RECOVERY OF COEFFICIENTS IN A SCHRÖDINGER SYSTEM WITH NONLINEAR COUPLING

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Abstract. We consider the inverse problem of recovering time-dependent coefficients in a system of Schrödinger equations with coupled nonlinear terms. We first address the well-posedness of the system for small boundary data and trivial initial data. Then, we show that the linearized Neumann data stably determines the nonlinear coefficients provided that, at any given time, they are known in some neighborhood of the boundary. The proof relies on linearization and complex geometric optics (CGO) solutions for the linear Schrödinger equation. We conclude by outlining an approach based on the direct use of Carleman estimates in the setting of trivial boundary data but non-trivial initial data.

1. Introduction

Letting Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n = 2, 3)$ and denoting the linear Schrödinger operator by $\mathcal{T}_j := i\partial_t + \Delta + p_j(x,t)$, we seek to study the following system of coupled nonlinear Schrödinger equations:

$$\begin{cases} \mathcal{T}_1 u_1 + q_1(x,t)\left|u_1\right|^2 u_1 + \tilde{q}_1(x,t)|u_2|^2 u_1 = 0 \text{ on } \Omega \times (0,T), \\ \mathcal{T}_2 u_2 + q_2(x,t)\left|u_2\right|^2 u_2 + \tilde{q}_2(x,t)|u_1|^2 u_2 = 0 \text{ on } \Omega \times (0,T), \\ u_1(x,t) = g_1(x,t),\ u_2(x,t) = g_2(x,t) \text{ on } \partial\Omega \times (0,T), \\ u_1(x,0) = u_2(x,0) = 0 \text{ on } \Omega \times \{0\}. \end{cases} \tag{1.1}$$

When the coefficients $q_j, \tilde{q}_j$ $(j = 1, 2)$ are real constants, it is noted in [2] that (1.1) could model a two-component Bose-Einstein condensate with wavefunction $\mathbf{u} = (u_1, u_2)$ or coupled modes in birefringent media, where u_1 and u_2 are the electric field intensities along each birefringence axis. It is also noted in [37] that variable coefficients can arise in the nonlinear Schrödinger equation when modelling optical pulses through nonlinear optical fibers.

We begin by recording some notation used in the paper. We will denote $Q := \Omega \times (0,T)$ and $\Sigma := \partial\Omega \times (0,T)$. Taking Γ_j $(j = 1, 2)$ arbitrary relatively open subsets of $\partial\Omega$, we also denote $\Sigma_j := \Gamma_j \times (0,T)$. Throughout the paper, we will use the Sobolev space

$$H^{r,s}(S \times (0,T)) := H^r(0,T; L^2(S)) \cap L^2(0,T; H^s(S)). \tag{1.2}$$

For a given Banach space B, we will consider when needed the product space B^l for $l \in \mathbb{N}$ and equip it with the norm given by

$$\|(b_1, \dots, b_l)\|_{B^l} := \sum_{i=1}^{l} \|b_i\|_B . \tag{1.3}$$

We will also use the permutation $\iota : \{1,2\} \longrightarrow \{1,2\}$ defined by $\iota(1) = 2$ and $\iota(2) = 1$.

We are interested in the following inverse problem:

(IP) Do the coefficients $q_j, \tilde{q}_j$ $(j = 1, 2)$ depend continuously on the Neumann data

$$\left(\partial u_1/\partial\nu\big|_{\Sigma_1}, \partial u_2/\partial\nu\big|_{\Sigma_2}\right)? \tag{1.4}$$

That is, we seek to stably recover the nonlinear coefficients of (1.1) from the Neumann data (1.4). This question is related to the Calderón problem for the conductivity equation $\nabla \cdot (\gamma \nabla u) = 0$, which seeks to determine the coefficient γ from the knowledge of the Dirichlet-to-Neumann map (DN-map) $f \longmapsto \gamma\, (\partial u/\partial \nu)\,\big|_{\partial\Omega}$, where f is the Dirichlet data. In [41], the unique determination of γ from the DN-map was addressed in the affirmative for a smooth γ with positive lower bound and since then, a large body of work has emerged to address the Calderón problem for other equations. We briefly mention references for the dynamical (time-dependent) Schrödinger equation and coupled PDE systems.

The proof of the uniqueness result in [41] relies on a transformation of the conductivity equation into a time-independent Schrödinger equation. However, the Calderón problem for the dynamical linear Schrödinger equation began with the work of [4]. In this work, the time-independent electromagnetic potential was stably recovered by the Neumann data measured on a part of the boundary satisfying a geometric constraint. Subsequent work on the recovery of coefficients in the linear dynamical Schrödinger equation includes [18, 39, 7, 44, 23, 13, 8, 21, 24, 6]. Inverse problems for the dynamical Schrödinger equation with polynomial nonlinearities began with the work of [32], where the time-dependent linear and nonlinear coefficients were recovered from the source-to-solution map. Polynomial nonlinearities were also considered in [29, 30]. In [29], the authors considered the recovery of a nonlinear coefficient from the DN-map. The authors of [30] treated a nonlinear magnetic Schrödinger equation and recovered both the linear (including the magnetic and electric potentials) and nonlinear coefficients from the DN-map. The results in [32, 29, 30] apply to the setting of time-dependent coefficients and trivial initial data. In [3], a modified setting (with trivial boundary data but nontrivial initial data) was considered for the Schrödinger equation with locally analytic nonlinearities and the time-independent nonlinear coefficient was recovered from the linearized Neumann data. We mention that while the studies [29, 30, 3] only require measurements of the Neumann data on an arbitrary relatively open subset of the boundary, they require the coefficients to be known *a priori* in a neighborhood of the entire boundary.

Previous work on the Calderón problem for PDE systems with coupling through linear terms includes [36, 43, 16, 22, 17, 20, 35, 42, 9, 10]. We also mention [33] which concerns an inverse problem for a single nonlinear elliptic equation but whose analysis relies on boundary determination and unique continuation for a coupled linear elliptic system. Studies on the Calderón problem for PDE systems with coupling through nonlinear terms include [15, 14, 25, 12]. The work in [38] uses a neural network to numerically reconstruct the coefficients in a reaction-diffusion system with nonlinear coupling.

The studies in [16, 17, 22, 20] are concerned with systems of linear Schrödinger equations and they establish the stable recovery of coefficients through measurements of the solution in a subset of the interior of the domain or by measurements of the Neumann data on a subset of the boundary. In either case, the proof relies on an appropriate Carleman estimate requiring geometric conditions on the set over which measurements take place. The use of Carleman estimates in the style of [16, 17, 22, 20] (and for example [4, 39, 7, 44, 21, 3] in the case of a single equation) goes back to A. L. Bukhgeim and M. V. Klibanov in [11]. To our knowledge, the present paper is the first to study the recovery of coefficients in a system of Schrödinger equations with coupled nonlinear terms.

We conclude our literature review by noting that the strategy used in [32, 29, 14, 30, 3] to study the inverse problem for nonlinear equations is to consider derivatives of the solution with respect to small parameters in the source, boundary, or initial data. The differentiated solution satisfies a linear equation for which the inverse problem is more tractable. This linearization technique goes back to V. Isakov in [28] and was further developed in [27, 26, 19, 31].

In order to address **(IP)**, we introduce small parameters $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}$ into the boundary data of

(1.1) to consider

$$
\begin{cases} \mathcal{T}_j u_{j,k} + q_{j,k}(x,t)\left|u_{j,k}\right|^2 u_{j,k} + \tilde{q}_{j,k}(x,t)|u_{\iota(j),k}|^2 u_{j,k} = 0 \text{ on } Q, \\ u_{j,k}(x,t) = \left(\epsilon_1 f_1^{(j)} + \epsilon_2 f_2^{(j)} + \epsilon_3 f_3^{(j)}\right)(x,t) \text{ on } \Sigma, \\ u_j(x,0) = 0 \text{ on } \Omega\times\{0\}, \end{cases} \tag{1.5}
$$

where $j \in \{1,2\}$, $k \in \{1,2\}$, and

$$
f_l^{(j)} \in \mathcal{H}^{9/2,9/2}(\Sigma) := \left\{ g \in H^{9/2,9/2}(\Sigma) \,\middle|\, \partial_t^m g(\cdot,0) = 0 \text{ for integers } 0 \le m < 9/2 \right\} \tag{1.6}
$$

for $l \in \{1,2,3\}$ is real-valued. For (1.5), we assume that $p_j \in W^{11,\infty}(Q), q_{j,k} \in H^3(Q), \tilde{q}_{j,k} \in H^3(Q)$ are real-valued and that there exist constants $N_j, M_j > 0$ such that

$$
\begin{aligned} \|p_j\|_{W^{11,\infty}(Q)} &\le N_j, \\ \|q_{j,k}\|_{H^3(Q)}, \|\tilde{q}_{j,k}\|_{H^3(Q)} &\le M_j. \end{aligned} \tag{1.7}
$$

We also suppose that there exists a relatively open neighborhood $\mathcal{O} \subset \Omega$ of $\partial\Omega$ such that on $\mathcal{O}\times(0,T)$, we have

$$
\begin{aligned} p_j(x,t) &= p_j(x), \\ q_{j,1} &\equiv q_{j,2}, \\ \tilde{q}_{j,1} &\equiv \tilde{q}_{j,2}. \end{aligned} \tag{1.8}
$$

That is, the coefficient p_j is time-independent on $\mathcal{O}\times(0,T)$. Then, our main result is as follows.

Theorem 1.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n = 2,3)$ and let $\mathcal{O} \subset \Omega$ be a relatively open neighborhood of $\partial\Omega$ such that $\partial\mathcal{O}\setminus\partial\Omega$ is C^2. For* (1.5)*, suppose $p_j \in W^{11,\infty}(Q), q_{j,k} \in H^3(Q), \tilde{q}_{j,k} \in H^3(Q)$ are real-valued and that the coefficients satisfy* (1.7)-(1.8)*. Suppose that the boundary data of* (1.5) *are real-valued and satisfy* (1.6)*.*

Then, for $j \in \{1,2\}$, $l \in \{1,2,3\}$, and any $T^* \in (0,T)$, there exists $\delta_{*,j} \in (0,1)$ such that if (1.5) *is well-posed for all sufficiently small $\epsilon_l \in \mathbb{R}$, and*

$$
\left\| \partial^3_{\epsilon_1\epsilon_2\epsilon_3}\left(\partial_\nu\left(u_{j,1} - u_{j,2}\right)\right)\Big|_{\epsilon_1=\epsilon_2=\epsilon_3=0} \right\|_{L^2(\Sigma_j)} \le \delta \in (0,\delta_{*,j}), \tag{1.9}
$$

we have that

$$
\begin{aligned} \|q_{j,1} - q_{j,2}\|^2_{L^2(\Omega\setminus\mathcal{O}\times(0,T^*))} &\le C_j\left[\left(\ln\delta^{-1/2}\right)^{-1/2} + \delta^{1/2}\right], \\ \|\tilde{q}_{j,1} - \tilde{q}_{j,2}\|^2_{L^2(\Omega\setminus\mathcal{O}\times(0,T^*))} &\le \tilde{C}_j\left[\left(\ln\delta^{-1/2}\right)^{-1/2} + \delta^{1/2}\right] \end{aligned} \tag{1.10}
$$

for constants $C_j, \tilde{C}_j > 0$ that depend on $\Omega, \Gamma_j, \mathcal{O}, T, T^*, N_j, M_j$.

The proof of Theorem 1.1 relies on the linearization of (1.5) with respect to the small parameters $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}$. In particular, we consider the first-order linearization

$$
v_{j,k}^{(l)} := \partial_{\epsilon_l} u_{j,k}\big|_{\epsilon_l=0} \tag{1.11}
$$

and the third-order linearization

$$
w_{j,k}^{(1,2,3)} := \partial^3_{\epsilon_1\epsilon_2\epsilon_3} u_{j,k}\big|_{\epsilon_1=\epsilon_2=\epsilon_3=0}. \tag{1.12}
$$

The partial derivatives in (1.11)-(1.12) are taken in the sense of the mapping from the boundary data of (1.5) to the unique solution of (1.5) (the existence of these derivatives will be justified through Corollary 3.2). We will also consider $v_{0,j}$ the solution to the adjoint problem corresponding to the first-order linearization. From here, we apply the method developed in [29] to recover nonlinear coefficients. In particular, we begin by multiplying

$$w_j^{(1,2,3)} := w_{j,1}^{(1,2,3)} - w_{j,2}^{(1,2,3)} \tag{1.13}$$

with a smooth cut-off function $\chi(x)$ that will restrict the linearized equation to $\mathcal{O} \times (0,T)$. Then, we use the equation satisfied by $\chi w_j^{(1,2,3)}$ to derive an integral identity involving the products of $q_1 - q_2, \tilde{q}_1 - \tilde{q}_2$ with the functions $\left\{v_{j,k}^{(l)}, \overline{v_{0,j}}\right\}$ and the product of $[\Delta, \chi] w_j^{(1,2,3)}$ with $\overline{v_{0,j}}$. We then substitute complex geometric optics (CGO) solutions constructed for the linear Schrödinger equation in [29] in place of the functions $v_{j,k}^{(l)}$ and $v_{0,j}$ appearing in the integral identity. By carefully picking the vectors defining the exponential phases in the CGO solutions, we can derive an estimate of the Fourier transform of either $q_1 - q_2$ or $\tilde{q}_1 - \tilde{q}_2$ in terms of an integral involving $[\Delta, \chi] w_j^{(1,2,3)}$. This step uses an observation from [30] that allows one to separately recover the coefficients of polynomials of the same order. Then, we use a unique continuation estimate from [7] to introduce the linearized Neumann data defined by (1.9) through the $[\Delta, \chi] w_j^{(1,2,3)}$ term.

Remark 1.1.

(a) *Our requirement that $q_j, \tilde{q}_j \in H^3(Q)$ is related to the fact that $H^3(Q)$ is a Banach algebra*[1] *when Ω is a smooth bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. For example, we will use the Banach algebra property in the proof of the well-posedness of* (1.1) *for small boundary data. The assumption $p_j \in W^{11,\infty}(Q)$ will later ensure that the CGO solutions we use for the linear Schrödinger equation have sufficient regularity.*

(b) *The restriction that p_j is real-valued is used in the proof of an integral identity stated in Proposition 4.1. Moreover, we stipulate $p_j(x,t) = p_j(x)$ on $\mathcal{O} \times (0,T)$ in order to use a unique continuation result from* [7]. *The assumption that $p_j, q_{j,k}, \tilde{q}_{j,k}, f_l^{(j)}$ are real-valued along with the fact that $\partial\mathcal{O} \setminus \partial\Omega$ is C^2 are used in the proof of this unique continuation result. Further details about the unique continuation result and its proof are provided in Proposition 4.3 and Remark 4.3-4.4.*

(c) *One can uniquely determine the coefficient p_j by applying existing results in the literature to the first-order linearization defined by* (1.11). *This is discussed in Remark 3.1.*

(d) *The effect of the nonlinear coupling in* (1.5) *is present in the quantity δ defined by* (1.9). *Indeed, we will see in Section 3 that the function defined by* (1.12) *satisfies an equation (namely* (3.3)*) whose source term involves derivatives of the nonlinearity*

$$q_{j,k}(x,t)\,|u_{j,k}|^2\, u_{j,k} + \tilde{q}_{j,k}(x,t)|u_{\iota(j),k}|^2 u_{j,k}$$

that appears in (1.5). *As a result, δ implicitly depends on both $u_{1,k}$ and $u_{2,k}$.*

Remark 1.2. *In* (1.1), *we consider nonlinear terms that are third degree polynomials in $\{u_1, \overline{u_1}, u_2, \overline{u_2}\}$. However, the methods in the present paper can be carried over to the case of more general nonlinear terms that are locally analytic in $\{u_1, \overline{u_1}, u_2, \overline{u_2}\}$. We refer the reader to* [3] *for a treatment of such nonlinearities for a single equation.*

The rest of the paper is organized as follows. We begin by establishing the well-posedness of (1.1) for small boundary data in Section 2. In Section 3, we linearize (1.5) with respect to the small parameters $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}$. In Section 4, we prove the main result by using the previously discussed method from [29] to recover nonlinear coefficients via an integral identity for the linearized equation restricted to $\mathcal{O} \times (0,T)$. In Section 5, we outline a method from [3] based on the direct application

[1] See Theorem 4.39 in [1].

of Carleman estimates to obtain a stability estimate similar to that of Theorem 1.1 in the setting of trivial boundary data but non-trivial initial data for (1.1).

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2. The Forward Problem

We seek to show the well-posedness of (1.1) in $\left(H^3(Q)\right)^2$ in Theorem 2.3. Our argument will rely on Theorem 2.2, which addresses the well-posedness of the linear Schrödinger equation in $H^m(Q)$ for integers $m \geq 1$. We begin by noting the following result from Lemma 2.2 and Remark 2.3 in [24].

Lemma 2.1. *Suppose $p \in W^{1,\infty}(Q)$ and $F \in H^1(0,T;L^2(\Omega))$ satisfies $F(\cdot,0)=0$. Then,*

$$\begin{cases} (i\partial_t + \Delta + p(x,t))v = F(x,t) \text{ on } Q, \\ v(x,t) = 0 \text{ on } \Sigma, \\ v(x,0) = 0 \text{ on } \Omega \times \{0\}. \end{cases} \tag{2.1}$$

has a unique solution $v \in C\left([0,T];H^2(\Omega) \cap H_0^1(\Omega)\right)$ satisfying the estimates

$$\begin{aligned} \|v\|_{L^\infty(0,T;L^2(\Omega))} &\leq C\,\|F\|_{L^2(Q)}\,, \\ \|v\|_{C([0,T];H^2(\Omega))} + \|v\|_{C^1([0,T];L^2(\Omega))} &\leq C\,\|F\|_{H^1(0,T;L^2(\Omega))} \end{aligned} \tag{2.2}$$

for some constant $C = C\left(\Omega, T, \|p\|_{W^{1,\infty}(Q)}\right) > 0$.

Now we note a linear well-posedness result that is similar to Proposition 2.1 from [29] and Lemma 4 from [32]. We still provide the details of the proof to emphasize that it suffices to take $p \in W^{m,\infty}(Q)$ (whereas the authors of [29, 32] assume smoothness for the linear coefficient p).

Theorem 2.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ ($n \geq 2$). For $m \geq 1$ an integer, suppose that $p \in W^{m,\infty}(Q)$, $h \in H^m(Q)$ satisfies $\partial_t^k h(\cdot,0) = 0$ for integers $0 \leq k \leq m$, and $g \in H^{m+(3/2),m+(3/2)}(\Sigma)$ satisfies $\partial_t^m g(x,0) = 0$ for integers $0 \leq k < m+(3/2)$. Then,*

$$\begin{cases} (i\partial_t + \Delta + p(x,t))v = h(x,t) \text{ on } Q, \\ v(x,t) = g(x,t) \text{ on } \Sigma, \\ v(x,0) = 0 \text{ on } \Omega \times \{0\}. \end{cases} \tag{2.3}$$

has a unique solution $v \in H^m(Q)$ satisfying the estimate

$$\|v\|_{H^m(Q)} \leq C\left(\|g\|_{H^{m+(3/2),m+(3/2)}(\Sigma)} + \|h\|_{H^m(Q)}\right) \tag{2.4}$$

for some constant $C = C\left(\Omega, T, \|p\|_{W^{m,\infty}(Q)}\right) > 0$. Moreover, we have that $\partial_t^k v(\cdot,0) = 0$ for integers $0 \leq k \leq m$.

Proof. Using Theorem 2.3 in Chapter 4 of [34], we have that there exists $v_2 \in H^{m+2,m+2}(Q)$ such that

$$\partial_t^k v_2(\cdot,0) = 0 \tag{2.5}$$

for integers $0 \leq k < m+(3/2)$ and that $g = v_2|_\Sigma$. The same theorem in tandem with the Bounded Inverse Theorem (e.g. Theorem 8.34 in [40]) implies the inequality

$$\|v_2\|_{H^{m+2,m+2}(Q)} \leq C\|g\|_{H^{m+(3/2),m+(3/2)}(\Sigma)} \tag{2.6}$$

for some constant $C = C(\Omega, T) > 0$. We also have from Proposition 2.3 in Chapter 4 of [34] the fact that

$$\|v_2\|_{H^{m+2}(Q)} \leq C\|v_2\|_{H^{m+2,m+2}(Q)} \left(\leq C\|g\|_{H^{m+(3/2),m+(3/2)}(\Sigma)}\right) \tag{2.7}$$

for some constant $C = C(\Omega, T) > 0$. Since $p \in W^{m,\infty}(Q)$, we have that $-(i\partial_t + \Delta + p)v_2 \in H^m(Q)$. We will consider the solution $v_1(x,t)$ to

$$\begin{cases} (i\partial_t + \Delta + p(x,t))v_1 = F(x,t) := h(x,t) - (i\partial_t + \Delta + p)v_2 \text{ on } Q, \\ v_1(x,t) = 0 \text{ on } \Sigma, \\ v_1(x,0) = 0 \text{ on } \Omega \times \{0\} \end{cases} \tag{2.8}$$

and show that $v := v_1 + v_2$ satisfies the estimate in (2.4).

From Lemma 2.1, we know that (2.8) has a unique solution in $L^\infty(0,T;L^2(\Omega))$ satisfying the estimate

$$\|v_1\|_{L^\infty(0,T;L^2(\Omega))} \leq C\|F\|_{L^2(Q)} \tag{2.9}$$

for some constant $C = C\left(\Omega, T, \|p\|_{W^{1,\infty}(Q)}\right) > 0$ provided that $F(\cdot,0) = 0$. In fact, recall that we have assumed $\partial_t^k h(\cdot,0) = 0$ for integers $0 \leq k \leq m$. So, it follows from (2.5) that $\partial_t^k F(\cdot,0) = 0$ for integers $0 \leq k \leq m$.

We will now show that

$$\begin{aligned} \left\|\partial_t^\alpha \partial_x^\beta v_1\right\|_{L^2(Q)} &\leq C\left\|F\right\|_{H^m(Q)} \text{ for } 0 \leq \alpha + |\beta| \leq m, \\ \partial_t^\alpha \partial_x^\beta v_1(\cdot,0) &= 0 \text{ for } 0 \leq \alpha + |\beta| \leq m, \end{aligned} \tag{2.10}$$

where β is a multi-index and $C = C(\Omega, T, \|p\|_{W^{m,\infty}(Q)}) > 0$. Indeed, (2.10) holds when $\alpha + |\beta| = 0$ by the estimate in (2.9) and the initial data in (2.8). Seeking to apply strong induction, suppose that the statements in (2.10) hold for $0 \leq \alpha + |\beta| \leq m-1$. By differentiating (2.8) and rearranging terms, we have that when $\alpha + |\beta| = m$,

$$\partial_t^\alpha \partial_x^\beta v_1 = i\Delta\left(\partial_t^{\alpha-1}\partial_x^\beta\right)v_1 + i\left(\partial_t^{\alpha-1}\partial_x^\beta\right)(pv_1) + \partial_t^{\alpha-1}\partial_x^\beta F. \tag{2.11}$$

Thus, the induction hypothesis implies that $\partial_t^\alpha \partial_x^\beta v_1(\cdot,0) = 0$. By differentiating (2.8) again, we have (with $\alpha + |\beta| = m$) that

$$(i\partial_t + \Delta + p)\left(\partial_t^\alpha \partial_x^\beta v_1\right) = \partial_t^\alpha \partial_x^\beta F + \sum_{\alpha_0 + |\beta_0| = 0}^{m-1} C(\alpha_0, \beta_0)\left(\partial_t^{\alpha-\alpha_0}\partial_x^{\beta-\beta_0} p\right)\partial_t^{\alpha_0}\partial_x^{\beta_0} v_1. \tag{2.12}$$

The estimate in (2.10) now follows from (2.9) and the induction hypothesis. We conclude that

$$\|v_1\|_{H^m(Q)} \leq C\|F\|_{H^m(Q)} \tag{2.13}$$

for a constant $C = C(\Omega, T, \|p\|_{W^{m,\infty}(Q)}) > 0$, which combined with (2.7) yields (2.4). The fact that $\partial_t^k v(\cdot,0) = 0$ for integers $0 \leq k \leq m$ is a result of (2.5) and (2.10). $\square$

Using Theorem 2.2, we can establish the following result for the well-posedness of (1.1) with small boundary data. For later use, we define for $r > 0$ the neighborhood of zero

$$\mathcal{B}_r(0) = \mathcal{H}^{9/2,9/2}(\Sigma) \cap \left\{ g \in H^{9/2,9/2}(\Sigma) \mid \|g\|_{H^{9/2,9/2}(\Sigma)} \leq r \right\}, \tag{2.14}$$

where we recall that $\mathcal{H}^{9/2,9/2}(\Sigma)$ was defined in (1.6). We work in the space $H^m(Q)$ with $m = 3$ since this is the smallest integer for which $H^m(Q)$ is a Banach algebra when Ω is a smooth bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Theorem 2.3. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n = 2, 3)$. In (1.1), suppose that $p_j \in W^{3,\infty}(Q)$ and that $q_j, \tilde{q}_j \in H^3(Q)$. Then, there exists $r \in (0, 1)$ such that whenever $(g_1, g_2) \in (\mathcal{B}_r(0))^2$, there is a unique solution $(u_1, u_2) \in \left(H^3(Q)\right)^2$ to (1.1) satisfying*

$$\|u_j\|_{H^3(Q)} \leq C\|g_j\|_{H^{9/2,9/2}(\Sigma)} \tag{2.15}$$

for some $C > 0$. The constants C and r depend on $\Omega, T, \|p_j\|_{W^{3,\infty}(Q)}, \|q_j\|_{H^3(Q)}, \|\tilde{q}_j\|_{H^3(Q)}$. Moreover, we have that $\partial_t u_j(\cdot, 0) = \partial_t^2 u_j(\cdot, 0) = 0$.

Proof. The proof relies on a standard argument based on the Contraction Mapping Principle. We refer the readers to [3, 29, 30, 32] for related examples.

Step 1 (A Fixed Point Equation).

We seek to show that there is a solution of the form $(u_1, u_2) := (\tilde{u}_1 + w_1, \tilde{u}_2 + w_2)$ where $\tilde{u}_j$ satisfies

$$\begin{cases} \mathcal{T}_j \tilde{u}_j = 0 \text{ on } Q, \\ \tilde{u}_j(x,t) = g_j(x,t) \text{ on } \Sigma, \\ \tilde{u}_j(x,0) = 0 \text{ on } \Omega \times \{0\}. \end{cases} \tag{2.16}$$

and w_j satisfies

$$\begin{cases} \mathcal{T}_j w_j = f_j(u_1, u_2) := -q_j(x,t)\left|u_j\right|^2 u_j - \tilde{q}_j(x,t)|u_{\iota(j)}|^2 u_j \text{ on } Q, \\ w_j(x,t) = 0 \text{ on } \Sigma, \\ w_j(x,0) = 0 \text{ on } \Omega \times \{0\}. \end{cases} \tag{2.17}$$

For $g_j \in \mathcal{B}_r(0)$, Theorem 2.2 (with $m = 3$) implies that (2.16) has a unique solution in $H^3(Q)$. Moreover, we have that

$$\partial_t \tilde{u}_j(\cdot, 0) = \partial_t^2 \tilde{u}_j(\cdot, 0) = 0. \tag{2.18}$$

Now, in addition to $g_j \in \mathcal{B}_r(0)$, suppose that

$$w_j \in \mathcal{C}_r(0) := \left\{ w \in H^3(Q) \,\middle|\, \|w\|_{H^3(Q)} \leq r,\ w(\cdot,0) = \partial_t w(\cdot,0) = \partial_t^2 w(\cdot,0) = 0 \right\}, \tag{2.19}$$

where $r > 0$ will be specified later. One then has that

$$f_j(u_1(\cdot,0), u_2(\cdot,0)) = \partial_t f_j(u_1(\cdot,0), u_2(\cdot,0)) = \partial_t^2 f_j(u_1(\cdot,0), u_2(\cdot,0)) = 0 \tag{2.20}$$

and as a result, we can apply the results of Theorem 2.2 to (2.17).

Fix $g_j \in \mathcal{B}_r(0)$ and consider the unique solution to (2.17), which we denote by $S_j := S(f_j) \in H^3(Q)$. We seek to find a fixed point (w_1, w_2) such that

$$(w_1, w_2) = S(\tilde{u}_1 + w_1, \tilde{u}_2 + w_2) := (S_1(\tilde{u}_1 + w_1, \tilde{u}_2 + w_2), S_2(\tilde{u}_1 + w_1, \tilde{u}_2 + w_2)) \tag{2.21}$$

We will show that (2.21) has a fixed point by showing that the solution map

$$(w_1, w_2) \longmapsto S(\tilde{u}_1 + w_1, \tilde{u}_2 + w_2) \tag{2.22}$$

is a contraction map on $(\mathcal{C}_r(0))^2$ when $r > 0$ is sufficiently small.

Step 2 (Closure of the Solution Map).

We now check that if $(g_1, g_2) \in (\mathcal{B}_r(0))^2$ and $(w_1, w_2) \in (\mathcal{C}_r(0))^2$ for $r > 0$ sufficiently small, then $S(f_1, f_2) \in (\mathcal{C}_r(0))^2$. Now, observe from the estimate in Theorem 2.2 and the fact that $H^3(Q)$ is a Banach algebra that

$$\|S_j\|_{H^3(Q)} \leq C\||\tilde{u}_j + w_j|^2(\tilde{u}_j + w_j) + |\tilde{u}_{\iota(j)} + w_{\iota(j)}|^2(\tilde{u}_j + w_j)\|_{H^3(Q)} \leq Cr^3 \leq r \tag{2.23}$$

for an appropriate choice of $r \in (0, 1)$.

Step 3 (Verifying the Contraction Mapping Property).

Now, we will show that if $r \in (0,1)$ is sufficiently small and

$$\begin{aligned}(g_1, g_2) &\in (\mathcal{B}_r(0))^2,\\ (w_1, w_2), (W_1, W_2) &\in (\mathcal{C}_r(0))^2,\end{aligned} \tag{2.24}$$

then there exists $C \in (0,1)$ such that

$$\begin{aligned}&\|S(\tilde{u}_1 + w_1, \tilde{u}_2 + w_2) - S(\tilde{u}_1 + W_1, \tilde{u}_2 + W_2)\|_{(H^3(Q))^2}\\ &\qquad \leq C\left(\|(w_1, w_2) - (W_1, W_2)\|_{(H^3(Q))^2}\right).\end{aligned} \tag{2.25}$$

When we apply Theorem 2.2 to (2.17), we obtain the following estimate for each component of the solution map:

$$\begin{aligned}&\|S_j(\tilde{u}_1 + w_1, \tilde{u}_2 + w_2) - S_j(\tilde{u}_1 + W_1, \tilde{u}_2 + W_2)\|_{H^3(Q)}\\ &\leq C\|f_j(\tilde{u}_1 + w_1, \tilde{u}_2 + w_2) - f_j(\tilde{u}_1 + W_1, \tilde{u}_2 + W_2)\|_{H^3(Q)}.\end{aligned} \tag{2.26}$$

To further develop the estimate in (2.26), we recall the definition of f_j in (2.17) and note for example that

$$\begin{aligned}&\left\||\tilde{u}_{\iota(j)} + w_{\iota(j)}|^2(\tilde{u}_j + w_j) - |\tilde{u}_{\iota(j)} + W_{\iota(j)}|^2(\tilde{u}_j + W_j)\right\|_{H^3(Q)}\\ &\leq \left\||\tilde{u}_{\iota(j)} + w_{\iota(j)}|^2(\tilde{u}_j + w_j) - |\tilde{u}_{\iota(j)} + w_{\iota(j)}|^2(\tilde{u}_j + W_j)\right\|_{H^3(Q)}\\ &+ \left\||\tilde{u}_{\iota(j)} + w_{\iota(j)}|^2(\tilde{u}_j + W_j) - |\tilde{u}_{\iota(j)} + W_{\iota(j)}|^2(\tilde{u}_j + W_j)\right\|_{H^3(Q)}\\ &\leq Cr^2\left\|w_j - W_j\right\|_{H^3(Q)} + Cr\left\||\tilde{u}_{\iota(j)} + w_{\iota(j)}|^2 - |\tilde{u}_{\iota(j)} + W_{\iota(j)}|^2\right\|_{H^3(Q)}.\end{aligned} \tag{2.27}$$

In (2.27), we used the assumptions in (2.24) along with the Banach algebra property of $H^3(Q)$. Now, by adding and subtracting $(\tilde{u}_{\iota(j)} + w_{\iota(j)})(\overline{\tilde{u}_{\iota(j)} + W_{\iota(j)}})$ inside the last term on the right-hand side of (2.27), one can show that

$$\left\||\tilde{u}_{\iota(j)} + w_{\iota(j)}|^2 - |\tilde{u}_{\iota(j)} + W_{\iota(j)}|^2\right\|_{H^3(Q)} \leq Cr\left\|w_{\iota(j)} - W_{\iota(j)}\right\|_{H^3(Q)}. \tag{2.28}$$

Taking $r \in (0,1)$ sufficiently small and similarly estimating the other terms in (2.26) resulting from the definition of f_j, the inequality in (2.25) follows. So, the Contraction Mapping Principle implies that there is a unique fixed point $(w_1, w_2) \in (\mathcal{C}_r(0))^2$ satisfying (2.21) and thus a unique solution of the form $(u_1, u_2) = (\tilde{u}_1 + w_1, \tilde{u}_2 + w_2)$ to (1.1). From (2.18) and (2.19), we have the fact that $\partial_t u_j(\cdot, 0) = \partial_t^2 u_j(\cdot, 0) = 0$.

Step 4 (Proof of the Stability Estimate).

Note that we can use the first inequality in (2.23) along with the assumptions that

$$\begin{aligned}(g_1, g_2) &\in (\mathcal{B}_r(0))^2,\\ (w_1, w_2) &\in (\mathcal{C}_r(0))^2\end{aligned} \tag{2.29}$$

to write

$$\|w_j\|_{H^3(Q)} = \|S_j\|_{H^3(Q)} \leq Cr^2\left(\|\tilde{u}_j\|_{H^3(Q)} + \|w_j\|_{H^3(Q)}\right). \tag{2.30}$$

Upon taking $r \in (0,1)$ sufficiently small and applying Theorem 2.2 to estimate $\|\tilde{u}_j\|_{H^3(Q)}$, we obtain

$$\|w_j\|_{H^3(Q)} \leq C\|g_j\|_{H^{9/2,9/2}(\Sigma)}. \tag{2.31}$$

Now, the fact that $u_j = \tilde{u}_j + w_j$ yields the stability estimate in (2.15). We see from Theorem 2.2 and the use of the Banach algebra property to absorb contributions from $\|q_j\|_{H^3(Q)}, \|\tilde{q}_j\|_{H^3(Q)}$ that the constants C, r depend on the parameters $\Omega, T, \|p_j\|_{W^{3,\infty}(Q)}, \|q_j\|_{H^3(Q)}, \|\tilde{q}_j\|_{H^3(Q)}$. $\square$

Remark 2.1. *The estimate* (2.15) *is a special case of the more general inequality*

$$\|(u_1, u_2)\|_{(H^3(Q))^2} \leq C\|(g_1, g_2)\|_{(H^3(Q))^2}. \tag{2.32}$$

That the stability estimate (2.15) *involves only g_j rather than (g_1, g_2) is a consequence of the structure of the nonlinear term f_j in* (2.17)*. In a general coupled system, both g_1 and g_2 contribute to the solution. As an example, suppose that we instead considered a system where*

$$f_j(u_1, u_2) \coloneqq -q_j|u_j|^2 u_j + \tilde{q}_j u_{\iota(j)}^3. \tag{2.33}$$

We will now replace the nonlinearity in (1.1) *with the one defined by* (2.33) *and try to prepare an estimate similar to* (2.30)*. We begin with an estimate similar to* (2.23)*, namely*

$$\|w_j\|_{H^3(Q)} = \|S_j\|_{H^3(Q)} \leq C\||\tilde{u}_j + w_j|^2(\tilde{u}_j + w_j) + (\tilde{u}_{\iota(j)} + w_{\iota(j)})^3\|_{H^3(Q)}. \tag{2.34}$$

Now, we seek to factor r^2 as in (2.30) *using the assumptions $(g_1, g_2) \in (\mathcal{B}_r(0))^2$ and $(w_1, w_2) \in (\mathcal{C}_r(0))^2$. This yields*

$$\|w_j\|_{H^3(Q)} \leq Cr^2 \left(\|\tilde{u}_j\|_{H^3(Q)} + \|w_j\|_{H^3(Q)} + \|\tilde{u}_{\iota(j)}\|_{H^3(Q)} + \|w_{\iota(j)}\|_{H^3(Q)}\right), \tag{2.35}$$

where we majorized the terms $|\tilde{u}_j + w_j|^2$ and $(\tilde{u}_{\iota(j)} + w_{\iota(j)})^2$ from the right-hand side of (2.34) *by r^2. Applying Theorem 2.2 to $\|\tilde{u}_j\|_{H^3(Q)}$ and $\|\tilde{u}_{\iota(j)}\|_{H^3(Q)}$, we see from* (2.35) *that both g_1 and g_2 contribute to the solution in this case. Repeating the argument yields the same estimate for $\|w_{\iota(j)}\|_{H^3(Q)}$. So, adding the estimates for $\|\tilde{u}_j + w_j\|_{H^3(Q)}$ $(j = 1, 2)$ yields* (2.32).

3. Linearization for the Coupled Schrödinger System

In this section and the rest of paper, we assume that Ω is a smooth bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. We also take $j \in \{1, 2\}$, $k \in \{1, 2\}$, $l \in \{1, 2, 3\}$, and $\epsilon_l \in \mathbb{R}$. We now define $v_j^{(l)}(x, t)$ the unique solution in $H^3(Q)$ to

$$\begin{cases} (i\partial_t + \Delta + p_j(x, t))\, v_j^{(l)}(x, t) = 0 \text{ on } Q, \\ v_j^{(l)}(x, t) = f_l^{(j)}(x, t) \text{ on } \Sigma, \\ v_j^{(l)}(x, 0) = 0 \text{ on } \Omega \times \{0\}. \end{cases} \tag{3.1}$$

By Theorem 2.2, we have that

$$\partial_t v_j^{(l)}(\cdot, 0) = \partial_t^2 v_j^{(l)}(\cdot, 0) = 0. \tag{3.2}$$

We also define $w_{j,k}^{(\alpha,\beta,\gamma)}(x, t)$ the unique solution in $H^3(Q)$ to

$$\begin{cases} (i\partial_t + \Delta + p_j(x, t))\, w_{j,k}^{(\alpha,\beta,\gamma)}(x, t) = \mathcal{S}_{j,k}^{(\alpha,\beta,\gamma)}(x, t) + \tilde{\mathcal{S}}_{j,k}^{(\alpha,\beta,\gamma)}(x, t) \text{ on } Q, \\ w_{j,k}^{(\alpha,\beta,\gamma)}(x, t) = 0 \text{ on } \Sigma, \\ w_{j,k}^{(\alpha,\beta,\gamma)}(x, 0) = 0 \text{ on } \Omega \times \{0\}. \end{cases} \tag{3.3}$$

The source terms of (3.3) are given by

$$\begin{aligned} \mathcal{S}_{j,k}^{(\alpha,\beta,\gamma)}(x, t) &\coloneqq -q_{j,k} \sum_{\pi \in S_3} v_j^{(\pi(\alpha))} v_j^{(\pi(\beta))} \overline{v_j^{(\pi(\gamma))}}, \\ \tilde{\mathcal{S}}_{j,k}^{(\alpha,\beta,\gamma)}(x, t) &\coloneqq -\tilde{q}_{j,k} \sum_{\pi \in S_3} v_j^{(\pi(\alpha))} v_{\iota(j)}^{(\pi(\beta))} \overline{v_{\iota(j)}^{(\pi(\gamma))}}, \end{aligned} \tag{3.4}$$

where S_3 is the set of all permutations of $\{1,2,3\}$. Note that (3.2) ensures that we can apply Theorem 2.2 to (3.3).

Denoting

$$\mathcal{L}_j(x,t) \coloneqq \left(\epsilon_1 v_j^{(1)} + \epsilon_2 v_j^{(2)} + \epsilon_3 v_j^{(3)}\right)(x,t), \tag{3.5}$$

we note the following result which is analogous to Proposition 3.3 from [29].

Lemma 3.1. *Suppose that $p_j \in W^{3,\infty}(Q)$ and the coefficients in (1.5) satisfy (1.7). Recalling the neighborhood of $0 \in H^{9/2,9/2}(\Sigma)$ defined in (2.14), suppose that the nonlinear equation (1.5) is equipped with boundary data*

$$\epsilon_1 f_1^{(j)} + \epsilon_2 f_2^{(j)} + \epsilon_3 f_3^{(j)} \in \mathcal{B}_r(0) \tag{3.6}$$

for $r \in (0,1)$ small enough to ensure a unique solution in $H^3(Q)$ satisfying

$$\partial_t u_{j,k}(\cdot,0) = \partial_t^2 u_{j,k}(\cdot,0) = 0 \tag{3.7}$$

and the estimate (2.15) in Theorem 2.3. Then, the solution to (1.5) can be written in the following form:

$$u_{j,k}(x,t) = \mathcal{L}_j(x,t) + \frac{1}{6}\left(\sum_{\{\alpha,\beta,\gamma\}\in\{1,2,3\}} \epsilon_\alpha\epsilon_\beta\epsilon_\gamma w_{j,k}^{(\alpha,\beta,\gamma)}(x,t)\right) + \mathcal{R}_{j,k}(x,t), \tag{3.8}$$

where $\mathcal{R}_{j,k}$ satisfies the estimate

$$\|\mathcal{R}_{j,k}\|_{H^3(Q)} \le C(|\epsilon_1|+|\epsilon_2|+|\epsilon_3|)^5 \cdot \left[\sum_{l=1}^{3} \left\|f_l^{(j)}\right\|_{H^{9/2,9/2}(\Sigma)}\right]^5 \tag{3.9}$$

for some constant $C = C\left(\Omega, T, \|p_j\|_{W^{3,\infty}(Q)}, M_j\right) > 0$.

Proof. Using (3.1)-(3.4), one can check that if (3.8) holds, then $R_{j,k}$ satisfies

$$\begin{cases} \mathcal{T}_j R_{j,k} = -q_{j,k}\left(|u_{j,k}|^2 u_{j,k} - |\mathcal{L}_j|^2\mathcal{L}_j\right) - \tilde{q}_{j,k}\left(|u_{\iota(j),k}|^2 u_{j,k} - \left|\mathcal{L}_{\iota(j)}\right|^2 \mathcal{L}_j\right) \text{ on } Q, \\ R_{j,k}(x,t) = 0 \text{ on } \Sigma, \\ R_{j,k}(x,0) = 0 \text{ on } \Omega\times\{0\}. \end{cases} \tag{3.10}$$

Since $q_{j,k}, \tilde{q}_{j,k}, u_{j,k}, \mathcal{L}_j \in H^3(Q)$, the Banach algebra property of $H^3(Q)$ implies that the source term in (3.10) belongs to $H^3(Q)$. In view of (3.2) and (3.7), (3.10) has a unique solution in $H^3(Q)$ by Theorem 2.2.

The calculation leading to (3.9) makes use of the fact that $u_{j,k} - \mathcal{L}_j$ satisfies (1.5) but with trivial initial and boundary data. Then, for example in the first source term in (3.10), we can add and subtract terms to obtain

$$\begin{aligned} -q_{j,k}\left(|u_{j,k}|^2 u_{j,k} - |\mathcal{L}_j|^2\mathcal{L}_j\right) &= -q_{j,k}\Big(|u_{j,k}|^2(u_{j,k}-\mathcal{L}_j) \\ &\quad + \mathcal{L}_j u_{j,k}\left(\overline{u_{j,k}} - \mathcal{L}_j\right) + (\mathcal{L}_j)^2\left(u_{j,k} - \overline{\mathcal{L}_j}\right)\Big). \end{aligned} \tag{3.11}$$

The estimates from Theorem 2.2-2.3 say that

$$\begin{aligned}
\|u_{j,k}\|_{H^3(Q)} &\le C\sum_{l=1}^{3}|\epsilon_l|\left\|f_l^{(j)}\right\|_{H^{9/2,9/2}(\Sigma)},\\
\|\mathcal{L}_j\|_{H^3(Q)} &\le C\sum_{l=1}^{3}|\epsilon_l|\left\|f_l^{(j)}\right\|_{H^{9/2,9/2}(\Sigma)},\\
\|u_{j,k}-\mathcal{L}_j\|_{H^3(Q)} &\le C\left(\|u_{j,k}\|^3_{H^3(Q)}+\left\|u_{\iota(j),k}\right\|^2_{H^3(Q)}\|u_{j,k}\|_{H^3(Q)}\right)
\end{aligned} \tag{3.12}$$

for a constant $C=C\left(\Omega,T,\|p_j\|_{W^{3,\infty}(Q)},M_j\right)>0$. The Banach algebra property of $H^3(Q)$ and (1.7) were applied to obtain the last inequality in (3.12). If we use (3.12) to estimate the right-hand side of (3.11) while similarly treating the second source term in (3.10), we obtain the inequality in (3.9). $\square$

As a consequence of (3.8), we have the following result.

Corollary 3.2. *Suppose that the conditions of Lemma 3.1 are satisfied. Then, we have that*

$$v_j^{(l)}=\partial_{\epsilon_l}u_{j,k}\big|_{\epsilon_l=0}\quad \textit{and}\quad w_{j,k}^{(\alpha,\beta,\gamma)}=\partial^3_{\epsilon_\alpha\epsilon_\beta\epsilon_\gamma}u_{j,k}\big|_{\epsilon_\alpha=\epsilon_\beta=\epsilon_\gamma=0}, \tag{3.13}$$

where the partial derivatives are taken in the sense of the mapping from the boundary data $\epsilon_1f_1^{(j)}+\epsilon_2f_2^{(j)}+\epsilon_3f_3^{(j)}\in\mathcal{B}_r(0)$ *to the unique solution of* (1.5) *in* $H^3(Q)$.

This justifies the definitions of the first and third-order linearizations in (1.11)-(1.12).

Remark 3.1. *In view of Corollary 3.2, one can apply Theorem* 1.1 *from* [6] *to* (3.1) *in order to show that the linearized Dirichlet-to-Neumann map defined by*

$$f_l^{(j)}\in\mathcal{B}_r(0)\longmapsto\partial_\nu v_j^{(l)}\big|_{\Sigma_j} \tag{3.14}$$

uniquely determines the coefficient p_j.

4. Proof of the Inverse Stability Estimate

We will now use the linearized equation (3.3) and the method developed in [29] for the recovery of nonlinear coefficients in a single equation to prove the inverse stability estimate in Theorem 1.1.

4.1. **An Integral Identity.** We will now prepare an integral identity that will enable us to derive a Fourier transform estimate for the coefficients in (1.5). To prepare the integral identity and also to eventually apply a unique continuation result, we need to restrict the solution of (3.3) to a subset of $\mathcal{O}$ (previously defined in relation to (1.8)) away from Γ_j and to a subinterval $(0,T^*)$ where $0<T^*<T$.

To this end, we define the relatively open subsets $\mathcal{O}_l$ $(l=1,2,3)$ satisfying

$$\overline{\mathcal{O}_l}\subset\mathcal{O}_{l-1}\subset\mathcal{O}, \tag{4.1}$$

where we denote $\mathcal{O}_0:=\mathcal{O}$ in the above. We also denote $\Omega_l:=\Omega\setminus\overline{\mathcal{O}_l}$. Consider the smooth cut-off function

$$\chi(x):=\begin{cases}1,\ x\in\overline{\Omega}\setminus\mathcal{O}_2\\ 0,\ x\in\mathcal{O}_3\end{cases} \tag{4.2}$$

satisfying $0 \le \chi \le 1$.

Define $v_{0,j}(x,t)$ to be the unique solution in $H^3(Q)$ to the adjoint problem

$$\begin{cases} (i\partial_t + \Delta + p_j(x,t))\, v_{0,j} = 0 \text{ on } \Omega \times (0,T^*), \\ v_{0,j}(x,t) = f_{0,j}(x,t) \text{ on } \partial\Omega \times (0,T^*), \\ v_{0,j}(x,0) = 0 \text{ on } \Omega \times \{T^*\}, \end{cases} \tag{4.3}$$

where $f_{0,j}$ will be specified later.

Proposition 4.1. *Recalling the functions* $\mathcal{S}^{(1,2,3)}_{j,k}, \tilde{\mathcal{S}}^{(1,2,3)}_{j,k}, w^{(1,2,3)}_{j,k} \in H^3(Q)$ *defined by* (3.3)-(3.4), *consider*

$$\begin{aligned} \mathcal{S}^{(1,2,3)}_{j} &:= \mathcal{S}^{(1,2,3)}_{j,1} - \mathcal{S}^{(1,2,3)}_{j,2}, \\ \tilde{\mathcal{S}}^{(1,2,3)}_{j} &:= \tilde{\mathcal{S}}^{(1,2,3)}_{j,1} - \tilde{\mathcal{S}}^{(1,2,3)}_{j,2}, \\ w^{(1,2,3)}_{j} &:= w^{(1,2,3)}_{j,1} - w^{(1,2,3)}_{j,2}. \end{aligned} \tag{4.4}$$

We have that[2]

$$\int_0^{T^*} \int_\Omega \left[\chi \left(\mathcal{S}^{(1,2,3)}_{j} + \tilde{\mathcal{S}}^{(1,2,3)}_{j}\right) + \left([\Delta,\chi]\, w^{(1,2,3)}_{j}\right)\right] \overline{v_{0,j}}(x,t)\, dx\, dt = 0. \tag{4.5}$$

Proof. The function $\chi(x) w^{(1,2,3)}_{j}(x,t)$ satisfies (3.3) with source term

$$\chi \left(\mathcal{S}^{(1,2,3)}_{j} + \tilde{\mathcal{S}}^{(1,2,3)}_{j}\right)(x,t) + \left([\Delta,\chi]\, w^{(1,2,3)}_{j}\right)(x,t). \tag{4.6}$$

Let us now multiply both sides of the equation satisfied by $\chi w^{(1,2,3)}_{j}$ by $\overline{v_{0,j}}$ and integrate both sides over $\Omega \times (0,T^*)$. Integrating by parts on the resulting left-hand side, we obtain

$$\begin{aligned} &\int_{\Omega\times(0,T^*)} \left[(i\partial_t + \Delta + p_j)\left(\chi w^{(1,2,3)}_{j}\right)\right] \overline{v_{0,j}}(x,t)\, dx\, dt \\ &= i \int_0^{T^*} \int_\Omega \partial_t \left(\chi w^{(1,2,3)}_{j} \overline{v_{0,j}}\right) dx\, dt - i \int_0^{T^*} \int_\Omega \chi w^{(1,2,3)}_{j} \left(\partial_t \overline{v_{0,j}}\right)(x,t)\, dx\, dt \\ &+ \int_0^{T^*} \int_{\partial\Omega} \partial_\nu \left(\chi w^{(1,2,3)}_{j}\right) \overline{v_{0,j}}(x,t)\, dS\, dt - \int_0^{T^*} \left[\nabla\left(\chi w^{(1,2,3)}_{j}\right) \cdot \nabla \overline{v_{0,j}}\right](x,t)\, dx\, dt \\ &+ \int_0^{T^*} \int_\Omega \chi w^{(1,2,3)}_{j} \left(p_j \overline{v_{0,j}}\right)(x,t)\, dx\, dt. \end{aligned} \tag{4.7}$$

The boundary integral in (4.7) vanishes since $w^{(1,2,3)}_{j} \equiv 0$ on $\mathcal{O} \times (0,T^*)$ (owing to the fact that $q_{j,1} - q_{j,2} \equiv \tilde{q}_{j,1} - \tilde{q}_{j,2} \equiv 0$ on $\mathcal{O} \times (0,T^*)$ from the assumption in (1.8)).

Again integrating by parts, note that

$$\begin{aligned} \int_0^{T^*} \left[\nabla\left(\chi w^{(1,2,3)}_{j}\right) \cdot \nabla \overline{v_{0,j}}\right](x,t)\, dx\, dt &= \int_0^{T^*} \int_{\partial\Omega} \chi w^{(1,2,3)}_{j} \left(\partial_\nu \overline{v_{0,j}}\right)(x,t)\, dS\, dt \\ &- \int_0^{T^*} \int_\Omega \chi w^{(1,2,3)}_{j} \left(\Delta \overline{v_{0,j}}\right) dx\, dt. \end{aligned} \tag{4.8}$$

[2]Note that $[\Delta,\chi] = \Delta\chi + 2(\nabla\chi\cdot\nabla)$.

The boundary integral in (4.8) vanishes again using the fact that $w_j^{(1,2,3)} \equiv 0$ on $\mathcal{O} \times (0, T^*)$. Then, since $w_j^{(1,2,3)}(\cdot, 0) \equiv 0$ and $v_{0,j}(\cdot, T^*) \equiv 0$, (4.7) becomes

$$\begin{aligned} &\int_{\Omega\times(0,T^*)} \left[(i\partial_t + \Delta + p_j)\left(\chi w_j^{(1,2,3)}\right)\right] \overline{v_{0,j}}(x,t)\, dx\, dt \\ &= \int_{\Omega\times(0,T^*)} \left(\chi w_j^{(1,2,3)}\right)(-i\partial_t + \Delta + p_j)\, \overline{v_{0,j}}(x,t)\, dx\, dt. \end{aligned} \tag{4.9}$$

The right-hand side of (4.9) vanishes by (4.3) assuming that p_j is real-valued. As a result, we obtain the identity in (4.5). We point out that the calculations in (4.7)-(4.9) are justified since we had taken $\mathcal{S}_j^{(1,2,3)}, \tilde{\mathcal{S}}_j^{(1,2,3)}, w_j^{(1,2,3)} \in H^3(Q)$. □

4.2. **CGO Solutions for the Linear Schrödinger Equation.** We seek to substitute the CGO solutions for the linear Schrödinger equation

$$(i\partial_t + \Delta + p(x,t))u = 0 \tag{4.10}$$

(with vanishing data at either $t = 0$ or $t = T^*$) constructed in Section 4 of [29] into the integral identity (4.5). Denoting

$$\Phi(x,t;\, \omega) := \rho(x\cdot\omega - \rho|\omega|^2 t) \tag{4.11}$$

for a fixed parameter $\rho > 0$ and some fixed $\omega \in \mathbb{R}^n$ $(n = 2, 3)$, these solutions are of the form

$$u(x,t) = e^{i\Phi(x,t;\,\omega)} \sum_{k'=0}^{N} \rho^{-k'} a_{k'}(x,t) + r(x,t), \tag{4.12}$$

where $N > 0$ is a fixed integer.

We introduce a smooth cut-off function

$$\theta_h(t) = \begin{cases} 0, \ t \in [0,h] \cup [T^* - h, T^*] \\ 1, \ t \in [2h, T^* - 2h] \end{cases} \tag{4.13}$$

satisfying $0 \le \theta_h(t) \le 1$, where $h \in (0, \min\{1, T^*/4\})$ is chosen to ensure the estimate

$$\|\theta_h\|_{W^{j,\infty}(\mathbb{R})} \le C_j h^{-j} \tag{4.14}$$

for $j \in \mathbb{N}$. Then, in (4.12), we either take leading order amplitude

$$a_0(x,t) = \theta_h(t) \quad \text{or} \quad a_0(x,t) = \theta_h(t) e^{i(t\tau + x\cdot\xi)} \tag{4.15}$$

for some fixed $\xi \in \omega^\perp$.

In the case of $a_0 = \theta_h(t)$, we take

$$a_1(x,t) = -(1/2)|\omega|^{-2}(\partial_t\theta_h)(x\cdot\omega). \tag{4.16}$$

Now, taking a fixed $y \in \partial\Omega$, we define

$$a_{k'}(x + s\omega, t) = \frac{i}{2}\int_0^s (i\partial_t + \Delta + p) a_{k'-1}(x + \tilde{s}\omega, t)\, d\tilde{s} \quad (2 \le k' \le N) \tag{4.17}$$

for

$$x \in L := \{x \in \mathbb{R}^n \,|\, \omega\cdot(x - y) = 0\}. \tag{4.18}$$

In the case of $a_0(x,t) = \theta_h(t)e^{i(t\tau + x\cdot\xi)}$, we use (4.17) to define the higher order amplitudes for $1 \le k' \le N$. We observe from (4.13) that in any case,

$$a_{k'}(\cdot, 0) = a_{k'}(\cdot, T^*) = 0 \ (0 \le k' \le N). \tag{4.19}$$

The remainder $r(x,t)$ satisfies

$$\begin{cases} (i\partial_t + \Delta + p)r = -\rho^{-N} e^{i\Phi}(i\partial_t + \Delta + p)a_N \text{ on } Q, \\ r = 0 \text{ on } \Sigma, \\ r = 0 \text{ on } \Omega \times \{0\}, \end{cases} \tag{4.20}$$

We now establish estimates in $L^\infty(Q)$ and $H^m(Q)$, where $m \geq 1$ is an integer, for the solution constructed in (4.12). The $H^m(Q)$ estimates that we record have already been proven in Proposition 4.1 and Remark 4.1 from [29] under the assumption that the linear coefficient p is smooth. We provide the details again to emphasize that it suffices to take $p \in W^{2N+m,\infty}(Q)$, where $N > 0$ is the fixed integer in (4.12). In the rest of the paper, we shall denote

$$\langle \tau, \xi \rangle = \sqrt{1 + \tau^2 + |\xi|^2}. \tag{4.21}$$

Proposition 4.2. *Fix an integer $N > 0$ in (4.12), an integer $m \geq 1$, and let $p \in W^{2N+m,\infty}(Q)$. Further consider a fixed vector $\omega \in \mathbb{R}^n$ along with fixed parameters $h \in (0, \min\{1, T^*/4\})$ and $\tau \in \mathbb{R}$. For $a_0(x,t) = \theta_h(t)$, we have the estimates*

$$\begin{aligned} \|a_{k'}\|_{L^\infty(Q)} &\leq C h^{-k'} \ (0 \leq k' \leq N), \\ \|r\|_{L^\infty(Q)} &\leq C \rho^{-N+2} h^{-N-2} \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} \|a_{k'}\|_{H^m(Q)} &\leq \tilde{C} h^{-k'-m} \ (0 \leq k' \leq N), \\ \|r\|_{H^m(Q)} &\leq \tilde{C} \rho^{-N+2m} h^{-N-m-1}. \end{aligned} \tag{4.23}$$

For $a_0(x,t) = \theta_h(t) e^{i(t\tau + x\cdot\xi)}$, we have the estimates

$$\begin{aligned} \|a_{k'}\|_{L^\infty(Q)} &\leq C \langle \tau, \xi \rangle^{2k'} h^{-k'} \ (0 \leq k' \leq N), \\ \|r\|_{L^\infty(Q)} &\leq C \rho^{-N+2} \langle \tau, \xi \rangle^{2N+2} h^{-N-2} \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} \|a_{k'}\|_{H^m(Q)} &\leq \tilde{C} \langle \tau, \xi \rangle^{2k'+m} h^{-k'-m} \ (0 \leq k' \leq N), \\ \|r\|_{H^m(Q)} &\leq \tilde{C} \rho^{-N+2m} \langle \tau, \xi \rangle^{2N+m+2} h^{-N-m-1}. \end{aligned} \tag{4.25}$$

For the $L^\infty(Q)$ estimates in (4.22) and (4.24), we have $C = C\left(\Omega, T, \|p\|_{W^{2N+1,\infty}(Q)}\right) > 0$. For the $H^m(Q)$ estimates in (4.23) and (4.25), we have $\tilde{C} = \tilde{C}(\Omega, T, \|p\|_{W^{2N+m,\infty}(Q)}) > 0$.

Proof. Our primary tool will be the estimate of $\|\theta_h\|_{W^{j,\infty}(\mathbb{R})}$ provided in (4.14).

(I) We first consider $a_0 = \theta_h(t)$ (whence $a_1 = -(1/2)|\omega|^{-2}(\partial_t\theta_h)(x\cdot\omega)$ from (4.16)). In this case, when $k' = 0$ or 1, the amplitude estimate in (4.22) follows directly from (4.14). For $2 \leq k' \leq N$, we observe from (4.17)-(4.18) that we can write

$$a_{k'}(x,t) = \left(\frac{i}{2}\right)^{k'-1} \int_0^{s_1} \cdots \int_0^{s_{k'-1}} (i\partial_t + \Delta + p)^{k'-1} a_1(x - s_1\omega + s_{k'}\omega, t)\, ds_{k'} \cdots ds_2, \tag{4.26}$$

where $s_1 \in \mathbb{R}$ is such that $x - s_1\omega \in L$ (recall that L was defined in (4.18)). In (4.26), we note that the highest order derivative of p arises from $\Delta^{k'-2}p$ while the highest order (nonzero) derivative of

θ_h arises from $(i\partial_t)^{k'-1}a_1$. Since $h \in (0, \min\{1, T^*/4\})$, it follows from (4.14) that

$$\begin{aligned} \|a_{k'}\|_{L^\infty(Q)} &\leq Ch^{-k'}, \\ \left\|\partial_t^\alpha \partial_x^\beta a_{k'}\right\|_{L^2(Q)} &\leq \tilde{C} h^{-k'-m} \end{aligned} \tag{4.27}$$

for $0 \leq \alpha + |\beta| \leq m$ and constants $C = C\left(\Omega, \|p\|_{W^{2(N-2),\infty}(Q)}\right) > 0, \tilde{C} = \tilde{C}\left(\Omega, \|p\|_{W^{2(N-2)+m,\infty}(Q)}\right) > 0$. This completes the proofs of the amplitude estimates in (4.22)-(4.23).

To prove the remainder estimates in (4.22)-(4.23), we apply Lemma 2.1 and Theorem 2.2 to (4.20), which (when combined with the Sobolev Embedding Theorem) yield the inequalities

$$\begin{aligned} \|r\|_{L^\infty(Q)} \leq C\|r\|_{C([0,T];H^2(\Omega))} &\leq C\rho^{-N}\|e^{i\Phi}(i\partial_t + \Delta + p)a_N\|_{H^1(0,T;L^2(\Omega))}, \\ \|r\|_{H^m(Q)} &\leq C\rho^{-N}\|e^{i\Phi}(i\partial_t + \Delta + p)a_N\|_{H^m(Q)} \end{aligned} \tag{4.28}$$

for some constant $C = C\left(\Omega, T, \|p\|_{W^{m,\infty}(Q)}\right) > 0$. Upon substituting (4.26) (with $k' = N$) into $\|e^{i\Phi}(i\partial_t + \Delta + p)a_N\|_{H^1(0,T;L^2(\Omega))}$, we see that the highest order derivative of p arises from $\partial_t(\Delta a_N)$ (that is, $\partial_t\Delta(\Delta^{N-2}p)$) while the highest order (nonzero) derivative of θ_h arises from $\partial_t\left(i\partial_t\,(i\partial_t)^{N-1}a_1\right)$. We also obtain a factor of ρ^2 from $\partial_t\left(e^{i\Phi}\right)$. Likewise, in $\|e^{i\Phi}(i\partial_t + \Delta + p)a_N\|_{H^m(Q)}$, the highest order derivative of p arises from $\partial_t^\alpha\partial_x^\beta(\Delta a_N)$ $(\alpha + |\beta| = m)$ (thus from $\partial_t^\alpha\partial_x^\beta\Delta(\Delta^{N-2}p)$ $(\alpha + |\beta| = m)$) while the highest order (nonzero) derivative of θ_h arises from $\partial_t^m\left(i\partial_t\,(i\partial_t)^{N-1}a_1\right)$. We obtain a factor of ρ^{2m} from $\partial_t^m\left(e^{\Phi}\right)$. These considerations, along with (4.14) and the fact that $h \in (0, \min\{1, T^*/4\})$, yield the estimates

$$\begin{aligned} \|r\|_{L^\infty(Q)} &\leq C\rho^{-N+2}h^{-(N+2)}, \\ \|r\|_{H^m(Q)} &\leq \tilde{C}\rho^{-N+2m}h^{-(N+m+1)} \end{aligned} \tag{4.29}$$

for constants $C = C\left(\Omega, T, \|p\|_{W^{2N-1,\infty}(Q)}\right) > 0, \tilde{C} = \tilde{C}\left(\Omega, T, \|p\|_{W^{2(N-1)+m,\infty}(Q)}\right) > 0$.

(II) The proofs of (4.24)-(4.25) when $a_0 = \theta_h(t)e^{i(t\tau + x\cdot\xi)}$ are similar except that we obtain factors of $\langle\tau, \xi\rangle$ from the exponential term. In what follows, we discuss the proofs of the $\|r\|_{L^\infty(Q)}$ and $\|r\|_{H^m(Q)}$ estimates in (4.24)-(4.25). First, we observe from (4.17) that in this case

$$a_{k'}(x,t) = \left(\frac{i}{2}\right)^{k'}\int_0^{s_0}\cdots\int_0^{s_{k'-1}}(i\partial_t + \Delta + p)^{k'}a_0(x - s_0\omega + s_{k'}\omega, t)\,ds_{k'}\cdots ds_1, \tag{4.30}$$

where $s_0 \in \mathbb{R}$ is such that $x - s_0\omega \in L$.

Let us substitute (4.30) (with $k' = N$) into the first and second inequalities listed in (4.28). Note that we obtain factors of ρ^2, ρ^{2m} due to $\partial_t\left(e^{i\Phi}\right), \partial_t^m\left(e^{\Phi}\right)$ in the first and second inequalities from (4.28) respectively. In $\|e^{i\Phi}(i\partial_t + \Delta + p)a_N\|_{H^1(0,T;L^2(\Omega))}$, we have that the highest order derivative of p arises from $\partial_t(\Delta a_N)$ (thus from $\partial_t\Delta(\Delta^{N-1}p)$). The highest order derivative of a_0 with respect to t arises from $\partial_t(i\partial_t a_N)$ (thus from $\partial_t(i\partial_t^{N+1}a_0)$) while the highest order derivative of x arises from Δa_N (thus from $\Delta^{N+1}a_0$). In $\|e^{i\Phi}(i\partial_t + \Delta + p)a_N\|_{H^m(Q)}$, the highest order derivative of p arises from $\partial_t^\alpha\partial_x^\beta\Delta a_N$ $(\alpha + |\beta| = m)$ (thus from $\partial_t^\alpha\partial_x^\beta\Delta(\Delta^{N-1}p)$ $(\alpha + |\beta| = m)$). The highest order derivative of a_0 with respect to t arises from $\partial_t^m(i\partial_t a_N)$ (thus from $\partial_t^m(i\partial_t^{N+1}a_0)$) while the highest order derivative with respect to x arises from $\partial_x^\beta\Delta a_N$ $(|\beta| = m)$ (thus from $\partial_x^\beta\Delta^{N+1}a_0$ $(|\beta| = m)$). Now, by majorizing each factor of τ or ξ that appears in these derivatives by $\langle\tau, \xi\rangle$ and invoking

(4.14), we can develop the estimates in (4.28) into

$$\begin{aligned} \|r\|_{L^\infty(Q)} &\leq C\rho^{-N+2} \langle \tau, \xi \rangle^{2N+2} h^{-N-2}, \\ \|r\|_{H^m(Q)} &\leq \tilde{C}\rho^{-N+2m} \langle \tau, \xi \rangle^{2N+m+2} h^{-(N+m+1)} \end{aligned} \tag{4.31}$$

for constants $C = C\left(\Omega, T, \|p\|_{W^{2N+1,\infty}(Q)}\right) > 0$ and $\tilde{C} = \tilde{C}\left(\Omega, T, \|p\|_{W^{2N+m,\infty}(Q)}\right)$. □

Remark 4.1. *We comment on our motivation for recording $L^\infty(Q)$ estimates. We observe from (4.23) and (4.25) that an $\mathcal{O}(\rho^{-1})$ estimate for the remainder $r(x,t)$ requires $N \geq 2m+1$. On the other hand, such a remainder estimate can be fulfilled by taking $N = 3$ in (4.22) and (4.24). This smaller choice of N allows us to reduce the regularity required for p_j in the forthcoming arguments.*

4.3. **A Fourier Transform Estimate.** In what follows for the rest of the paper, we illustrate the proof of the inverse stability estimate (1.10) for $\tilde{q}_{j,k}$, where we fix some $j \in \{1,2\}$. A similar argument holds for the other nonlinear coefficients in (1.5).

We will now substitute CGO solutions of the form (4.12) for $v_j^{(l)}$ and $v_{0,j}$ into the integral identity (4.5):

$$\int_0^{T^*} \int_\Omega \left[\chi\left(\mathcal{S}_j^{(1,2,3)} + \tilde{\mathcal{S}}_j^{(1,2,3)}\right) + \left([\Delta, \chi]\, w_j^{(1,2,3)}\right)\right] \overline{v_{0,j}}\, dx\, dt = 0, \tag{4.32}$$

which implies

$$\int_0^{T^*} \int_{\Omega \setminus \mathcal{O}_3} \chi\left(\mathcal{S}_j^{(1,2,3)} + \tilde{\mathcal{S}}_j^{(1,2,3)}\right) \overline{v_{0,j}}\, dx\, dt \leq \left| \int_0^{T^*} \int_\Omega \left([\Delta, \chi]\, w_j^{(1,2,3)}\right) \overline{v_{0,j}}\, dx\, dt \right|. \tag{4.33}$$

Recall from (3.4) and (4.4) that

$$\begin{aligned} \mathcal{S}_j^{(1,2,3)}(x,t) &= -2(q_{j,1} - q_{j,2})(x,t)\left[v_j^{(1)} v_j^{(2)} \overline{v_j^{(3)}} + v_j^{(1)} v_j^{(3)} \overline{v_j^{(2)}} + v_j^{(2)} v_j^{(3)} \overline{v_j^{(1)}}\right], \\ \tilde{\mathcal{S}}_j^{(1,2,3)}(x,t) &= -(\tilde{q}_{j,1} - \tilde{q}_{j,2})(x,t)\Big[v_j^{(1)} v_{\iota(j)}^{(2)} \overline{v_{\iota(j)}^{(3)}} + v_j^{(2)} v_{\iota(j)}^{(1)} \overline{v_{\iota(j)}^{(3)}} + v_j^{(1)} v_{\iota(j)}^{(3)} \overline{v_{\iota(j)}^{(2)}} \\ &\qquad + v_j^{(3)} v_{\iota(j)}^{(1)} \overline{v_{\iota(j)}^{(2)}} + v_j^{(2)} v_{\iota(j)}^{(3)} \overline{v_{\iota(j)}^{(1)}} + v_j^{(3)} v_{\iota(j)}^{(2)} \overline{v_{\iota(j)}^{(1)}}\Big]. \end{aligned} \tag{4.34}$$

Also recall the definition of $\Phi(x,t;\, \omega)$ in (4.11) and $\theta_h(t)$ in (4.13). Denoting

$$\begin{aligned} a_j^{(l)}(x,t) &:= e^{i\Phi\left(x,t;\, \omega_j^{(l)}\right)} \theta_h(t), \\ a_{0,j}(x,t) &:= e^{i\Phi(x,t;\, \omega_{0,j})} \theta_h(t) e^{i(t\tau + x\cdot\xi_j)}, \end{aligned} \tag{4.35}$$

we shall fix $N = 3$ and put

$$v_j^{(l)}(x,t) = a_j^{(l)}(x,t) + e^{i\Phi\left(x,t;\, \omega_j^{(l)}\right)} \sum_{k'=1}^{3} \rho^{-k'} a_{k'}^{(l),j}(x,t) + r_j^{(l)}(x,t) \tag{4.36}$$

for some choice of $\omega_j^{(l)} \in \mathbb{R}^n$. For the adjoint IBVP (4.3), we put

$$v_{0,j}(x,t) = a_{0,j}(x,t) + e^{i\Phi(x,t;\, \omega_{0,j})} \sum_{k'=1}^{3} \rho^{-k'} a_{k'}^{0,j}(x,t) + r_{0,j}(x,t) \tag{4.37}$$

for some choice of $\omega_{0,j} \in \mathbb{R}^n$ and $\xi_j \in \omega_{0,j}^{\perp}$. With $N = 3$, we observe from Proposition 4.2 that

$$\begin{aligned} \left\| a_j^{(l)} \right\|_{L^\infty(Q)}, \|a_{0,j}\|_{L^\infty(Q)} &\leq C, \\ \left\| a_{k'}^{(l),j} \right\|_{L^\infty(Q)} &\leq C h^{-k'} \ (0 \leq k' \leq 3), \\ \left\| r_j^{(l)} \right\|_{L^\infty(Q)} &\leq C\rho^{-1} h^{-5}, \\ \left\| a_{k'}^{0,j} \right\|_{L^\infty(Q)} &\leq C \langle \tau, \xi_j \rangle^{2k'} h^{-k'} \ (0 \leq k' \leq 3), \\ \|r_{0,j}\|_{L^\infty(Q)} &\leq C\rho^{-1} \langle \tau, \xi_j \rangle^8 h^{-5} \end{aligned} \tag{4.38}$$

for a constant $C = C\left(\Omega, T, \|p_j\|_{W^{7,\infty}(Q)}\right) > 0$.

In what follows, we refer to $a_j^{(l)}$ and $a_{0,j}$ as *leading order amplitudes* and the other terms in (4.36)-(4.37) as *non-leading order amplitudes*. Now, observe that if we stipulate

$$\omega_j^{(1)} + \omega_{\iota(j)}^{(2)} - \omega_{\iota(j)}^{(3)} - \omega_{0,j} = 0 \quad \text{and} \quad -\left|\omega_j^{(1)}\right|^2 - \left|\omega_{\iota(j)}^{(2)}\right|^2 + \left|\omega_{\iota(j)}^{(3)}\right|^2 + |\omega_{0,j}|^2 = 0, \tag{4.39}$$

the exponential phase for the $v_j^{(1)} v_{\iota(j)}^{(2)} \overline{v_{\iota(j)}^{(3)}}\,\overline{v_{0,j}}$ term in left-hand side of (4.33) vanishes. That is, denoting $\Phi\left(\omega_j^{(l)}\right) := \Phi\left(x, t;\ \omega_j^{(l)}\right)$, we have

$$\Phi\left(\omega_j^{(1)}\right) + \Phi\left(\omega_{\iota(j)}^{(2)}\right) - \Phi\left(\omega_{\iota(j)}^{(3)}\right) - \Phi\left(\omega_{0,j}\right) = 0. \tag{4.40}$$

This results in the leading order amplitudes for that term combining to yield

$$\int_0^{T^*} \int_{\Omega \setminus \mathcal{O}_3} \chi\left(\tilde{q}_{j,2} - \tilde{q}_{j,1}\right) \theta_h^4 e^{-i(\tau t + x\cdot\xi_j)}\, dx\, dt. \tag{4.41}$$

The idea of the subsequent calculation is to isolate the above term in the left-hand side of (4.33) and to show that the other terms are of order ρ^{-1}.

We will next look at the other leading order amplitude products that result from substituting (4.36)-(4.37) into the left-hand side of (4.33). We demonstrate with an example that such terms are of order ρ^{-1}. Let us consider the product of the leading order amplitudes in the $v_j^{(2)} v_{\iota(j)}^{(1)} \overline{v_{\iota(j)}^{(3)}}\,\overline{v_{0,j}}$ term from (4.33). Denoting

$$\omega' := \omega_j^{(2)} + \omega_{\iota(j)}^{(1)} - \omega_{\iota(j)}^{(3)} - \omega_{0,j} \quad \text{and} \quad W := \left|\omega_j^{(2)}\right|^2 + \left|\omega_{\iota(j)}^{(1)}\right|^2 - \left|\omega_{\iota(j)}^{(3)}\right|^2 - |\omega_{0,j}|^2, \tag{4.42}$$

we obtain

$$\int_0^{T^*} \int_{\Omega \setminus \mathcal{O}_3} \chi\left(\tilde{q}_{j,2} - \tilde{q}_{j,1}\right) \theta_h^4 e^{i\rho(x\cdot\omega' - \rho W t)} e^{-i(\tau t + x\cdot\xi_j)}\, dx\, dt \tag{4.43}$$

for the product of the leading order amplitudes. Observe that we can guarantee $\omega' \neq 0$ if we stipulate in addition to (4.39)

$$\omega_j^{(2)} + \omega_{\iota(j)}^{(1)} \neq \omega_j^{(1)} + \omega_{\iota(j)}^{(2)}. \tag{4.44}$$

Now, denoting $\nabla' = (\partial_{x_1}, \ldots, \partial_{x_n}, \partial_t)$, we observe (as noted in Section 5 of [30]) that

$$e^{i\rho(x\cdot\omega' - \rho W t)} = \nabla' e^{i\rho(x\cdot\omega' - \rho W t)} \cdot \frac{(-i\rho\omega', i\rho^2 W)}{|(\rho\omega', \rho^2 W)|^2}. \tag{4.45}$$

Substituting (4.45) into (4.43), we may integrate by parts to obtain

$$\begin{aligned}&\int_{\partial((\Omega\setminus\mathcal{O}_3)\times(0,T^*))}\left(\nu(x)\cdot\frac{(-i\rho\omega',i\rho^2W)}{|(\rho\omega',\rho^2W)|^2}\right)\chi\left(\tilde{q}_{j,2}-\tilde{q}_{j,1}\right)[\theta_h(t)]^4\,e^{i\rho(x\cdot\omega'-\rho Wt)}e^{-i(\tau t+x\cdot\xi_j)}\,dS\\&-\int_{(\Omega\setminus\mathcal{O}_3)\times(0,T^*)}\frac{(-i\rho\omega',i\rho^2W)}{|(\rho\omega',\rho^2W)|^2}\cdot\nabla'\left[\chi\left(\tilde{q}_{j,2}-\tilde{q}_{j,1}\right)[\theta_h(t)]^4\,e^{-i(\tau t+x\cdot\xi_j)}\right]e^{i\rho(x\cdot\omega'-\rho Wt)}\,dx\,dt.\end{aligned}\tag{4.46}$$

Due to the assumption that $\tilde{q}_{j,1}\equiv\tilde{q}_{j,2}$ on $\mathcal{O}\times(0,T)$ and the definition of $\theta_h(t)$ in (4.13), the boundary integral in (4.46) vanishes. Denoting the non-boundary term in (4.46) by I, we have from the fact that $\omega'\neq 0$ that

$$|I|\le\frac{\rho^{-1}}{|(\omega',0)|}\int_{(\Omega\setminus\mathcal{O}_3)\times(0,T^*)}\left|\nabla'\left[\chi\left(\tilde{q}_{j,2}-\tilde{q}_{j,1}\right)[\theta_h(t)]^4\,e^{-i(\tau t+x\cdot\xi_j)}\right]\right|\,dx\,dt\tag{4.47}$$

By (1.7) and (4.14), we can further say that

$$|I|\le C\rho^{-1}\left(h^{-1}+\langle\tau,\xi_j\rangle\right),\tag{4.48}$$

which is what we sought to show. In (4.48), the factor of h^{-1} resulted from applying (4.14) to $\nabla'\theta_h$, the factor $\langle\tau,\xi_j\rangle$ came from $\nabla' e^{-i(\tau t+x\cdot\xi_j)}$, and the contribution from $\nabla'\left(\tilde{q}_{j,2}-\tilde{q}_{j,1}\right)$ was absorbed into the constant by (1.7). We also absorbed the contribution of $\nabla'\chi$, which depends on $\mathcal{O}_2$ and $\mathcal{O}_3$ in view of (4.2), into the constant. The leads to the parameter dependence $C=C(\omega',\mathcal{O},T^*,M_j)>0$ in (4.48).

The other leading order amplitude products in the left-hand side of (4.33) can be treated as in (4.42)-(4.48) by stipulating a condition similar to (4.44) for all the terms (except the $v_j^{(1)}v_{\iota(j)}^{(2)}\overline{v_{\iota(j)}^{(3)}}\,\overline{v_{0,j}}$ term) that appear in the left-hand side of (4.33). This is possible by stipulating (4.39) and

$$\operatorname{span}\left\{\omega_j^{(1)},\omega_{\iota(j)}^{(2)}\right\}\cap\operatorname{span}\left\{\omega_j^{(2)},\omega_j^{(3)},\omega_{\iota(j)}^{(1)},\omega_{\iota(j)}^{(3)}\right\}=\emptyset.\tag{4.49}$$

To see that this condition can be satisfied, first pick some linearly independent unit vectors $\eta_1,\eta_2\in\mathbb{R}^n$. Then, define

$$\begin{aligned}\omega_j^{(1)}&\coloneqq\eta_1,\ \omega_{\iota(j)}^{(2)}\coloneqq-\eta_1,\\\omega_j^{(2)}=\omega_j^{(3)}=\omega_{\iota(j)}^{(1)}=\omega_{\iota(j)}^{(3)}&\coloneqq\eta_2,\ \omega_{0,j}=-\eta_2.\end{aligned}\tag{4.50}$$

Remark 4.2. *Henceforth, we will indicate the dependence of the constant C on vector combinations resulting from the product of exponential phases (for example the expression $|(\omega',0)|$ in (4.47)) by writing C_j, with the understanding that we have chosen a specific set of vectors for the purpose of recovering the coefficient $\tilde{q}_j$. If we were concerned with recovering the coefficient q_j for a fixed $j\in\{1,2\}$, one could stipulate*

$$\omega_j^{(1)}+\omega_j^{(2)}-\omega_j^{(3)}-\omega_{0,j}=0\quad\text{and}\quad-\left|\omega_j^{(1)}\right|^2-\left|\omega_j^{(2)}\right|^2+\left|\omega_j^{(3)}\right|^2+|\omega_{0,j}|^2=0,\tag{4.51}$$

which ensures that the exponential phase in the $v_j^{(1)}v_j^{(2)}\overline{v_j^{(3)}}\,\overline{v_{0,j}}$ term vanishes. In order to ensure that the exponential phase for the other terms doesn't vanish, we can additionally stipulate a condition similar to (4.49) and then pick

$$\begin{aligned}\omega_j^{(1)}&\coloneqq\eta_1,\ \omega_j^{(2)}\coloneqq-\eta_1,\\\omega_{\iota(j)}^{(2)}=\omega_j^{(3)}=\omega_{\iota(j)}^{(1)}=\omega_{\iota(j)}^{(3)}&\coloneqq\eta_2,\ \omega_{0,j}=-\eta_2\end{aligned}\tag{4.52}$$

for some linearly independent vectors $\eta_1,\eta_2\in\mathbb{R}^n$.

Next, we seek to estimate terms that result from products with non-leading order amplitudes after substituting (4.36)-(4.37) into (4.33). One may use the estimates in (4.38) to majorize such terms by expressions of the form

$$C\rho^{-1}h^{-\alpha}\langle\tau,\xi_j\rangle^{\beta} \tag{4.53}$$

for $\alpha,\beta>0$ sufficiently large. Indeed, we see from the exponents in (4.38) that all such products can be majorized in magnitude by

$$C\left(\rho^{-1}h^{-5}\langle\tau,\xi_j\rangle^{8}\right)^4 \tag{4.54}$$

for a constant $C=C\left(\Omega,T,T^*,\|p_j\|_{W^{7,\infty}(Q)},M_j\right)>0$. Combining the considerations in (4.42)-(4.50), Remark 4.2, and (4.53)-(4.54), we can develop (4.33) into the estimate

$$\begin{aligned}&\int_0^{T^*}\int_{\Omega\setminus\mathcal{O}_3}\chi\left(\tilde q_{j,2}-\tilde q_{j,1}\right)\theta_h^4 e^{-i(\tau t+x\cdot\xi_j)}\,dx\,dt\\ &\le C_j\rho^{-1}h^{-20}\langle\tau,\xi_j\rangle^{32}+\left|\int_0^{T^*}\int_\Omega\left([\Delta,\chi]\,w_j^{(1,2,3)}\right)\overline{v_{0,j}}\,dx\,dt\right|\end{aligned} \tag{4.55}$$

for some constant $C_j=C_j(\Omega,\mathcal{O},T,T^*,\|p_j\|_{W^{7,\infty}(Q)},M_j)>0$. Extending χ, $\tilde q_{j,k}$, and θ_h to be zero outside Q, we obtain the estimate

$$\mathcal{F}\left[\chi\left(\tilde q_{j,2}-\tilde q_{j,1}\right)\theta_h^4\right]\le C_j\rho^{-1}h^{-20}\langle\tau,\xi_j\rangle^{32}+\left|\int_0^{T^*}\int_\Omega\left([\Delta,\chi]\,w_j^{(1,2,3)}\right)\overline{v_{0,j}}\,dx\,dt\right|, \tag{4.56}$$

where "$\mathcal{F}$" denotes the Fourier transform in the conjugate variables (τ,ξ_j).

We now seek to introduce the Neumann data through the commutator term $\left([\Delta,\chi]\,w_j^{(1,2,3)}\right)$ appearing in (4.56). For this, we will use a unique continuation result that can be established by following the proof of Lemma 2.1 in [7]. To state the result, we need to recall the subsets $\mathcal{O}_l\subset\mathcal{O}$ $(l=1,2,3)$ (and $\Omega_l=\Omega\setminus\overline{\mathcal{O}_l}$) defined in (4.1).

Proposition 4.3. *Suppose $\mathcal{O}$ is a relatively open neighborhood of $\partial\Omega$ such that $\partial\mathcal{O}\setminus\partial\Omega$ is C^2. Let $p(x,t)\in L^\infty(Q)$ with $p(x,t)=p(x)$ on $\mathcal{O}\times(0,T)$ and Γ a fixed relatively open subset of $\partial\Omega$. Suppose $u(x,t)\in H^1(0,T;H^1(\Omega))$ is a solution to*

$$\begin{cases}(i\partial_t+\Delta+p(x,t))u=g(x,t)\ \text{on } Q,\\ u(x,t)=0\ \text{on }\Sigma,\\ u(x,0)=0\ \text{on }\Omega\times\{0\},\end{cases} \tag{4.57}$$

where $g(x,t)\in L^2(Q)$ is supported in $(\Omega\setminus\mathcal{O})\times(0,T)$. Then, for any $T^*\in(0,T)$, there exist constants $C=C(\Omega,\Gamma,\mathcal{O},T^*,T)>0,\gamma_*>0$, and $m>0$ such that

$$\|u\|_{L^2(0,T^*;H^1(\Omega_3\setminus\Omega_2))}\le C\left(\gamma^{-1}+e^{m\gamma}\|\partial_\nu u\|_{L^2(\Gamma\times(0,T))}\right) \tag{4.58}$$

for any $\gamma\ge\gamma_*$.

Remark 4.3.

(a) *The proof of Proposition 4.3 is based on considering the linear Schrödinger equation satisfied by χu on $\mathcal{O}\times(0,T)$, where χ is defined by (4.2). The proof also requires extending χu to the interval $(-T,T)$ in order to later apply a Carleman estimate with a non-vanishing weight function at zero. This can done by defining*

$$u(x,t)=\overline{u(x,-t)}\quad\text{and}\quad g(x,t)=\overline{g(x,-t)}$$

for $(x,t) \in \mathcal{O} \times (-T,0)$. *Since* $p(x,t) = p(x)$ *and* $g(x,t) = 0$ *on* $\mathcal{O} \times (0,T)$, *the equation satisfied by* χu *then reduces to equation* (4.13) *in* [7]. *The rest of the proof then follows as in equations* (4.12) − (4.51) *in* [7]. *In fact, the proof in* [7] *holds under the weaker condition that* $\mathcal{O}$ *is a relatively open neighborhood of* Γ *rather than the entire boundary. However, the fact that* $\mathcal{O}$ *is a neighborhood of the entire boundary is essential for other steps towards the proof of the main result in* [7] *and our proof of Theorem 1.1. In particular, we need to know that* $q_{j,k}$ *and* $\tilde{q}_{j,k}$ *are determined near the entire boundary in order to derive the integral identity* (4.5). *The reason for the regularity requirement of* $\partial\mathcal{O} \setminus \partial\Omega$ *is explained in Remark* 4.5 *in* [3].

(b) *Strictly speaking, the inequality in Lemma* 4.6 *of* [7] *estimates* $\|u\|_{L^2(0,T_1;H^1(\Omega_3\setminus\Omega_2))}$ *for a specific choice of* $T_1 \in (0,T)$ *such that the constant in* (4.58) *is independent of* T_1. *However, the proof in* [7] *still holds for any choice of* $T^* \in (0,T)$ *and in this case, the constant that results in* (4.58) *now depends on* T^*. *Also, the estimate in* [7] *is in terms of* $\|\partial_\nu u\|_{L^2(\Gamma\times(-T,T))}$ *on the right-hand side. However, this can be majorized by a constant times* $\|\partial_\nu u\|_{L^2(\Gamma\times(0,T))}$ *in view of how we defined the extended solution in (a).*

(c) *Regarding* (4.58), *the dependence of the constant on* Γ *is not explicitly mentioned in Lemma* 2.1 *from* [7]. *However, such a dependence is apparent from the Carleman estimate (Lemma* 4.2 *in* [7]*) used to establish the Lemma* 2.1. *We refer the reader to Remark* 4.6 *in* [3] *for further details.*

Remark 4.4. *We seek to apply the estimate in* (4.58) *to the commutator term in* (4.56). *In view of the discussion in Remark 4.3(a), we seek to extend* $w_j^{(1,2,3)}$ *to* $\mathcal{O} \times (-T,T)$ *while ensuring that* (3.3) *is still satisfied. We can use the extension defined in Remark 4.3(a) if* $p_j, q_{j,k}, \tilde{q}_{j,k}$ *are real-valued and if* $v_j^{(l)}$ *(which defines the source terms in* (3.4)*) satisfies*

$$v_j^{(l)}(x,t) = \overline{v_j^{(l)}(x,-t)}$$

for $(x,t) \in \mathcal{O} \times (-T,0)$. *This requires the boundary data* $f_l^{(j)}$ *to be real-valued.*

Let us now estimate the commutator term in (4.55):

$$\begin{aligned}
&\left|\int_0^{T^*}\int_\Omega \left([\Delta,\chi]\, w_j^{(1,2,3)}\right)\overline{v_{0,j}}(x,t)\,dx\,dt\right| \\
&\leq C\left\|[\Delta,\chi]\, w_j^{(1,2,3)}\right\|_{L^2(\Omega\times(0,T^*))}\|v_{0,j}\|_{L^\infty(\Omega\times(0,T^*))} \\
&\leq C\left\|w_j^{(1,2,3)}\right\|_{L^2(0,T^*;H^1(\Omega_3\setminus\Omega_2))}\|v_{0,j}\|_{L^\infty(\Omega\times(0,T^*))} \\
&\leq C\left(\gamma_j^{-1} + e^{m_j\gamma_j}\left\|\partial_\nu w_j^{(1,2,3)}\right\|_{L^2(\Sigma_j)}\right)\|v_{0,j}\|_{L^\infty(\Omega\times(0,T^*))} \quad (\gamma_j \geq \gamma_{*,j})
\end{aligned} \tag{4.59}$$

for some constants $C = C(\Omega,\Gamma_j,\mathcal{O},T^*,T) > 0, \gamma_{*,j} > 0, m_j > 0$. The last line resulted from substituting the estimate (4.58) from Proposition 4.3.

Next, observe from (4.37) and (4.38) that

$$\begin{aligned}
\|v_{0,j}\|_{L^\infty(\Omega\times(0,T^*))} &\leq \|a_{0,j}\|_{L^\infty(\Omega\times(0,T^*))} + \sum_{k'=1}^{3}\rho^{-k'}\left\|a_{k'}^{0,j}\right\|_{L^\infty(\Omega\times(0,T^*))} + \|r_{0,j}\|_{L^\infty(\Omega\times(0,T^*))} \\
&\leq C\left(1 + \rho^{-1}h^{-5}\langle\tau,\xi_j\rangle^8\right),
\end{aligned} \tag{4.60}$$

where $C = C\left(\Omega, T, \|p\|_{W^{7,\infty}(Q)}\right) > 0$. In (4.60), we majorized each term by $\rho^{-1}h^{-\alpha}\left\langle \tau, \xi_j \right\rangle^{\beta}$ for $\alpha, \beta > 0$ sufficiently large. Substituting (4.60) into (4.59) and then substituting (4.59) into (4.56) yields the following Fourier transform estimate:

$$\begin{aligned} &\mathcal{F}\left[\chi\left(\tilde{q}_{j,2} - \tilde{q}_{j,1}\right)\theta_h^4\right] \le C_j \rho^{-1} h^{-20}\left\langle \tau, \xi_j \right\rangle^{32} \\ &+ C_j\left(\gamma_j^{-1} + e^{m_j\gamma_j}\left\|\partial_\nu w_j^{(1,2,3)}\right\|_{L^2(\Gamma\times(0,T))}\right)\left(1 + \rho^{-1}h^{-5}\left\langle \tau, \xi_j \right\rangle^8\right) \end{aligned} \tag{4.61}$$

for some constant $C_j = C_j\left(\Omega, \Gamma_j, \mathcal{O}, T, T^*, \|p_j\|_{W^{7,\infty}(Q)}, M_j\right) > 0$.

4.4. **Completing the Proof.** In (4.61), let us fix $\gamma_j > 1$ and choose $\rho := \gamma_j$. Also, pick $\tilde{m}_j \ge m_j > 0$ such that

$$e^{m_j\gamma_j}\rho^{-1} = e^{m_j\gamma_j}\gamma_j^{-1} \le e^{\tilde{m}_j\gamma_j}. \tag{4.62}$$

With these choices, (4.61) becomes

$$\begin{aligned} &\mathcal{F}\left[\chi\left(\tilde{q}_{j,2} - \tilde{q}_{j,1}\right)\theta_h^4\right] \le C_j\gamma_j^{-1}h^{-20}\left\langle \tau, \xi_j \right\rangle^{32} \\ &+ C_j\left(\gamma_j^{-1} + e^{\tilde{m}_j\gamma_j}\left\|\partial_\nu w_j^{(1,2,3)}\right\|_{L^2(\Sigma_j)}\right)\left(1 + \gamma_j^{-1}h^{-5}\left\langle \tau, \xi_j \right\rangle^8\right) \\ &\le C_j\left(\gamma_j^{-1} + e^{\tilde{m}_j\gamma_j}\left\|\partial_\nu w_j^{(1,2,3)}\right\|_{L^2(\Sigma_j)}\right)h^{-20}\left\langle \tau, \xi_j \right\rangle^{32}. \end{aligned} \tag{4.63}$$

We will now use (4.63) to estimate the $H^{-1}(\mathbb{R}^{n+1})$-norm and subsequently use interpolation to pass to the $L^2(\Omega\times(0,T^*))$-norm. To begin with, we have the following for some constant $M > 1$ that will be fixed later:

$$\begin{aligned} \left\|\chi\left(\tilde{q}_{j,1} - \tilde{q}_{j,2}\right)\theta_h^4\right\|_{H^{-1}(\mathbb{R}^{n+1})}^2 &= \int_{\langle\tau,\xi_j\rangle\le M}\left\langle \tau, \xi_j \right\rangle^{-2}\left|\mathcal{F}\left[\chi\left(\tilde{q}_{j,1} - \tilde{q}_{j,2}\right)\theta_h^4\right]\right|^2 d\tau\, d\xi_j \\ &+ \int_{\langle\tau,\xi_j\rangle> M}\left\langle \tau, \xi_j \right\rangle^{-2}\left|\mathcal{F}\left[\chi\left(\tilde{q}_{j,1} - \tilde{q}_{j,2}\right)\theta_h^4\right]\right|^2 d\tau\, d\xi_j. \end{aligned} \tag{4.64}$$

Denote the first and second integrals in (4.68) by I_1 and I_2 respectively. Upon substituting (4.63) into I_1, we have that

$$\begin{aligned} |I_1| &\le CM^{n+1}\cdot C_j\left(\gamma_j^{-1} + e^{\tilde{m}_j\gamma_j}\left\|\partial_\nu w_j^{(1,2,3)}\right\|_{L^2(\Sigma_j)}\right)^2 h^{-40}M^{64-2} \\ &\le CM^4\cdot C_j\left(\gamma_j^{-1} + e^{\tilde{m}_j\gamma_j}\left\|\partial_\nu w_j^{(1,2,3)}\right\|_{L^2(\Sigma_j)}\right)^2 h^{-40}M^{62}, \end{aligned} \tag{4.65}$$

where the factor of CM^{n+1} arises from the measure of

$$\left\{(\tau,\xi_j)\in\mathbb{R}^{n+1} \mid \left\langle \tau, \xi_j \right\rangle \le M\right\}\left(\subset\left\{(\tau,\xi_j)\in\mathbb{R}^{n+1} \mid \tau^2 + |\xi_j|^2 \le 2M^2\right\}\right). \tag{4.66}$$

Since we are in the setting of $\mathbb{R}^n$ ($n = 2, 3$), we have majorized M^{n+1} by M^4. From (1.7) and Plancherel's Theorem, we have that

$$|I_2| \le M^{-2}\cdot\left\|\chi(\tilde{q}_{j,1} - \tilde{q}_{j,2})\theta_h^4\right\|_{L^2(\mathbb{R}^{n+1})} \le CM^{-2} \tag{4.67}$$

for a constant $C = C(\Omega, \mathcal{O}, T^*, M_j) > 0$. Combining (4.65) and (4.67), we have that

$$\begin{aligned} &\left\|\chi\left(\tilde{q}_{j,1} - \tilde{q}_{j,2}\right)\theta_h^4\right\|_{H^{-1}(\mathbb{R}^{n+1})} \\ &\le \left(C_j h^{-40}M^{66}\left(\gamma_j^{-1} + e^{\tilde{m}_j\gamma_j}\left\|\partial_\nu w_j^{(1,2,3)}\right\|_{L^2(\Sigma_j)}\right)^2 + CM^{-2}\right)^{1/2}. \end{aligned} \tag{4.68}$$

Now, interpolation yields

$$\begin{aligned}\left\|(\tilde{q}_{j,1}-\tilde{q}_{j,2})\,\theta_h^4\right\|^2_{L^2(\Omega\setminus\mathcal{O}\times(0,T^*))} &\leq \left\|\chi\,(\tilde{q}_{j,1}-\tilde{q}_{j,2})\,\theta_h^4\right\|^2_{L^2(\Omega\times(0,T^*))}\\ &\leq \left\|\chi\,(\tilde{q}_{j,1}-\tilde{q}_{j,2})\,\theta_h^4\right\|_{H^{-1}(\mathbb{R}^{n+1})}\left\|\chi\,(\tilde{q}_{j,1}-\tilde{q}_{j,2})\,\theta_h^4\right\|_{H^1(\mathbb{R}^{n+1})}\\ &\leq C_jh^{-20}M^{33}\left(\gamma_j^{-1}+e^{\tilde{m}_j\gamma_j}\left\|\partial_\nu w_j^{(1,2,3)}\right\|_{L^2(\Sigma_j)}\right)\cdot h^{-1}+CM^{-1}h^{-1},\end{aligned}\tag{4.69}$$

where we used (1.7) and (4.14) in order estimate

$$\left\|\chi\,(\tilde{q}_{j,1}-\tilde{q}_{j,2})\,\theta_h^4\right\|_{H^1(\mathbb{R}^{n+1})}\leq Ch^{-1}\tag{4.70}$$

for a constant $C=C(\Omega,\mathcal{O},T^*,M_j)>0$. Thus, we have that

$$\begin{aligned}&\|\tilde{q}_{j,1}-\tilde{q}_{j,2}\|^2_{L^2(\Omega\setminus\mathcal{O}\times(0,T^*))}\\ &\leq C\left\|(\tilde{q}_{j,1}-\tilde{q}_{j,2})\,\theta_h^4\right\|^2_{L^2(\Omega\setminus\mathcal{O}\times(0,T^*))}+C\left\|(\tilde{q}_{j,1}-\tilde{q}_{j,2})\,(1-\theta_h^4)\right\|^2_{L^2(\Omega\setminus\mathcal{O}\times(0,T^*))}\\ &\leq C_jh^{-21}M^{33}\left(\gamma_j^{-1}+e^{\tilde{m}_j\gamma_j}\left\|\partial_\nu w_j^{(1,2,3)}\right\|_{L^2(\Sigma_j)}\right)+CM^{-1}h^{-1}+C_0h,\end{aligned}\tag{4.71}$$

where we estimated

$$\left\|(\tilde{q}_{j,1}-\tilde{q}_{j,2})\,(1-\theta_h^4)\right\|^2_{L^2(\Omega\setminus\mathcal{O}\times(0,T^*))}\leq C_0h\tag{4.72}$$

for a constant $C_0=C_0(T^*,M_j)>0$. This uses (1.7) and the fact from (4.13) that $1-\theta_h^4$ is supported on $[0,2h]\cup[T^*-2h,T^*]$.

Setting $h:=M^{-1/2}$ (for $M>1$ large enough to ensure $h\in(0,\min\{1,T^*/4\})$), the estimate in (4.71) becomes

$$\begin{aligned}&\|\tilde{q}_{j,1}-\tilde{q}_{j,2}\|^2_{L^2(\Omega\setminus\mathcal{O}\times(0,T^*))}\\ &\leq C_jM^{87/2}\left(\gamma_j^{-1}+e^{\tilde{m}_j\gamma_j}\left\|\partial_\nu w_j^{(1,2,3)}\right\|_{L^2(\Sigma_j)}\right)+CM^{-1/2}\end{aligned}\tag{4.73}$$

for constants $C_j=C_j\left(\Omega,\Gamma_j,\mathcal{O},T,T^*,\|p_j\|_{W^{7,\infty}(Q)},M_j\right)>0$ and $C=C(\Omega,\mathcal{O},T^*,M_j)>0$. We will also pick $M>1$ so that it satisfies

$$M^{87/2}\gamma_j^{-1}=\gamma_j^{-1/2}\iff M:=\gamma_j^{1/87}.\tag{4.74}$$

As a result, (4.73) becomes

$$\|\tilde{q}_{j,1}-\tilde{q}_{j,2}\|^2_{L^2(\Omega\setminus\mathcal{O}\times(0,T^*))}\leq C_j\left(\gamma_j^{-1/2}+e^{\tilde{m}_{1,j}\gamma_j}\left\|\partial_\nu w_j^{(1,2,3)}\right\|_{L^2(\Sigma_j)}\right)\tag{4.75}$$

where $\tilde{m}_{1,j}\geq\tilde{m}_j$ is such that $\gamma_j^{1/2}e^{\tilde{m}_j\gamma_j}\leq e^{\tilde{m}_{1,j}\gamma_j}$ and $C_j=C_j\left(\Omega,\Gamma_j,\mathcal{O},T,T^*,\|p_j\|_{W^{7,\infty}(Q)},M_j\right)>0$.

We finally consider $\delta_{*,j}\in(0,1)$ such that when

$$\left\|\partial_\nu w_j^{(1,2,3)}\right\|_{L^2(\Sigma_j)}\leq\delta\in(0,\delta_{*,j}),\tag{4.76}$$

we have that

$$\gamma_j:=\tilde{m}_{1,j}^{-1}\ln\left(\delta^{-1/2}\right)\geq\gamma_{*,j}\quad\text{and}\quad h:=M^{-1/2}=\gamma_j^{-1/174}\in(0,\min\{1,T^*/4\}).\tag{4.77}$$

Then, (4.75) becomes

$$\|\tilde{q}_{j,1}-\tilde{q}_{j,2}\|^2_{L^2(\Omega\setminus\mathcal{O}\times(0,T^*))}\leq C_j\left[\left(\ln\left(\delta^{-1/2}\right)\right)^{-1/2}+\delta^{1/2}\right],\tag{4.78}$$

when (4.76) holds. We have $C_j = C_j \left(\Omega, \Gamma_j, \mathcal{O}, T, T^*, \|p_j\|_{W^{7,\infty}(Q)}, M_j\right) > 0$ in (4.78).

The proof of (4.78) relied on the CGO solutions constructed in (4.36)-(4.37) for $v_j^{(l)}$ and $v_{0,j}$ respectively. In relation to equations (3.1) and (4.3), we shall define

$$f_l^{(j)} := v_j^{(l)}\big|_\Sigma,\ f_{0,j} := v_{0,j}|_\Sigma \tag{4.79}$$

and check that the boundary traces have sufficient regularity to ensure well-posedness by Theorem 2.2. Indeed, we have from Theorem 2.3 in Chapter 4 of [34], equations (4.36)-(4.37), the Banach algebra property of $H^m(Q)\,(m \geq 3)$, and the estimates in Proposition 4.2 that

$$\begin{aligned} \left\|f_l^{(j)}\right\|_{H^{9/2,9/2}(\Sigma)} \leq C\left\|v_j^{(l)}\right\|_{H^{5,5}(Q)} \leq C\left\|v_j^{(l)}\right\|_{H^5(Q)} \\ \leq C\left(\left\|a_j^{(l)}\right\|_{H^5(Q)} + \left\|\exp\left\{i\Phi\left(\cdot,\cdot;\omega_j^{(l)}\right)\right\}\right\|_{H^5(Q)}\sum_{k'=1}^{3}\rho^{-k'}\left\|a_{k'}^{(l),j}\right\|_{H^5(Q)} + \left\|r_j^{(l)}\right\|_{H^5(Q)}\right) \\ \leq C'\rho^{10}h^{-9} = C'\gamma_j^{1749/174} \end{aligned} \tag{4.80}$$

where we substituted $\rho = \gamma_j, h = \gamma_j^{-1/174}$, and $C' = C'(\Omega, T, N_j) > 0$ (recall that N_j is defined in (1.7)). Likewise, we can show that

$$\|f_{0,j}\|_{H^{9/2,9/2}(\Sigma)} \leq C\rho^{10}\langle\tau,\xi_j\rangle^{13}h^{-9} = C\langle\tau,\xi_j\rangle^{13}\gamma_j^{1749/174} \tag{4.81}$$

for a constant $C = C(\Omega, T, N_j) > 0$. This shows that $\{f_l^{(j)}, f_{0,j}\} \subseteq H^{9/2,9/2}(\Sigma)$. The construction of the CGO amplitude and remainder terms in Section 4.2 (in particular, recall equations (4.15), (4.17), and (4.20)) ensure that $\{f_l^{(j)}, f_{0,j}\} \subseteq \mathcal{H}^{9/2,9/2}(\Sigma)$.

We lastly check that given $\delta \in (0, \delta_{*,j})$, we can pick $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}$ sufficiently small to ensure that

$$\epsilon_1 f_1^{(j)} + \epsilon_2 f_2^{(j)} + \epsilon_3 f_3^{(j)} \in \mathcal{B}_r(0), \tag{4.82}$$

where $\mathcal{B}_r(0)$ is defined by (2.14). By (4.80), it suffices to take

$$\epsilon_l \leq \frac{r}{3C'}\gamma_j^{-1749/174} = \frac{r}{3C'}\left[\tilde{m}_{1,j}^{-1}\ln\left(\delta^{-1/2}\right)\right]^{-1749/174} \quad (l = 1, 2, 3). \tag{4.83}$$

This choice of ϵ_l ensures the well-posedness of (1.5) by Theorem 2.3.

The argument in Sections 4.3-4.4 can be repeated in a similar manner for the other nonlinear coefficients in (1.5) and this completes the proof of Theorem 1.1.

Remark 4.5. *By passing derivatives onto test functions in $C^\infty(\partial\Omega)$, we can exchange derivatives in the weak sense for the Banach space-valued mapping*

$$t \longmapsto \partial_\nu w_j^{(1,2,3)}(\cdot, t)\big|_{\partial\Omega} \in H^{3/2}(\partial\Omega),$$

which justifies (1.9).

5. Recovery by Carleman Estimates

In this section, we consider the setting in which (1.5) is equipped with trivial boundary data but non-trivial initial data and outline the main idea for deriving an estimate similar to (4.78) using the method from [3]. However, the method in [3] is only able to treat time-independent coefficients and as a result, we assume in what follows that the coefficients $p_j, q_{j,k}, \tilde{q}_{j,k}$ in (1.5) depend only on

$x \in \Omega$.

Denoting $\mathcal{T}_j := i\partial_t + \Delta + p_j(x)$, we now consider

$$\begin{cases} \mathcal{T}_j u_{j,k} + q_{j,k}(x)\left|u_{j,k}\right|^2 u_{j,k} + \tilde{q}_{j,k}(x)|u_{\iota(j),k}|^2 u_{j,k} = 0 \text{ on } Q, \\ u_{j,k}(x,t) = 0 \text{ on } \Sigma, \\ u_j(x,0) = \left(\epsilon_1 f_1^{(j)} + \epsilon_2 f_2^{(j)} + \epsilon_3 f_3^{(j)}\right)(x) \text{ on } \Omega \times \{0\}, \end{cases} \tag{5.1}$$

where $j \in \{1,2\}$, $k \in \{1,2\}$, and $f_l^{(j)} \in H^4(\Omega) \cap H_0^3(\Omega)$ for $l \in \{1,2,3\}$. We take $p_j, q_{j,k}, \tilde{q}_{j,k} \in H^2(\Omega)$. We further have the regularity assumptions

$$\begin{aligned} \|p_j\|_{H^2(\Omega)} &\leq M_0, \\ \|q_{j,1} - q_{j,2}\|_{H^4(\Omega)} &\leq M_1, \\ \|\tilde{q}_{j,1} - \tilde{q}_{j,2}\|_{H^4(\Omega)} &\leq M_2, \\ \left\|f_l^{(j)}\right\|_{H^4(\Omega)} &\leq \gamma_+ \end{aligned} \tag{5.2}$$

for some constants $M_0, M_1, M_2, \gamma_+ > 0$. Letting $\mathcal{O} \subset \Omega$ be a relatively open neighborhood of $\partial\Omega$, we also assume that

$$\begin{aligned} q_{j,1} &\equiv q_{j,2} \text{ on } \mathcal{O}, \\ \tilde{q}_{j,1} &\equiv \tilde{q}_{j,2} \text{ on } \mathcal{O}, \\ \left|f_l^{(j)}(x)\right| &\geq \gamma_- \text{ on } \Omega \setminus \mathcal{O} \end{aligned} \tag{5.3}$$

for some constant $\gamma_- > 0$. Then, we have the following result.

Theorem 5.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n = 2,3)$. Consider Γ_j $(j = 1,2)$ relatively open subsets of $\partial\Omega$ and let $\mathcal{O} \subset \Omega$ be a relatively open neighborhood of $\partial\Omega$ such that $\partial\mathcal{O} \setminus \partial\Omega$ is C^2. For* (5.1)*, suppose we have real-valued $f_l^{(j)} \in H^4(\Omega) \cap H_0^3(\Omega)$ and the conditions in* (5.2)*-*(5.3) *satisfied.*

Then, for $j \in \{1,2\}$, there exists $\delta_{*,j} > 0$ such that when

$$\delta_{j,\Gamma_j} := \left\|\partial_t \left(\partial^3_{\epsilon_1\epsilon_2\epsilon_3} \partial_\nu (u_{j,1} - u_{j,2})\Big|_{\epsilon_1=\epsilon_2=\epsilon_3=0}\right)\right\|_{L^2(\Gamma_j \times (0,T))} \leq \delta_{*,j}, \tag{5.4}$$

we have

$$\|\tilde{q}_{j,1} - \tilde{q}_{j,2}\|^2_{L^2(\Omega\setminus\mathcal{O})} \leq C_j \left[\left(\ln\left(\delta^{-1}_{j,\Gamma_j}\right)\right)^{-1} + \delta_{j,\Gamma_j}\right] \tag{5.5}$$

for a constant $C_j = C_j\left(\Omega, \Gamma_j, \mathcal{O}, M_0, M_1, M_2, T, \gamma_-, \gamma_+\right) > 0$.

Remark 5.1.

(a) *The conditions $p_j, q_{j,k}, \tilde{q}_{j,k} \in H^2(\Omega)$ and $f_l^{(j)} \in H^4(\Omega) \cap H_0^3(\Omega)$ ensure that the solutions to linearized versions of* (5.1) *(namely equations* (5.8)*-*(5.10)*) have unique solutions in $C\left([0,T]; H^2(\Omega) \cap H_0^1(\Omega)\right)$. Moreover, the conditions in* (5.2) *ensure that we can prove estimates similar to those of Lemma* 4.2 *in* [3] *for the solution to* (5.10)*.*

(b) *The reason we need the conditions in* (5.3) *is related to* (5.15)*-*(5.18) *in which we need to estimate*

$$\left\|(\tilde{q}_{j,1} - \tilde{q}_{j,2})\left[f_3^{(j)} f_1^{(\iota(j))} f_2^{(\iota(j))}\right]\right\|^2_{L^2(\Omega_1)}, \tag{5.6}$$

where we recall that Ω_l $(l = 1,2)$ was defined in relation to (4.1)*. By the Sobolev Embedding Theorem, $f_l^{(j)} \in H^4(\Omega) \cap H_0^3(\Omega)$ becomes arbitrarily small near the boundary. As a result, we* *need to stipulate that $q_{j,k}$ and $\tilde{q}_{j,k}$ are determined near the boundary. Moreover, converting (5.6) into an estimate of $\|\tilde{q}_{j,1}-\tilde{q}_{j,2}\|^2_{L^2(\Omega_1)}$ requires a lower bound for the initial data $f_l^{(j)}$ away from the boundary.*

When $\epsilon_l \in \mathbb{R}$ is sufficiently small, one can show as in [3] that (5.1) has a unique solution in $C\left([0,T];H^2(\Omega)\cap H_0^1(\Omega)\right)$. Denoting this unique solution to (5.1) by $u_{j,k}\left(x,t;\, \epsilon_1 f_1^{(j)}+\epsilon_2 f_2^{(j)}+\epsilon_3 f_3^{(j)}\right)$, one can also show that the map

$$(\epsilon_1,\epsilon_2,\epsilon_3)\in\mathbb{R}^3 \longmapsto u_{j,k}\left(x,t;\, \epsilon_1 f_1^{(j)}+\epsilon_2 f_2^{(j)}+\epsilon_3 f_3^{(j)}\right) \tag{5.7}$$

is differentiable to third order at $(\epsilon_1,\epsilon_2,\epsilon_3)=(0,0,0)$. So, we can consider

$$v_j^{(l)}(x,t) := \partial_{\epsilon_l} u_{j,k}\big|_{\epsilon_l=0} \tag{5.8}$$

and

$$w_j^{(1,2,3)}(x,t) := \partial^3_{\epsilon_1\epsilon_2\epsilon_3}\left(u_{j,1}-u_{j,2}\right)\big|_{\epsilon_1=\epsilon_2=\epsilon_3=0} \tag{5.9}$$

as in Section 3. We can show that $r_j(x,t)=\partial_t w_j^{(1,2,3)}$ satisfies

$$\begin{cases} \mathcal{T}_j r_j = \partial_t\left[\left(\mathcal{S}_{j,1}^{(1,2,3)}-\mathcal{S}_{j,2}^{(1,2,3)}\right)+\left(\tilde{\mathcal{S}}_{j,1}^{(1,2,3)}-\tilde{\mathcal{S}}_{j,2}^{(1,2,3)}\right)\right] \text{ on } Q, \\ r_j = 0 \text{ on } \Sigma, \\ r_j(x,0) = -i\left[\left(\mathcal{S}_{j,1}^{(1,2,3)}-\mathcal{S}_{j,2}^{(1,2,3)}\right)+\left(\tilde{\mathcal{S}}_{j,1}^{(1,2,3)}-\tilde{\mathcal{S}}_{j,2}^{(1,2,3)}\right)\right](x,0) \text{ on } \Omega\times\{0\}, \end{cases} \tag{5.10}$$

where we recall the notation in (3.4) for the source terms.

Remark 5.2. *We can define the solution for $t\in(-T,0)$ by taking $v_j^{(l)}(x,t)=\overline{v_j^{(l)}(x,-t)}$ (this is why we assumed $f_l^{(j)}$ was real-valued in Theorem 5.1) and $r_j(x,t)=-\overline{r_j(x,-t)}$. This allows one to bound $\|\partial r/\partial\nu\|_{L^2(\Gamma_j\times(-T,T))}$ by a constant times $\|\partial r/\partial\nu\|_{L^2(\Sigma_j)}$.*

We also need the following Carleman estimate from [7] (also see Proposition 3.3 in [3]):

Proposition 5.2. *Let $p(x)\in L^\infty(\Omega)$ with $\|p\|_{L^\infty(\Omega)}\le M$ for some $M>0$. Picking a point $x_0\notin\overline{\Omega}$ and $T^*\in(0,T)$, define $\psi(x)=|x-x_0|^2$ and*

$$\phi(x,t)=\frac{e^{2\lambda\|\psi\|_\infty}-e^{\lambda\psi(x)}}{(T^*)^2-t^2} \tag{5.11}$$

on $\Omega\times(-T^*,T^*)$, where $\|\psi\|_\infty=\|\psi\|_{L^\infty(\Omega)}$. Now, there exist $\lambda_*=\lambda_*(\Omega,T^*,\Omega_1,\Omega_2)>0$, $s_*=s_*(\Omega,T^*,\Omega_1,\Omega_2)>0$ such that for all $\lambda\ge\lambda_*$, $s\ge s_*$, and a constant $C=C(M,\Omega,T^*,s_*,\lambda_*)>0$,

$$\begin{aligned} s^3\lambda^4\|e^{-s\phi}u\|^2_{L^2(\Omega_2\times(-T^*,T^*))}+s\lambda\|e^{-s\phi}\nabla u\|^2_{L^2(\Omega_2\times(-T^*,T^*))} &\le C\|e^{-s\phi}Lu\|^2_{L^2(\Omega\times(-T^*,T^*))} \\ &+ C\left(s^3\lambda^4\|e^{-s\phi}u\|^2_{L^2(\Omega_3\setminus\Omega_2\times(-T^*,T^*))}+s\lambda\|e^{-s\phi}\nabla u\|^2_{L^2(\Omega_3\setminus\Omega_2\times(-T^*,T^*))}\right). \end{aligned} \tag{5.12}$$

The above holds for $u\in L^2\left(-T^*,T^*;\,H_0^1(\Omega)\right)$ such that $Lu\in L^2\left(\Omega\times(-T^*,T^*)\right)$.

Remark 5.3. *One can show that* (5.12) *also holds with Ω_3 in place of Ω_2 on the left-hand side (e.g. see Remark* 3.2 *in* [3]*).*

Now, note for example that if we pick initial data where $f_1^{(j)}\equiv f_2^{(j)}\equiv 0$, we have in (5.10) the initial data

$$r_j(x,0)=i\left(\tilde{q}_{j,1}-\tilde{q}_{j,2}\right)\left[f_3^{(j)}f_1^{(\iota(j))}f_2^{(\iota(j))}\right] \tag{5.13}$$

and also the fact that the source term can be written in the form $(\tilde{q}_{j,1} - \tilde{q}_{j,2})v_j$, where v_j depends only on $\tilde{\mathcal{S}}^{(1,2,3)}_{j,k}$.

We can then proceed as in the calculation after Remark 4.8 in [3] to obtain an estimate of $\|\tilde{q}_{j,1} - \tilde{q}_{j,2}\|_{L^2(\Omega)}$. In the following, we summarize the calculation while omitting the parameter dependence of the constants (this is discussed thoroughly in [3]). One can show that $e^{-s\phi}r_j(x,t)$ satisfies

$$\begin{aligned}\|e^{-s\phi(\cdot\,,0)}r_j(\cdot\,,0)\|^2_{L^2(\Omega_2)} \le C\Big(s^{-2}\left\|e^{-s\phi(\cdot,0)}\left(\tilde{q}_{j,1} - \tilde{q}_{j,2}\right)\right\|^2_{L^2(\Omega_3\times(-T^*,T^*))}\|v_j\|_{L^\infty(-T^*,T^*;H^2(\Omega_3))} \\ + s^2\|e^{-s\phi}r_j\|^2_{L^2(\Omega_3\times(-T^*,T^*))} + \|e^{-s\phi}\nabla r_j\|^2_{L^2(\Omega_3\times(-T^*,T^*))}\Big)\end{aligned} \tag{5.14}$$

for $s \ge s_0$, where $s_0 > 1$ and $C > 0$ are constants. Using estimates similar to those of Lemma 4.2 in [3] to estimate $\|v_j\|_{L^\infty(-T^*,T^*;H^2(\Omega_2))}$ and the initial data lower bound in (5.3), we further obtain

$$\begin{aligned}\left\|e^{-s\phi(\cdot,0)}\left(\tilde{q}_{j,1} - \tilde{q}_{j,2}\right)\right\|^2_{L^2(\Omega_2)} \le C\Big(s^{-2}\left\|e^{-s\phi(\cdot,0)}\left(\tilde{q}_{j,1} - \tilde{q}_{j,2}\right)\right\|^2_{L^2(\Omega_3\times(-T^*,T^*))} \\ + s^2\left\|e^{-s\phi}r_j\right\|^2_{L^2(\Omega_3\times(-T^*,T^*))} + \|e^{-s\phi}\nabla r_j\|^2_{L^2(\Omega_3\times(-T^*,T^*))}\Big)\end{aligned} \tag{5.15}$$

for $s \ge s_0$. Taking $s \ge s_0$ large enough to absorb the first term on the right-hand side of (5.15) into the left-hand side, we now obtain

$$\left\|e^{-s\phi(\cdot,0)}\left(\tilde{q}_{j,1} - \tilde{q}_{j,2}\right)\right\|^2_{L^2(\Omega_2)} \le C\Big(s^2\left\|e^{-s\phi}r_j\right\|^2_{L^2(\Omega_3\times(-T^*,T^*))} + \|e^{-s\phi}\nabla r_j\|^2_{L^2(\Omega_3\times(-T^*,T^*))}\Big). \tag{5.16}$$

Let us note the estimate

$$\phi(x,0) \le \frac{e^{2\lambda\|\psi\|_\infty}}{(T^*)^2}. \tag{5.17}$$

Now, applying (5.17), the Carleman estimate in Proposition 5.2 with fixed $\lambda \ge \max\{\lambda_*, 1\}$, the unique continuation estimate in Proposition 4.3, and the assumption that $\tilde{q}_{j,1} \equiv \tilde{q}_{j,2}$ on $\mathcal{O}$ yield

$$\begin{aligned}\|\tilde{q}_{j,1} - \tilde{q}_{j,2}\|^2_{L^2(\Omega\setminus\mathcal{O})} \le C\Big(s^{-1}\|\tilde{q}_{j,1} - \tilde{q}_{j,2}\|^2_{L^2(\Omega\setminus\mathcal{O})}\Big) \\ + Cs^2\left(\gamma_j^{-1} + e^{m_j\gamma_j}\|\partial r_j/\partial\nu\|^2_{L^2(\Sigma_j)}\right)\end{aligned} \tag{5.18}$$

for constants $C, \gamma_{*,j}, m_j > 0$ and sufficiently large $s \ge s_0, \gamma_j \ge \gamma_{*,j}$. We can again take $s \ge s_0$ large enough to absorb the first term on the right-hand side of (5.18) into the left-hand side. Now, the required result follows by defining

$$\gamma_j = m_j^{-1}\ln\|\partial r_j/\partial\nu\|^{-1}_{L^2(\Sigma_j)} \tag{5.19}$$

for

$$\|\partial r_j/\partial\nu\|_{L^2(\Sigma_j)} \le \delta_{*,j}, \tag{5.20}$$

which is chosen small enough to ensure that $\gamma_j \ge \gamma_{*,j}$. Through considerations similar to those of Remark 4.5, we can interchange the order of the derivatives in $\partial r_j/\partial\nu$ and this completes the proof of Theorem 5.1. Similar reasoning can be used to obtain estimates for the other nonlinear coefficients in (5.1).

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