \langle b, \cdot\rangle b & \langle a, \cdot\rangle \eta\otimes_s & \langle b, \cdot\rangle \eta\otimes \\ \iota^s_\eta a & 0 & 0 \\ \iota_\eta b & 0 & 0 \end{pmatrix},$$

where $\langle\xi\rangle = \sqrt{\xi^2 + F^2}$. It is easy to check that the first part of the symbol has a lower bound $O(\xi^2 + F^2 + |\eta|^2)$; by taking F large enough, it can absorb the second part of the symbol which is independent of F. Thus Δ^s_F is elliptic for large F. Moreover, letting $\nabla_F = e^{-F/x} \nabla e^{F/x}$ with ∇ being the gradient with respect to the scattering metric g_{sc}, we define $\nabla_F^s[\beta,\varphi] := [\nabla_F\beta, \sqrt{2}\nabla_F\varphi]$, which has the following principal symbol:

$$\begin{pmatrix} \xi+iF & 0 & 0 \\ \eta\otimes & 0 & 0 \\ 0 & \xi+iF & 0 \\ 0 & \eta\otimes & 0 \\ 0 & 0 & \sqrt{2}(\xi+iF) \\ 0 & 0 & \sqrt{2}\eta\otimes \end{pmatrix}.$$

So the principal symbol of $(\nabla_F^s)^*$, the adjoint under the scattering metric g_{sc}, is

$$\begin{pmatrix} \xi-iF & \iota_\eta & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi-iF & \iota_\eta^s & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}(\xi-iF) & \sqrt{2}\iota_\eta \end{pmatrix}$$

and the principal symbol of $(\nabla_F^s)^*\nabla_F^s$ is

$$\begin{pmatrix} \langle\xi\rangle^2+|\eta|^2 & 0 & 0 \\ 0 & \langle\xi\rangle^2+|\eta|^2 & 0 \\ 0 & 0 & 2(\langle\xi\rangle^2+|\eta|^2) \end{pmatrix}.$$

On the other hand, applying Lemma 2.1 again, we get the principal symbol of $\mathbf{d}_F\boldsymbol{\delta}_F$

$$\begin{pmatrix} \langle\xi\rangle^2 & (\xi+iF)\iota_\eta & 0 \\ (\xi-iF)\eta\otimes & |\eta|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $\Delta_F^s = \frac{1}{2}(\nabla_F^s)^*\nabla_F^s + \frac{1}{2}\mathbf{d}_F\boldsymbol{\delta}_F + R + \tilde{R}$ with $R \in \mathrm{Diff}_{sc}^1(T_{sc}^*\overline{U}\times\underline{U}, T_{sc}^*\overline{U}\times\underline{U})$ given by the second part of the principal symbol of Δ_F^s and $\tilde{R} \in x\,\mathrm{Diff}_{sc}^1(T_{sc}^*\overline{U}\times\underline{U}, T_{sc}^*\overline{U}\times\underline{U})$ containing all the lower order terms. We have proved [42, Lemma 4.1] under our settings; now Theorem 1.2 follows by an argument similar to that of [42, section 4].

5. Proof of the global result.

We prove part (a) of Theorem 1.3 based on the local result Theorem 1.1 in this section; part (b) follows in a similar way by applying Theorem 1.2. A similar argument for the geodesic ray transform can be found in [24]. We first prove the following weaker version of Theorem 1.3 up to a set of empty interior. We define $\Sigma_t := \tau^{-1}(t)$, $M_t := M \setminus \{\tau \le t\}$ and $\Omega_t := \partial M \setminus \{\tau \le t\}$.

LEMMA 5.1. *Under the assumptions of Theorem 1.3, part (a), there exists $v \in C^\infty(M_a)$ with $v|_{\Omega_a} = 0$ such that $f = \mathbf{d}v$ in M_a.*

Recall that the constant a in the lemma and the constant b below are the ones appearing in the definition of the global foliation condition before Theorem 1.3.

Proof. Let

$$\sigma := \inf\{t \le b : \exists\, v \in C^\infty(M_t), \text{ such that } v|_{\Omega_t} = 0, \text{ and } f = \mathbf{d}v \text{ in } M_t\}.$$

We claim that $\sigma \le a$ and we will argue by contradiction.

First we show that $\sigma < b$. It is not difficult to see that Σ_b is a compact subset of ∂M (in fact, if Σ_b contains interior points, $\{\tau \le b\}$ cannot cover M). Since ∂M is strictly magnetic convex, by Theorem 1.1, for each $p \in \Sigma_b$, there is a neighborhood

$O_p \subset M$ of p and $v_p \in C^\infty(O_p)$ with $v_p|_{O_p\cap\partial M} = 0$ such that $f = \mathbf{d}v_p$ in O_p. If $O_p \cap O_q \neq \emptyset$ for some $p, q \in \Sigma_b$, we have

$$\mathbf{d}(v_p - v_q)|_{O_p\cap O_q} = 0, \quad v_p - v_q|_{O_p\cap O_q\cap\partial M} = 0.$$

This implies that $v_p = v_q$ in $O_p \cap O_q$. Since $\tau^{-1}(b)$ is compact, there exist $t_0 < b$ and v smooth in M_{t_0} such that $f = dv$ in M_{t_0}; in particular $v = v_p$ in $M_{t_0} \cap O_p$. Thus $\sigma \leq t_0 < b$.

The infimum in the definition of σ is a minimum. Let $\{t_j\}_{j=1}^\infty \subset (\sigma, b]$ be a strictly decreasing sequence with $t_j \to \sigma$ as $j \to \infty$. For each j, there is v_j satisfying $f = \mathbf{d}v_j$ in M_{t_j} and $v_j|_{\Omega_{t_j}} = 0$. Since $\Sigma_t \cap M^{int}$ is strictly magnetic convex for any $t > a$, one can easily show that given arbitrary $k > 0$, $v_k = v_\ell$ on M_{t_k} for any $\ell > k$. This implies that the set $\{v_j\}$ defines a smooth function v_σ in M_σ with $v_\sigma|_{M_{t_j}\setminus M_{t_{j-1}}} = v_j$, $f = \mathbf{d}v_\sigma$ in M_σ, and $v_\sigma|_{\Omega_\sigma} = 0$, i.e., σ is a minimum.

Assume that $\sigma > a$ and consider the level set Σ_σ. There exists $v_\sigma \in C^\infty(M_\sigma)$ with $v_\sigma|_{\Omega_\sigma} = 0$ such that $f = \mathbf{d}v_\sigma$ in M_σ. We first extend v_σ a little bit near the boundary. Notice that ∂M is strictly magnetic convex and $\Sigma_\sigma \cap \partial M$ is compact; by Theorem 1.1 and an argument similar to the one showing $\sigma < b$, one can find a neighborhood O of $\Sigma_\sigma \cap \partial M$ and $v_O \in C^\infty(O)$ such that $f = \mathbf{d}v_O$ and $v_O|_{\partial M\cap O} = 0$. Moreover, on the overlap $O \cap M_\sigma$, one can similarly show that $v_\sigma = v_O$ by choosing O appropriately. This implies that we can actually define a smooth function u on $U := M_\sigma \cup O$. Thus now $f = \mathbf{d}u$ in U, $u|_{\partial U\cap\partial M} = 0$. This will allow us to avoid the set $\Sigma_\sigma \cap \partial M$ for the rest of the proof.

With U chosen as above, we see that $K := \partial U \cap M^{int} \cap \Sigma_\sigma$ is a compact subset of $\Sigma_\sigma \cap M^{int}$. Applying Theorem 1.1 again, there exist $c > 0$ small ($\sigma - c > a$) and an open neighborhood V of K in $\{\tau \leq \sigma\} \cap M^{int}$ such that the local invertibility of I holds on V and $(\{\sigma - c \leq \tau \leq \sigma\} \setminus O) \subset V$ (notice that O is an open neighborhood of $\Sigma_\sigma \cap \partial M$). In particular, the constant c (which is related to the definition of the neighborhood for the local theorem) is uniform for $p \in \Sigma_t$ close to K when t is sufficiently close to σ, e.g., $|\sigma - t| \ll c$. Thus we pick $\sigma' > \sigma$ with $\sigma' - \sigma \ll c$; then there exists an open neighborhood V' of $\Sigma_{\sigma'} \setminus O$ (compact) in $\{\tau \leq \sigma'\} \cap M^{int}$ such that the local invertibility holds in V' and $(\{\sigma' - c \leq \tau \leq \sigma'\} \setminus O) \subset V'$. Obviously $\sigma' - c < \sigma$.

Now let ϕ be a smooth cut-off function on M, which satisfies $\phi \equiv 1$ near $\overline{M}_{\sigma'}$, supp$\phi \subset M_\sigma$, so ϕu is well-defined on M. We denote $\tilde{f} = f - \mathbf{d}(\phi u)$, which is supported in $\{\tau < \sigma'\}$, by assumption $I\tilde{f} = 0$. So we apply Theorem 1.1 again to conclude that there is a smooth function $\tilde{v}$ defined in V', such that $\tilde{f} = \mathbf{d}\tilde{v}$ in V' and $\tilde{v}|_{V'\cap\Sigma_{\sigma'}} = 0$. Moreover, on the overlap $V' \cap M_\sigma$, since $(1-\phi)u = \tilde{v} = 0$ on $V' \cap \Sigma_{\sigma'}$, one easily obtains that $(1-\phi)u = \tilde{v}$ on the overlap too. Therefore, we get a smooth function w on $U \cup V'$ with $f = \mathbf{d}w$ there and $w|_{\partial M\cap U} = 0$. In particular, this implies that $\sigma \leq \sigma' - c < \sigma$, which is a contradiction. Thus $\sigma \leq a$ and the lemma is proved. ∎

Proof of Theorem 1.3(a). Note that the foliation condition implies that M_a is nontrapping. On the other hand, since $\{\tau \leq a\}$ is nontrapping too, $M = M_a \cup \{\tau \leq a\}$ is nontrapping. As ∂M is strictly magnetic convex, by an argument similar to [20, Proposition 5.2], which is for the geodesic case, there exists $u \in C^\infty(SM)$ satisfying the following transport equation:

$$\mathbf{G}_\mu u = -f, \quad u|_{\partial SM} = 0. \tag{23}$$

Thus by Lemma 5.1

$$\mathbf{G}_\mu(u+v)=0 \text{ in } M_a, \quad u+v|_{\partial SM^{\Omega_a}}=0, \tag{24}$$

where

$$\partial SM^{\Omega_a}:=\{(z,\xi)\in\partial SM : z\in\Omega_a\}.$$

Since $\Sigma_t\cap M^{int}$ is strictly magnetic convex for $t\in(a,b]$, given arbitrary $(z,\xi)\in SM_a$, we can find a magnetic geodesic segment $\gamma:[0,T]\to M$ connecting z with Ω_a, which is completely contained in M_a, such that (z,ξ) is either $(\gamma(0),\dot\gamma(0))$ or $(\gamma(T),\dot\gamma(T))$. Together with (24), this implies that $u+v=0$ in SM_a, i.e., $u=-v$ is a smooth function on M_a. However, as $u\in C^\infty(SM)$ and the set $\{\tau\leq a\}$ has empty interior, we conclude that $u\in C^\infty(M)$. To show this, we take use of the spherical harmonics expansion of u through the vertical Laplacian $\overset{v}{\Delta}$ on SM as

$$u=\sum_{k=0}^{\infty}u_k,$$

where each $u_k\in C^\infty(SM)$ satisfies $\overset{v}{\Delta}u_k=k(k+n-2)u_k$ ($n=\dim M$). Note that this is an orthogonal decomposition of u under the L^2 inner product; see, e.g., [23] for more details. In particular, if $u\in C^\infty(M)$, then $u_k\equiv 0$ for all $k>0$. Since $u=-v$ on M_a, we get that $u_k=0$ on SM_a for any $k>0$. Now given any $(z,v)\in S(M\setminus M_a)$, since $M\setminus M_a$ has empty interior, we can find a sequence $\{(z_j,v_j)\}_{j=1}^\infty\subset SM_a$ such that $(z_j,v_j)\to(z,v)$ as $j\to\infty$. Since $u_k(z_j,v_j)=0$ for any j and $k>0$, $u_k(z,v)=0$ too for any $k>0$. Thus $u=u_0$ on SM, i.e., $u\in C^\infty(M)$. By (23), $f=\mathbf{G}_\mu(-u)=\mathbf{d}(-u)$ on M with $u|_{\partial M}=0$, which completes the proof. ∎

Remark. It is possible to allow the existence of some type of trapped geodesics in the set $\{\tau\leq a\}$ under additional assumptions, which will still produce a smooth global solution to the transport equation (23); see, e.g., [8, Proposition 5.5].

Acknowledgments. The author wants to thank Prof. Gunther Uhlmann for suggesting this problem and reading an earlier version of the paper. Thanks are also due to Prof. Ting Zhou; part of the work was carried out during the author's visit to Zhou at Northeastern University in 2015. The author is also grateful to the referees for very helpful comments and suggestions.

REFERENCES

[1] G. Ainsworth, *The attenuated magnetic ray transform on surfaces,* Inverse Problems Imaging, 7 (2012), pp. 27–46.

[2] Yu. E. Anikonov and V. G. Romanov, *On uniqueness of determination of a form of first degree by its integrals along geodesics*, J. Inverse Ill-Posed Probl., 5 (1997), pp. 487–490.

[3] C. Croke, *Rigidity theorems in Riemannian geometry*, in Geometric Methods in Inverse Problems and PDE Control, IMA Vol. Math. Appl. 137, Springer, New York, 2004, pp. 47–72.

[4] N. S. Dairbekov, *Integral geometry problem for nontrapping manifolds,* Inverse Problems, 22 (2006), pp. 431–445.

[5] N. S. Dairbekov, G. P. Paternain, P. Stefanov and G. Uhlmann, *The boundary rigidity problem in the presence of a magnetic field*, Adv. Math., 216 (2007), pp. 535–609.

[6] J. H. Eschenburg, *Local convexity and nonnegative curvature—Gromovs proof of the sphere theorem,* Invent. Math., 84 (1986), pp. 507–522.

[7] R. E. Greene and H. Wu, *C^∞ convex functions and manifolds of positive curvature,* Acta Math., 137 (1976), pp. 209–245.

[8] C. Guillarmou, *Lens rigidity for manifolds with hyperbolic trapped set,* J. Amer. Math. Soc., 30 (2017), pp. 561–599.

[9] C. Guillarmou and F. Monard, *Reconstruction formulas for X-ray transforms in negative curvature,* Ann. Inst. Fourier, to appear.

[10] G. Herglotz, *Über die Elastizität der Erde bei Berücksichtigung ihrer variablen Dichte*, Z. Math. Phys., 52 (1905), pp. 275–299.

[11] S. Holman and P. Stefanov, *The weighted Doppler transform*, Inverse Problems Imaging, 4 (2010), pp. 111–130.

[12] S. Holman and G. Uhlmann, *On the microlocal analysis of the geodesic X-ray transform with conjugate points,* J. Differential Geom., 108 (2018), pp. 459–494.

[13] V. Krishnan, *A support theorem for the geodesic ray transform on functions,* J. Fourier Anal. Appl., 15 (2009), pp. 515–520.

[14] V. Krishnan and P. Stefanov, *A support theorem for the geodesic ray transform of symmetric tensor fields,* Inverse Problems Imaging, 3 (2009), pp. 453–464.

[15] R. B. Melrose, *Spectral and Scattering Theory for the Laplacian on Asymptotically Euclidean Spaces*, Marcel Dekker, New York, 1994.

[16] F. Monard, *Numerical implementation of geodesic X-ray transforms and their inversion,* SIAM J. Imaging Sci., 7 (2014), pp. 1335–1357.

[17] F. Monard, P. Stefanov, and G. Uhlmann, *The geodesic ray transform on Riemannian surfaces with conjugate points*, Communi. Math. Phys., 337 (2015), pp. 1491–1513.

[18] R. G. Mukhometov, *The reconstruction problem of a two-dimensional Riemannian metric, and integral geometry,* Dokl. Akad. Nauk SSSR, 232 (1977), pp. 32–35 (in Russian).

[19] R. G. Mukhometov and V. G. Romanov, *On the problem of finding an isotropic Riemannian metric in an n-dimensional space*, Dokl. Akad. Nauk SSSR, 243 (1978), pp. 41–44 (in Russian).

[20] G. Paternain, M. Salo, and G. Uhlmann, *The attenuated ray transform for connections and Higgs fields*, Geom. Funct. Anal., 22 (2012), pp. 1460–1489.

[21] G. Paternain, M. Salo, and G. Uhlmann, *Tensor tomography on simple surfaces,* Invent. Math., 193 (2013), pp. 229–247.

[22] G. Paternain, M. Salo, and G. Uhlmann, *Tensor tomography: progress and challenges*, Chin. Ann. Math. Ser. B, 35 (2014), pp. 399–427.

[23] G. Paternain, M. Salo, and G. Uhlmann, *Invariant distributions, Beurling transforms and tensor tomography in higher dimensions*, Math. Ann., 363 (2015), pp. 305–362.

[24] G. P. Paternain, M. Salo, G. Uhlmann, and H. Zhou, *The Geodesic Ray Transform with Matrix Weights*, arXiv:1605.07894, 2016.

[25] G. P. Paternain and H. Zhou, *Invariant distributions and the geodesic ray transform,* Anal. PDE, 9 (2016), pp. 1903–1930.

[26] L. Pestov and V. A. Sharafutdinov, *Integral geometry of tensor fields on a manifold of negative curvature,* Sibirsk. Mat. Zh., 29 (1988), pp. 114–130.

[27] L. Pestov and G. Uhlmann, *On Characterization of the Range and Inversion Formulas for the Geodesic X-ray Transform,* Int. Math. Res. Not. IMRN, 80 (2004), pp. 4331–4347.

[28] L. Pestov and G. Uhlmann, *Two dimensional simple Riemannian manifolds with boundary are boundary distance rigid*, Ann. of Math., 161 (2005), pp. 1089–1106.

[29] A. Ranjan and H. Shah, *Convexity of spheres in a manifold without conjugate points*, Proc. Indian Acad. Sci. Math. Sci., 112 (2002), 595–599.

[30] M. Salo and G. Uhlmann, *The attenuated ray transform on simple surfaces,* J. Differential Geom., 88 (2011), pp. 161–187.

[31] V. A. Sharafutdinov, *Integral Geometry of Tensor Fields*, Inverse Ill-posed Probl. Ser., VSP, Utrecht, 1994.

[32] V. A. Sharafutdinov, *Integral geometry of a tensor field on a surface of revolution*, Siberian Math. J., 38 (1997), pp. 603620.

[33] V. A. Sharafutdinov, *A problem in integral geometry in a nonconvex domain*, Siberian Math. J., 43 (2002), pp. 1159–1168.

[34] V. A. Sharafutdinov, *Variations of Dirichlet-to-Neumann map and deformation boundary rigidity of simple* 2*-manifolds,* J. Geom. Anal., 17 (2007), pp. 147–187.

[35] P. Stefanov, *A sharp stability estimate in tensor tomography*, J. Phys. Conf. Ser., 124 (2008), 012007.

[36] P. Stefanov and G. Uhlmann, *Stability estimates for the X-ray transform of tensor fields and boundary rigidity,* Duke Math. J., 123 (2004), pp. 445–467.

[37] P. Stefanov and G. Uhlmann, *Boundary rigidity and stability for generic simple metrics*, J. Amer. Math. Soc., 18 (2005), 975–1003.

[38] P. Stefanov and G. Uhlmann, *Boundary and lens rigidity, tensor tomography and analytic microlocal analysis*, in Algebraic Analysis of Differential Equations, T. Aoki, H. Majima, Y. Katei, and N. Tose, eds., Springer, New York, 2008, pp. 275–293.

[39] P. Stefanov and G. Uhlmann, *Integral geometry of tensor fields on a class of non-simple Riemannian manifolds,* Amer. J. Math., 130 (2008), pp. 239–268.

[40] P. Stefanov and G. Uhlmann, *The geodesic X-ray transform with fold caustics,* Anal. PDE, 5 (2012), pp. 219–260.

[41] P. Stefanov, G. Uhlmann, and A. Vasy, *Boundary rigidity with partial data*, J. Amer. Math. Soc., 29 (2016), pp. 299–332.

[42] P. Stefanov, G. Uhlmann, and A. Vasy, *Inverting the local geodesic X-ray transform on tensors*, J. Anal. Math., to appear, arXiv:1410.5145.

[43] G. Uhlmann and A. Vasy, *The inverse problem for the local geodesic ray transform*, Invent. Math., 205 (2016), pp. 83–120.

[44] G. Uhlmann and H. Zhou, *Journey to the Center of the Earth*, arXiv:1604.00630, 2016.

[45] E. Wiechert and K. Zoeppritz, *Über Erdbebenwellen*, Nachr. Koenigl. Geselschaft Wiss, Goettingen, 4 (1907), pp. 415–549.

[46] H. Zhou, *Lens Rigidity with Partial Data in the Presence of a Magnetic Field*, arXiv:1605.06257, 2016.